STUDENTS' UNDERSTANDING OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

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ABSTRACT

Exponential and logarithmic functions are pivotal mathematical concepts that play central roles in advanced mathematics. Unfortunately, these are also concepts that give students serious difficulty. In this report, we describe a theory of how students acquire an understanding of these functions by prescribing a set of mental constructions that a student can make to develop his or her understanding of these concepts.

We analyze students' understanding of these concepts within the context of our theory. Our main result is that while all of the students in our study could compute exponents in simple cases, few students could reason about the process of exponentiation. Thus, according to our theory, these students' knowledge of exponential and logarithmic functions will be limited.

We conclude by describing instructional activities based on our theoretical analysis designed to foster students' understanding of these concepts.

1. Introduction

Exponential and logarithms functions are important concepts that play crucial roles in college mathematics courses, including calculus, differential equations, and complex analysis. Unfortunately, these are also concepts that give students considerable difficulty.

Researchers and educators alike have recognized the need to improve the way we teach exponential and logarithmic functions; both have proposed alternative instructional techniques to supplement or replace traditional instruction. (For examples, see Confrey and Smith, 1995; Rahn and Berndes, 1994; Forster, 1998). Other than these instructional techniques, our literature search has found little research on exponential and logarithmic functions in the mathematics education literature (Confrey and Smith, 1995, is a notable exception). In particular, little is known on what mental constructions students can make to develop a meaningful understanding of exponents or logarithms. The purpose of this study is to describe a theory of how students might develop their understanding of these topics and to analyze students' understanding of these concepts within the context of this theory.

This paper is organized as follows: In section 2, we propose a set a theoretical constructions that a student could make to understand the concepts of exponents and logarithms. In our view, it is critical that students be capable of understanding exponentiation as a mathematical process and exponential expressions as mathematical objects that are the result of this process. In section 3, we report an empirical study in which we investigate students' understanding of these topics within the context of our theory. Our investigations reveal that students' understanding of exponents and logarithms is rather limited and that most students are incapable of understanding exponents and logarithms as processes. In section 4, we briefly describe instruction based on our theoretical analysis designed to teach students these concepts.

2. Theoretical analysis of exponents and logarithms

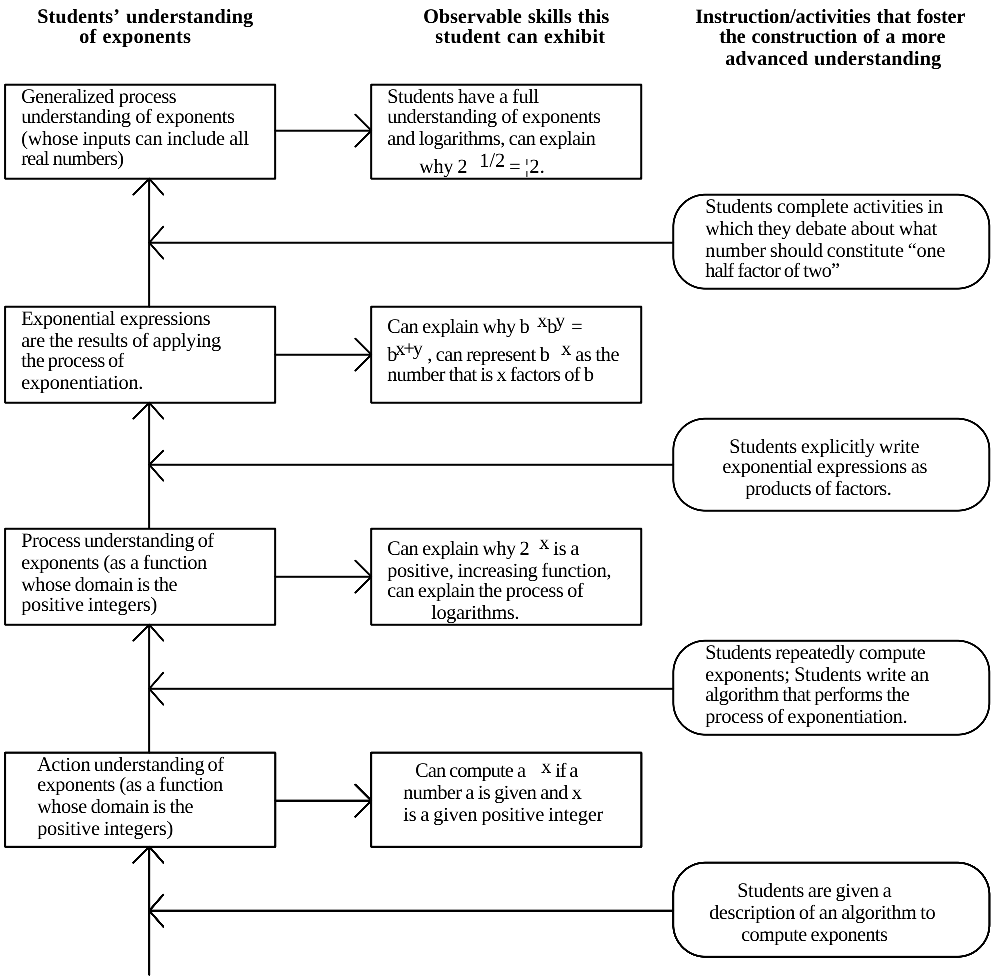
In this section, we propose a set of specific mental constructions a student might make to develop an understanding of exponents. In our view, the most plausible way that a student can learn to understand real-valued functions is to first understand exponential functions with their domain restricted to the natural numbers. The student must then generalize his or her understanding of this process to make sense of what it means to be "the product of x factors of a" when x is not a positive integer.

We present our theoretical analysis in Figure 1. In the leftmost column, we describe stages that we believe students progress through as they develop an understanding of exponential functions. In the middle column, we describe observable skills that students with each level of understanding can exhibit. We describe both of these columns in more detail below. In the right column, we propose instructional techniques to lead students to progress through these stages. We briefly describe these instructional techniques in section 4.

We use Dubinsky's APOS theory (Dubinsky, 1991) to understand how students develop their understanding of exponentiation and logarithms as functions. Our analysis of understanding exponentiation as actions and processes is very similar to Breidenbach, Dubinsky, Hawks, and Nichols (1992) analysis of how students view functions in general.

Exponentiation as an action- An action is a repeatable physical or mental transformation of objects that obtains other objects. In the case of exponents with powers that are specific positive integer coefficients, computing b^x involves repeatedly multiplying by b x times. A student limited to an action understanding of exponents will be able to evaluate exponential functions only in the cases when the power is a given positive integer. These students will not be able to do much with exponents besides compute these values and manipulate their formulas.

Figure 1. Stages of students' understanding as they develop an understanding of exponents



Exponentiation as a process- After an individual repeats an action and reflects upon it, the individual may interiorize the action as a process. Individuals with a process understanding of a concept can imagine the result of a transformation without actually performing the corresponding action, and can reverse the steps of the original transformation to obtain a new process. Students with a process understanding of exponentiation can view exponentiation as a function and reason about properties of this function (e.g. 2^x will be a positive function since you start with the integer one and repeatedly multiply this by a positive number; it will be an increasing function since every time x increases by one, 2^x doubles). They can also imagine the process obtained by reversing the steps of exponentiation to form the process of taking logarithms.

Exponential expressions as the result of a process- Terms such as 2^3 can be viewed in two distinct ways. On one hand, this can be interpreted as an external prompt for the student to compute two times two times two. However, this can also represent the output of applying exponentiation- that is, 2^3 represents the mathematical object that is the product of three factors of two. Research indicates that students are not capable of viewing 2^3 in this way (e.g. Sfard, 1991).

Representing b^x as the number that is the product of x factors of b is necessary for understanding laws of exponentiation such as $b^x b^y = b^{x+y}$. In a similar vein, students can think of computing $\log_b x$ as answering the question, "x is the product of how many factors of b?"

Generalization- Until this point, students' understanding of exponential functions only makes sense when their domain is restricted to the natural numbers. Of course, a full understanding of exponential functions involves interpreting situations where the number to be evaluated is a fraction, a negative number, or even an irrational number. To understand these situations, the student must generalize his or her understanding of b^x representing the number that is "the product of x factors of b". For instance, consider the function $f(x) = 2^x$. To interpret f(1/2), the student must make sense of what "one half factor of 2" would be. What is critical here is that students do not reason that $2^{1/2} = \ 2$ because of an arbitrary rule given by a teacher or a textbook. Rather, they should reason that 2 is the only logically consistent number that would qualify as "one half factor of 2".

3. Students' understanding of exponentiation and logarithms

In this section, we report the results of a study in which we analyze students' understanding of exponents and logarithms within the context of our theory. 15 students enrolled in a traditional pre-calculus course at a university in the southern United States volunteered to participate in this study. Three weeks after learning about exponential and logarithmic functions, the students agreed to be interviewed about these topics. In the interviews, students were asked a wide range of questions: Students were asked to recall properties of exponents and logarithms, explain why these properties were true, and to perform standard and non-standard computation. The students were also asked open-ended questions designed to probe their conceptual understanding of these topics.

Although this was not the point of this study, it should be noted that students' performance on the traditional questions was poor. For instance, when asked to simplify $b^x b^y$, only six students correctly recalled that this simplified to b^{x+y} and only six students recalled that $\log_b x + \log_b y = \log_b xy$. No student saw any connection between the previous two rules. Just eight students recalled that $x^{1/2} = \ x$ and no student could compute $\log_x x$. Not a single student could explain why any of the rules of exponents and logarithms were true.

Every participant in this study could compute 2^3 and was able to correctly specify how they would compute 7^6. Hence all students were capable of understanding exponentiation as an action. The main finding of this study was that most students could only understand exponentiation as an action and did not understand this concept as a process. We argue this point by presenting students' responses to some of our questions below.

What does the function $f(x) = a^x$ mean to you? What do you think of when you see this function?

This was an open-ended question designed to probe students' general understanding of exponential functions. One student noted that, "This is a multiplied by itself x times". Another student gave a similar response.

The rest of the responses were varied, and somewhat idiosyncratic. Examples of some of these responses are given below:

Student: This is a to the x^{th} power, where a is a constant.

Interviewer: Can you elaborate on that?

Student: No.

Student: It's a certain number raised to a certain power.

Interviewer: Can you elaborate on that?

Student: It's like suppose a was two. If x was two, it would be two times two.

Student: It's a variable raised to another variable.

Interviewer: Can you elaborate on that?

Student: Um, it's one variable taken to the power of another variable.

Interviewer: OK, can you expand on that?

Student: I don't think so. I don't know what you mean.

Besides the first two students, no students gave a response demonstrating any understanding of exponentiation as a process, or a^x as a function. In particular, unlike the first two students, no student explicitly stated how the term x was used in computing a^x without first assigning x a concrete value. If nothing else, this indicates that these students are not very articulate when speaking of exponential functions.

Is 5^{17} an even or an odd number?

Answering this question correctly requires a process understanding of exponentiation. Clearly this number cannot be explicitly computed, but one could reason that you are repeatedly multiplying by an odd number, and an odd number times an odd number is always an odd number.

Only three students answered this correctly, and they all did so by examining a small number of cases. A representative response is given below.

Student: 5 is odd. 25 is odd. 5 cubed would be… 125 which is odd. And it would keep being odd so it's odd.

Interviewer: Are you sure that it would keep being odd?

Student: Um, I think so, yeah.

Interviewer: Can you explain to me why it would keep being odd?

Student: [laughs] I don't know.

10 other students guessed that the answer was even, often conjecturing that an odd number raised to an odd number was odd and an odd number raised to an even number was even. Two students did not know how to approach this problem and refused to hazard a guess at all.

Is $f(x) = (1/2)^x$ an increasing function or a decreasing function?

All 15 students correctly answered that this was a decreasing function. Explaining why this was a decreasing function requires a process understanding of exponentiation- as x increases, you are multiplying by more factors of 1/2; hence, f(x) decreases. Only two students were able to give a mathematical explanation for why $(1/2)^x$ was a decreasing function. One student said, "Every time you multiply by 1/2, it keeps getting smaller and smaller". The other student correctly reasoned that the denominator of $(1/2)^x$ would grow as x increased, while the numerator remained constant.

10 students could not move beyond looking at specific cases (usually only x = 1 and x = 2) to determine the general behavior of $(1/2)^x$. A representative response is given below:

Student: It's a decreasing function.

Interviewer: OK, can you explain why it's decreasing?

Student: If it was like, 1/2 squared, it would be smaller than 1/2.

Interviewer: Will it always get smaller as x gets bigger?

Student: I think so.

Interviewer: Can you tell me why?

Student: I don't know.

The remaining three students appeared to know that a^x would be a decreasing function if a was a positive number less than one, but could not offer an explanation of why this was true. As one of these students said, "I'm not sure why this is decreasing. I think it has something to do with 1/2 being less than one, but don't quote me on that". (My apologies to this student for quoting him).

Suppose you didn't have a calculator. How would you go about computing $\log_5 78125$?

Answering this question requires a process understanding of exponentiation as it requires reversing its process. Correct responses might include continuously multiplying by five until you reached (or exceeded) 78,125. A more sophisticated response might involve dividing 78,125 by five repeatedly until 1 was reached (this is more akin to reversing the process of exponentiation). Unfortunately, no student gave responses of these types.

Four students knew that they must find an x such that 5^x equals 78,125, but were unable to find a way to determine what this x was. One student's response is given below:

Student: This involves solving $5^x = 78125$.

Interviewer: Do you have any ideas how you would solve such an equation?

Student: Um, a lot of trial and error?

Interviewer: OK, can you think of any other way to solve this equation?

Student: Um… no. Just trial and error.

Three students mistakenly believed that the answer would be the fifth root of 78125. The other eight students were unable to propose a way for computing its value. Clearly, the students' understanding of logarithms was quite limited.

4. Teaching suggestions and conclusions

In this section, we describe instructional designed to foster students' understanding of exponential and logarithmic functions. These activities are based primarily on our theoretical analysis reported in Section 2. As these activities have yet to be evaluated, we will mention them only briefly.

Understanding exponentiation as a process- An effective tool for leading students to interiorize an action as a process is to have them write a computer program that performs that action (Tall and Dubinsky, 1991). Our first activity involves having students program a graphing calculator to perform exponentiation (when the power is a positive integer). We do not anticipate this to be difficult, as the program is a basic "for loop". In the previous section, we report that students have difficulty explicating the role x plays in the function $f(x) = a^x$. Writing a program that performs this computation will require the students to reason about the role of the variable x. Our second activity involves having students answer basic questions which require students to view exponentiation as a process. (e.g. Why is $(-1)^x$ negative when x is odd? Why is 2^{x+1} twice as much as 2^x?) When students in our study were confronted with unfamiliar problems, they could only resort to crude symbolic techniques, such as looking at specific cases and trial-and-error. We hope that by completing these exercises, students will be introduced to a more powerful technique for thinking about exponents.

Exponential expressions as the result of a process- Students will be asked to write terms such as 2^3 as $2{\cdot}2{\cdot}2$ and "the product of three factors of two". The students will then have to use these representations to solve problems, such as to demonstrate that $2^3 2^4 = 2^7$.

Generalization- The class will discuss what it means to be "a half factor of 2". Students will propose possible values for "a half factor of 2" and analyze the validity of their choices. This will continue until students become convinced that "a half factor of 2" must be 2. Students will also discover why other properties of exponents and logarithms are true, such as why 2^{001} should be a number very close to one.

These instructional activities are currently being implemented in an experimental pre-calculus class. The effectiveness of these activities will be the subject of a future report.

In this paper, we proposed mental constructions that a student might make to develop his or her understanding of exponential and logarithmic functions. We also analyzed students' understanding of exponents and logarithms in the context of our theory only to find most students have not progressed beyond an action-level understanding of these topics. Understanding exponentiation as a function is required if one is to fully understand calculus and advanced mathematics. But understanding exponentiation as a function first requires understanding this concept as a process (Breidenbach et. al., 1992). As most students in our study were unable to view exponentiation as a process, their future in calculus is in jeopardy. Hopefully, employing our suggested instruction will better prepare our students to succeed in college mathematics.

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