NEW CHALLENGES IN THE TEACHING OF MATHEMATICS

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ABSTRACT

Does the manifold, but discrete, presence of Mathematics in many objects or services around us impose new constraints to the teaching of Mathematics? If citizens need to be comfortable in various situations with a variety of mathematical tools, the learning of Mathematics requires that one starts with simple concepts. How can one face this dilemma?

"L'école doit enseigner à analyser et à discuter les paramètres sur lesquels se fondent nos affirmations passionnelles."

Umberto ECO,
Le Monde, La Repubblica, 10.10.2001

The content of this lecture grew out of discussions with teachers and scientists. In the title one could replace "Mathematics" by other fields, as the following quote from the French philosopher Alain ETCHEGOYEN shows: *"Si la... apprend aux élèves à analyser les concepts, à raisonner de façon démonstrative et à argumenter, elle est une des disciplines qui façonnent l'"honnête homme" de jadis, le "citoyen" d'aujourd'hui, les deux étant liés."* For him of course the dots were to be replaced by *"philosophie"*. You can find Mathematics in the title of my lecture in good part because I am a professional mathematician. In putting my arguments in writing, my only ambition is to contribute to a debate. School is at risk in many societies because, in my opinion, not enough attention has been given neither to the variety of types of knowledge to which students have to be exposed there, nor to new links existing between Science and Society, nor finally to the need to position Mathematics as a human activity in the course of History.

1. How to Link Technical and Generic Knowledge?

a) Doing Mathematics and learning about Mathematics

Mathematicians tend to agree that one cannot study their discipline without actually "doing Math". This is why we are so keen on giving problems to our students. In doing so we hope to fight the misleading conception that Mathematics could be a new scholastics, when many of its concepts were born while taking up challenges coming from fields outside Mathematics.

To succeed in this, students need a certain familiarity with basic mathematical concepts and/or objects, and they must learn to manipulate them while getting some idea about their universality and their relevance. This last point needs to be further clarified since, as will be explained later, behind it lies a potentially annoying hiatus.

This very seldom leads students to the perception that beyond the mathematical exercises they struggle with lies hidden a profession. As mathematicians, we all had to face the (hard) question coming from relatives and/or friends: *"What can you do in a domain where facts do not change and everything has been known for thousands of years?"* Our situation is certainly very different from that of musicians. For them it is obvious to a wide public that the good practice of playing music can be learned through strenuous routines, and that music gets enriched through the contributions of creative composers. If one considers the percentage of our students who, later, will become mathematicians, this may appear a minor issue. For me, to the contrary, getting Mathematics recognized as a living science lies at the heart of the matter. I will say more on this later.

So far I have only touched upon technical knowledge in schools, about which of course there are very diverse opinions concerning its content, how to get it across to students and how to measure its appropriation by them. There is a lot to say on this but the point on which I would like to focus my attention is quite a different one, namely that the use of scientific knowledge in modern societies requires much more than this familiarity with simple concepts and tools. This is what I tentatively call "generic" knowledge. Because of the scientific underpinnings of many aspects of modern life, getting some understanding on how complex systems rely on knowledge has become of paramount importance for citizens to form enlightened opinions and to make independent decisions. How can school training help with that? The dilemma there is: How can one give the proper perspective on scientific issues relevant to the daily functioning of our society in the very constrained school world? This covers one specific issue: How to get the proper balance between "simplicity" and "complexity" at school?

b) Mathematics entertains special relations to language and truth

In this paragraph I will only raise two points with respect to which Mathematics appears special, namely its peculiar relation to language and its special link to truth. These two points are in some sense obvious but I do not think that they have been looked at in the proper perspective as far as the teaching of Mathematics in schools is concerned. There are mentioned here because they appear to me as possible obstacles to address the global challenge mentioned before.

Let us begin with the relation of Mathematics to language. It is well known that the mathematical language must be precise, for the good reason that the ultimate purpose of a mathematical development is to "prove" a statement. This is even sometimes the basis of jokes at the expense of mathematicians. Any imprecision opens the door to a misconception, and even the smallest one can destroy the whole edifice. One should be careful though with one point, namely that after all in a mathematical explanation one is often using ordinary language in a special way. Most of the time, ordinary and strictly mathematical expressions are mixed, forcing students to live a sort of "double life". Mathematicians should be aware that this situation is not without consequences, and does create a sense of frustration for a number of students, because they feel that their ability to express themselves has been substantially limited. This can be the basis of strong bad feelings about Mathematics on the part of a number of students. This potential handicap can even get worse in more advanced courses where names given to many concepts are purely conventional. It is a fact that most of the names are well chosen, but some choices may exaggerate the feeling that Mathematics is cut off from real life because students realize that practicing Mathematics may even require to give up the free use of language.

For the purpose of this lecture, I would like to limit the relation of Mathematics to truth to the fact that a student who masters an argument can win against his or her teacher and/or his or her classmates. Such an experience can play a major role in the structuring of the personality. It also forces students to practice the dialectics between doubt and certitude, a very healthy exercise. Other structuring effects can also be hoped for in relation with the strength of good argumentation. Evariste Galois put it in an interesting way. He proposed *"faire du raisonnement une seconde mémoire"* as possible motto for the great benefit of the mathematical training. All this has very important consequences for teachers. One of them is that their worst mistake can be to impose their views against those of students who are actually right. Mathematics has a major role to play in the training towards critical thinking. As a result, there are several instances in History where Mathematics, and/or mathematicians, were considered subversive.

2. New Links between Mathematics and Society

a) Making the link evident

It is not clear, even to some mathematicians that a great many of the mathematical notions are at work in Society around us. Moreover, our times are special. Indeed, there has never been so many instances where this happens. Very often this is through the use of a *mathematical model.* At the same time, they are very few cases at school where the notion of a model is properly introduced, and students invited to make use of it.

For me the variety of situations where mathematical notions are in action in objects and services of daily use justifies the claim that we are entering a new age for Mathematics in terms of its relations with Society. They are several aspects for this, some connected with Mathematics itself, some with the development of high technologies. Let us list some causes for this strengthening:

- the extraordinary increase in the power of *computers* now makes many more questions amenable to calculations via models;
- we are living in a society where *communications* play a major (if not dominant) role, and dealing with large amounts of data requires to think of them in mathematical terms. Mathematics needed for that purpose is sometimes sophisticated and can be of recent development; in some cases even, problems originating from dealing with these data do represent new challenges to mathematicians;
- more and more often *images* become the main object under consideration, and need to be stored, compressed and securely transmitted; this is new type of objects to be manipulated systematically by mathematicians;
- *stochastic aspects* of some phenomena have today to be taken into consideration and properly analysed, thanks to the progress of Probability Theory and of Statistics.

Let us give some specific examples, many of them having to do with complex systems (in which one must be careful with the fact that, in the long run, often secondary effects dominate primary ones):

- *telecommunication systems* are incorporating many different mathematical components to code messages, to compress data, to design cellular phone networks; etc.;
- *data collecting and accessing* have invaded, and will invade even more, our lives; think of the generalized presence of bar codes (fundamental to manage inventories), of GPS (Global Positioning System) which involves sophisticated Mathematics when one would naively think that, thanks to its satellite network, the problem to be solved is a mere Euclidean geometry one; the medical scanner is a machine whose principle is based on a mathematical theorem, the Radon transform;
- *automated systems* are hidden in very many objects of frequent uses, such as transportation means (planes, trains, buses, cars, elevators, etc.), telecommunications, and soon intelligent buildings or houses;
- *shape optimization* can be motivated either by technical reasons (improving the aerodynamics of a car, or a plane wing) or aesthetic ones. Dealing with shapes is very cumbersome experimentally. One needs to manufacture prototypes that have to be one by one tested in wind tunnels, hence the introduction of "numerical wind tunnels", i.e. pieces of software and combination of mathematical operations adjusting several parameters at once in order to improve the design.

This new situation is exemplified by the fact that today there are *mathematical products*, as they are chemical products. As professional we must acknowledge this new dimension. Mathematicians rarely do so, maybe because we are still facing the unpleasant situation that no industrial sector considers itself as a "mathematical" sector, although the finance industry is getting close to being one.

A good reason for the non obviousness of the presence of Mathematics around us is the fact that one is often tempted to focus one's attention on a concrete object, when its actual social use involves it as part of a network, something that is most of the time hidden, and rather invisible. This is typical in airplanes for example.

b) Learning about limits and detecting the impossible

Many dimensions of social life involve understanding the meaning and therefore the limits of the information one can draw from a given situation, even if it has some stochastic aspect. A typical example of this can be found in the *proper use of statistical data*. They are present in many different areas, from opinion polls to insurance estimations, from risks to forecasting. They do play an important role if one is to take seriously the task of helping citizens assume their responsibilities. The purpose is not at all to expect that a sophisticated technical training in statistics can be achieved at school, but rather to make sure that all citizens be ready to challenge some claims on the basis that they realize why these claims are either self-contradictory or impossible.

This can be coined as *a scientific approach to doubt*, which should be one of the targets given to the mathematical training at school. It has a technical side but putting it at a too technical level can obscure the issue, which is to improve the contribution school training can make to citizenship.

More broadly, school is also challenged to help future citizens to get a better apprehension of the impact of basic scientific knowledge in society. Indeed in the last part of the XXth century one could witness a number of *short-circuits*, direct connections between discoveries or innovations in research laboratories and new industrial fields. After all this is exactly how internet got started, or how the telecommunication industry boomed. There was no preexisting market. This forces to rethink the relationship between research and development, and to challenge the claim that the search for a concrete application has to be the driving force of a programme, since it is a misleading oversimplification of the real mechanism. Enough room must be kept for free thinking besides targeted research. Again such a goal will be easier to achieve if a larger number of people see more clearly how this mechanism works.

As a result, providing teachers with resources to illustrate their courses through concrete situations where notions they teach, and exercises they propose to students are put to work, becomes a very serious issue that has not yet been addressed properly in many countries, in particular at the secondary level.

3. Mathematical Sciences as Human Activities

a) How does, and did, knowledge form?

The resistance to some changes that we have been advocating in the previous paragraphs is likely to find some of its roots in a misconception on how Mathematics actually develops, and developed. A temporality was even claimed by some of us as a natural companion of the universality of Mathematics. I deeply disagree with such a statement. The need for a historic perspective on any technical knowledge is obvious. It dictates the introduction of the proper dose of History of Science in any science course, probably not as a subject in itself but rather as a facilitator of the acquisition of a new notion.

In this respect, an important role must be given to breaks in past conceptions. Indeed, they show that knowledge is not the result of a linear accumulation and requires some painful rediscussions of the heritage from the past. Such an approach is likely to provide opportunities to link the presentation of Mathematics with other disciplines, scientific or not, and to make more meaningful comparisons between the different methods at work in these disciplines.

b) What is known, what is not known and what cannot be known

As I said earlier, one of key features to hope and generate a different attitude towards Mathematics among new generations of students, is to make it perceptible to them that there are questions which presently do not have answers. Progress on them can be of different types: either they can be considered as non interesting (a highly subjective judgment of course), and as such not worthy of further investigation, or impossible to answer (realizing that some important statements in Mathematics can be proved to be non provable was one of the major achievements in XXth century Mathematics due to Kurt Gödel), or just beyond reach of present methods and concepts.

Giving some idea that there are challenges around us, and making them perceptible, and at the same time meaningful, is a challenge in itself. Today, to my knowledge, not much thought has been put towards this goal, and this lack of investment becomes a handicap in our societies where the relation of students to schools has changed a lot because of the huge amount of information on a variety of subjects they have access to outside the school system.

If we are to have a chance of convincing a large portion of the school population that Mathematics is a living science, the minimum we must achieve is to prove it has a future. We cannot take this for granted, and we have to design tools to do that.

c) The role and place of abstraction

One of the points that, in my opinion, needs to be addressed has to do with the process of *abstraction*. It has focused a lot of criticism, in fact the archetype of criticisms against Mathematics, when the nature of our science lies for the most part in it. Henri Poincaré went as far as saying *"Faire des mathématiques, c'est donner le même nom à des choses différentes."* For me, the request to make Mathematics less abstract is self-contradictory. It may be true though that we did not discuss enough, or at least make it enough evident, how the abstraction process functions within Mathematics. It does have several aspects: from realizing that a common structure is at work in different situations to coming up with the minimum formulation for it, therefore establishing[1] an ideal object.

The previous point is not at all separate from a discussion of the *axiomatic method*. Its widespread use in the teaching of Mathematics, especially at more advanced levels, confuses the issue concerning it. It is quite clear to me that its introduction is one of the achievements in the History of Mathematics. It clearly marked the independence of mathematical concepts, and forced to make precise the role that mathematical developments have to play in modelling a situation. It also made possible the fantastic expansion in the training of Mathematics that one could witness after the Second World War. Nevertheless, even if one makes the pedagogical choice of introducing some notions in a purely axiomatic manner, one is not freed from the obligation of making a connection, at some stage of the learning process, with what prompted this notion to be singled out, together with the interest or limitations of variants of it.

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[1]The choice I made of the word "establishing" in the previous sentence is deliberate in order not to take sides in the deep philosophical debate as to whether mathematical objects are "created" or "discovered", the long lasting dispute between Platonicists and Intuitionists. It is of course worthy of a thorough discussion, but to conduct it requires some technical philosophical tools that I do not want to introduce here. It could also divert us from the main points I want to discuss which are, I believe, independent from these philosophical stands.

4. A few Points as Conclusion

As said in the introduction, the main purpose of this address is to open a discussion. In my opinion, recent developments in Society require paying serious attention to new requests put to the teaching of Mathematics. Finding the best way to meet them will require many exchanges and attempts. Some of them will fail in certain circumstances and succeed in others. Understanding what makes this happen will probably force us to examine more thoroughly than we are used to the great diversity of pedagogical situations teachers face today.

In this conclusive paragraph I only offer some goals, which, I feel, have to be pursued a bit systematically. For none of them can I claim to have the right solution for achieving it.

a) Linking Mathematics to the rest of knowledge in schools

Isolating Mathematics from the rest of knowledge is for me the worst that can happen, especially in connection with other sciences. This does not mean that Mathematics does not have its own territory, specific methods, and peculiar requirements. Much to the contrary, it is in confronting the various approaches used in several areas that one has a chance of presenting Mathematics in the right perspective. Differences will stick out, and therefore an identity should emerge from this. Again, I cannot imagine that this will become a teaching in itself. It is by putting the right touches at the right moment that it is the more likely to be achieved.

At the same time, the worst would be that this link be made artificially or a necessary condition for the validation of a school work. There are indeed some topics worthy of attention at school that find their roots in Mathematics and whose development keeps you within the discipline. One must just make sure that the exposure of students to cross-disciplinary activities is big enough to make it perceptible to them that the various learning processes are indeed complementary. They all aim at understanding the world around us, and making it possible for them to put their knowledge to use in several different contexts.

b) Making sure that the knowledge relevant for all is properly integrated in curricula

Choosing material to be covered in curricula is a very delicate matter, but I feel sometimes too much attention is given to it at the expense of other aspects of the school environment that, in the long run, play an even more important role. At least this is the impression I got from participating in the elaboration of the curriculum for French high schools.

The need for coherent programmes compatible with the time allocated for the study is of course a big constraint. The introduction of new topics requires that teachers be trained, and proper documents be available. This should be thought in a much broader way as just having textbooks. One must also help teachers by providing them with documents for independent reading.

Nevertheless, efforts have to be made and competences gathered in order to make sure knowledge that has become pervasive in the understanding of how Society functions is taught at the right level. A typical example of this has to do with Statistics. Making it adequately connected to the traditional mathematical training requires some thought in the context of the present curricula in some countries. For these questions one should be careful in not taking a too technical approach, and be caught in a narrow pursuit of performance when what is at stakes is transmitting a basic, but very solid, understanding of the underpinnings and general ideas.

c) Working with teachers

None of this can be achieved if working and confident *contacts with teachers* are not established. It requires creating places where this working together can take place, forums where personal successful initiatives can be given the necessary resonance, and monographies and/or other media from which relevant information can be found.

Another issue, which can of course be the basis for a debate, is *the very purpose of the training in Mathematics in schools*. In my opinion it cannot be limited to giving the basis for future studies to those who will become professional mathematicians, when citizens, but also so many professionals, need more than ever to relate in confidence with Mathematics, even if their technical knowledge of it is limited. Having a good evaluation of what Mathematics does, and does not achieve has become very important.

This raises two questions about possible pedagogical methods. Involving students with personal projects, of a size appropriate to their level of sophistication, definitely gives them a chance to get a feeling of a more independent approach to work and, more important, to discover new connections by themselves. There is evidence that the learning effect of such experiences lasts longer than a more systematic and more technically oriented one but it can come only after a sufficient technical ability has been built. Again what is to be looked for is an optimal combination of the two. In this respect, it is sure that methods to *evaluate performances* at school have to be enriched and diversified. Much too often the teaching is completely geared by the *evaluation* schemes put in place. A political figure of the first half of last century in France, Edouard HERRIOT, is remembered for having said *"La culture, c'est ce qui reste quand on a tout oublié"*. I have the feeling that mathematicians have too often forgotten that building a mathematical culture is a responsibility that has been entrusted with them. It is indeed much broader than just training the new generation of people who are going to replace us as specialists. I am afraid that, at this moment, we, as a community, have not put enough thinking to our broad responsibilities.