IMPACT OF FORMATIVE FIELD RESEARCH WITH CHILDREN ON APPLICATIONS OF MODULO STRUCTURES UPON PREPARATION OF TEACHERS

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Key Words: Checking Arithmetic, Cast Outs, Educational Research, Preparation Teachers

ABSTRACT

Researching directly with children in school settings on the accessibility of mathematical ideas is analogous to the laboratory of a scientist where theory is discovered and validated. Discovering accessible ideas for children leads to researching potential applications for teachers and preparation of teachers. This has been central to the evolvement of research on applications of modulo structures to arithmetic over the past several years.

The authors have extended the idea of applications of modulo structures to checking arithmetic of rational numbers expressed in various numeral bases. The following problem, in base seven, has been chosen as an example because it represents a difficult problem to solve and check in rational number arithmetic: -6.0432 divided by 0.34 is –14.65 WR 0.0034, and checks with the cast out of 6 being 3. The authors have determined that the idea of cast outs is accessible to children as soon as they are conserving a one-to-one correspondence and can engage in developmental numeral structures. Also, the authors have determined several enhancing techniques for implementation of the idea as children become progressively more mathematically sophisticated, numerically. Not only can children access and apply the ideas, but also, the ideas/techniques impact conceptual understanding/applications of numeral structures. NCTM and others consider checking of arithmetic by application of calculators an abuse of technology as an educational tool.

Formative field research with teachers and pre-service teachers, based upon these applications of modulo structures, has led to significant changes in teacher preparation courses. Further, the implications suggest a renewed interest in modulo structures for pre-calculus.

The authors propose to share the fundamental accessibility of applications of modulo structures to arithmetic and how such has impacted the preparation of teachers, with implications for pre-calculus courses.

Formative Research Impacts Curriculum and Preparation of Teachers

Formative field researchers are often the avant-garde in curriculum innovations which impact teacher preparation programs. Sharing mathematical ideas directly with students in their classroom environment is the laboratory for formative research mathematics educators. Discovering accessible ideas and strategies for enhancing such accessibility requires knowing mathematics, having insights, intuition, imagination, and having lots of experience with sharing ideas with students. Some accessible ideas may be appropriate for curriculum. As professionals become aware of such ideas, then such ideas may have potential for impacting curriculum and teacher preparation programs. One such pioneer field researcher whose research impacted curriculum, the preparation of teachers and indirectly relates to the research of this paper is that of E. Glenadine Gibb (Gibb 1954, Van Engen and Gibb 1956). She perceived how children partition collections in relation to subtraction and repeated subtractions which not only impacted the teaching of subtraction but also the teaching of long division. This long division idea is related to the concept of cast outs. Field research by the authors on cast outs and applications of cast outs which impacts teacher preparation has occurred over several academic years and at all of the grade levels first through seventh. Research sessions were usually problem oriented involving guided discoveries by students with hands on manipulability, i.e. a constructionist mode. This mode is endorsed by the National Council of Teachers of Mathematics, "This constructive, active view of the learning process must be reflected in the way mathematics is taught." (NCTM 1989 p. 9). Aspects of the authors field research on applications of cast out structure have been reported over the years and recently (Edgell 2000, Edgell 2001, Edgell and Magnuson 2002).

Cast Out Structure and Applications

The cast out structure concept is imbedded in the idea of algebraic modulo structure. The cast out idea is independent of a specific base related numeral structure and can be applied at the verbal/word and tally numeral stages as well. Ordinarily though, the cast out idea is stated in terms of division by one less than a base when applied to historically and developmentally important base related numeral structures such as: simple grouping, multiplicative grouping, or base n (positional notation where co-factors are digits and powers of n which are determined by the position relative to a point which separates the non-negative powers from the negative powers) numerals. The cast out of a number, expressed in terms of a counting number base greater than two, is the remainder after dividing the number by one less than the base. This statement may symbolically be expressed:

$(B_n \overrightarrow{n-1} R) \Leftrightarrow (B_n = (n-1)Q + R, 0 \leq R < (n-1)$ and $n > 2)$. It is clear that the number expressed in terms of base n, B_n, when divided by one less than the base, $n-1$, has a non-negative remainder, R, which is the cast out of B_n. "The cast out of multiples of $(n-1)$ of B_n is R.", is represented by $B_n \overrightarrow{n-1}\, R$. When one is doing several problems in terms of one base the statement is often abbreviated to, "The cast out of B_n is R.", or $B_n \to R$. The arrow represents the equivalence relation "the cast out of multiples of $(n-1)$", which is directly related to the algebraic modulo equivalence relation of congruence. The cast out of any number expressed in base n is one of the fundamental digits of the base which is less than $n-1$.

There are lots of algorithmic strategies, other than the definition, for determining the cast out. As with algebraic modulo structure, there are important statements about cast out structure which apply to arithmetic operations such as: the cast out of the sum of the cast outs of addends is equal to the cast out of the sum, and, the cast out of the product of the cast outs of factors is equal to the cast out of the product. Since subtraction is defined directly in terms of addition and division is defined directly in terms of multiplication, the process of cast outs easily extends to these operations in terms of corresponding definitions. That is: the cast out of the sum of the cast outs of the subtrahend and difference is equal to the cast out of the minuend, and, the cast out of the product of the cast outs of the divisor and quotient is equal to the cast out of the dividend. Further, the cast out process can easily be extended to the operation of division with remainder. In the preceding statements one should be aware that the cast out of a number and another number may be the same, i.e. have the same remainder when divided by $n-1$. This implies that an incorrect result could have the same cast out as a correct result.

A Review of Alternatives for Checking Arithmetic

Traditional checking techniques include those that are based upon: re-doing the process with more focus, perhaps using more details, applying the commutative property of an operation, applying the definition of an operation in terms of another operation, applying an alternate algorithmic form of an operation, applying the cast out technique usually in terms of casting out nines (the cast out of nines process for checking addition or multiplication of counting numbers is documented to have been used prior to the ancient times of the Hindu arithmetic, possibly as early as the time of Euclid, (Boyer 1968, Cajori 1914, Eves 1953, Smith 1951, Smith 1953, and others)), and/or applying the calculator. In checking arithmetic with techniques such as demonstrating more details, increasing focus, applying the commutative property, applying a different operation, or using an alternate algorithm, the situation of having two different results can always occur. What are students then advised to do? There might have been a time when applying calculator/technological instruments had been considered as a checking technique. Influential mathematics education organizations such as the National Council of Teachers of Mathematics, the Mathematical Association of America, and others are not currently recommending the use of technology for the purposes of checking routine arithmetic. The underlying principle seems to be related to overuse/abuse of technology in general. The rationale is to reserve technology for situations where technology may be one of the best tools or even a unique tool for assisting a student in learning significant ideas of mathematics. Conversely, although calculators could be used for checking arithmetic of numbers expressed in base ten numeration, most are not programmed to perform base n arithmetic when n is not ten.

There are several issues which have emerged over the years which tend to influence mathematics educators to endorse more developmental strategies of introducing students to the idea of representing numbers, rather than starting with base ten numeration. Concerns about U.S. students not comparing well, internationally, on questions involving decimals and place value on the Third International Mathematics and Science Study, TIMSS, (TIMSS 1995) has researchers and educators considering viable alternatives. Such alternatives include those which tend to gradually incorporate the co-factor role of digits and exponential powers of a base. As a result, students may be introduced to the idea of representing number through a historical-developmental approach which could start with verbalization (number names directly associated with numbers of objects), enlarge to tally structure (a direct one-to-one correspondence between symbols and objects) experiences, incorporate simple grouping (trading power base groupings with addition) numeral structures, gradually introduce the need for digits and overt exponential co-factors as with multiplicative grouping numeral configurations, and finally incorporate base n numerals. Calculators would not ordinarily be used for checking arithmetic with such numeral structures.

As mentioned before, applying the cast out structure to checking arithmetic has a long and rich history which predicated the development of the idea of a congruence relation between numbers as described by Gauss in *Disquisitiones Arithmeticae*, (Reid 1992, pg 132). Further this eventually led to the law of quadratic reciprocity as proven by Gauss who described the law as the gem of arithmetic, (Reid 1992, pg 139). If one gets a different cast out in the final step of checking arithmetic by casting out nines (or in casting out $(n-1)$ as with base n numerals), then one is confident that an error has occurred in the initial operation. But, there are concerns about being sure that the initial result is correct when the final step of checking yields the same cast out, since two different numbers can have the same cast out. This can occur when digits get reversed, (for instance, the cast out of nines of fifty-one is the same as the cast out of nines of fifteen, or the cast out of nines of one hundred twenty-three is the same as the cast out of nines of three hundred twenty-one). Having errors such as these occurring very often can sometimes lead to diagnosis and modification. Alternately, the cast out of nines of sixty-seven is the same as the cast out of nines of thirty-one, which is not so likely to occur. The bottom line is that it seems to be an inherent property of all checking techniques to not be infallible or without confounding issues. It seems, with respect to casting out nines, that a major concern would be with the restrictions to checking just addition or multiplication of numbers in base ten. These restrictions are completely unfounded. The principal author has determined that cast out techniques can be applied to checking addition, subtraction, multiplication, and division of rational numbers, which include the counting numbers, integers, and rational numbers expressed in the base n related format, at least, and starting with verbalization.

Research Impacting Teacher Preparation

The principal author has been involved with the preparation of teachers of mathematics at all levels for the past forty or so years and has been involved in multiple field research programs focused upon several issues with students at the public school level for about twenty of those years. In general, impacting the preparation of prospective mathematics teachers with innovative ideas seems to be about as slow and difficult as incorporating innovative changes in public school mathematics education curriculum. Since the principal author directly teaches pre-service teachers there are opportunities to also use the classroom as a research laboratory from time to time, thus opportunities to incorporate the latest research findings can occur.

In sharing mathematical ideas with pre-professionals there are usually several options. In many instances, particularly when ideas relate directly to ideas to be shared at the public school level and when one has personal experience in formative field research with sharing the ideas with public school level students, one may decide to use techniques similar to those that have enhanced the accessibility of such ideas with public school students when sharing such with pre-service teachers. This has been the situation with respect to helping pre-professionals to discover the concept of cast outs and applications of cast out to checking arithmetic. The underlying principles for sharing have been the same for these pre-professionals as for students involved in the field research. These principles are essentially the constructionist perspective, that is, guided discoveries with active access to hands-on manipulative objects revolving around problems where students are expected to conceptualize and apply the concept of cast outs in terms of personal algorithms and as problem situations vary students are expected to modify their personal strategies accordingly.

Stages and strategies for acquiring and applying the idea of cast outs for public school students, which translate to strategies for the preparation of teachers as, determined by the authors are as follows.

1. One can introduce the ideas as soon as students are verbalizing numerals, refer to Example 1 for applications, in context with objects by physically removing groups of objects while describing the action in terms of casting out the number of objects removed from the group. Although a specific cast out group size is not required, one usually selects a group size that is consistent with an anticipated base associated with stage three. For instance, the cast out of fours of the number seven is three, because there are three objects left after physically removing four objects (the focus is upon the number of objects that are left after physically removing all possible groups of one less than the base). Some students start with a personal cast out algorithm which is essentially a form of repeated subtractions of the same number.
2. One continues the same kind of activity in terms of removing tally symbols, refer to Example 2 for applications which includes an example of division not appropriate for students usually at this stage of development but appropriate for potential teachers, again in context with also physically removing objects and verbalizing. Usually, since tally numeral expressions tend to represent relatively small numbers so as to not confuse students in understanding the numeral, students do not have to significantly modify their personal cast out strategy.
3. Simple grouping, the sum of multiple powers of a base should be introduced in conjunction with physical base power blocks. One should guide students to discover that the cast out of any power of the base is one. This might be aided by physically demonstrating trading a larger power base block for base of the next smaller power base blocks and removing base minus one of these smaller power blocks as related to casting out one less than the base. Also, one can introduce the graphing of the cast outs of consecutive counting numbers, see Graph 1, and help students to discover a geometrical as well as numerical pattern. Usually, since the numbers encountered tend to be larger numbers than those expressed in Stage 2, students start modifying their personal tactic for determining cast outs. Refer to Example 3 for applications of the cast out structure to checking arithmetic.

4. One builds upon the power of a base idea associated with simple grouping to incorporate co-factors with digits when introducing multiplicative grouping numerals. Students are led to discover relationships between the digit co-factors and the cast outs (there tends to be less focus upon the power co-factors since the cast out of such is always one). One continues to focus upon geometric and numeric patterns. Students, having gained a range of cast out experiences, tend to become rather sophisticated with personal algorithmic casting out strategies. Refer to Example 4 for applications of the cast out structure to checking arithmetic.

5. When one makes the jump to base n numeration, leaving out the overt expressions for powers of the base as co-factors, students are usually already focused upon the digits as primarily impacting the cast outs. Usually students have acquired sophisticated personal cast out strategies which need virtually no modification. And, they seem readily able to apply cast outs to the ordinary operations with counting numbers.

6. Students making the transition to integers and the cast out of integers seem to experience somewhat of a mental quantum jump. As before, one can usually help guide students to discover modifications of their personal cast out strategy by leading students to believe that the geometric and numeric pattern previously established for cast outs of consecutive counting numbers is consistent and discretely continues for integers. Having students involved in graphing some consecutive integers around zero while maintaining the discrete consistent geometrical pattern seems to be helpful to students in the transition, see Graph 2. Also, trial and error in conjunction with the geometric pattern evident in the graph and the emerging numeric pattern generally tends to assist students in modifying personal cast out strategies. Some students discover a principle such as: the cast out of the additive inverse of a counting number is equal to the difference of one less than the base and the cast out of the counting number, which is interesting.

7. Making the transition from determining the cast outs of integers to rational numbers expressed in digit-point numeration, refer to Example 6 for ordinary applications and refer to Example 7 for an application to division with remainder, is usually easy for students. Students recall that the co-factor powers involved with counting numbers and integers did not impact the cast outs. When they comprehend the role of the point in merely separating non-negative co-factor powers from the negative co-factor powers, they realize that whatever stratagem they were using to determine a cast out could be continued.

In the process of learning about cast outs students should also engage in applying cast outs to checking arithmetic at every stage, beginning with addition. Early in the process of checking addition students should make a verbal association with the discovery: the cast out of the sum is equal to the cast out of the sum of the cast outs of the addends. Similarly, with respect to the process of checking multiplication, students should make a verbal association with the discovery: the cast out of the product is equal to the cast out of the product of the cast outs of the factors. A group of thirteen first grade students progressed through the first three stages applying the cast out of fours in related base five simple grouping numerals to addition and multiplication over sixteen one hour research sessions, one per week (not consecutive weeks), over a period of an academic year. These same students as second graders seemed to have retained the information over a summer break and when combined with ten other second graders, new to the program, were able to share their previous experiences. This group of twenty-three students was able to continue through the next two stages in base five and enlarging the scope of operations to include subtraction. Other second grade groups were similar in scope of accomplishments. A group of fifth and sixth graders were able to start at Stage 1 and continue through the seven stages with more than one base of numeration. Fifth graders tend to have considerable difficulty with Stage 7. This was not with reference to determining cast outs or applying such, though. The issue for the fifth graders rested primarily with not having experience with rational numbers, which included the negative rational numbers, and also with inexperience with rational digit-point numeral expressions. A class of focused seventh graders taught by an innovative mathematics teacher had no problem with jumping directly into Stage 5 and using various bases for numeral expressions flexibly within a couple of hour sessions. Workshops of three hours with teachers having extensive mathematical backgrounds are usually required to share the cast out concept with limited applications at Stages 5 and 6 with bases other than just base ten. Ordinary teachers at the elementary school level involved in in-service training workshops usually require about a week at the first five or six stages to become functional and reasonably confident primarily with base ten. Since pre-professionals in this university system are required to have had prerequisite mathematics of at least college algebra recently, they are usually functional with the cast out concept and applications to checking the four fundamental operations (addition, subtraction, multiplication, division) starting with Stage 2 and continuing through Stage 6 in terms of base five, base ten and sometimes a couple of other bases after five or six class sessions. These sessions are of one hour fifteen minutes duration meeting twice a week. Elementary students and teachers tend to really feel mathematically empowered by these experiences and seemed to be enlightened or more enlightened as to the role of digits as co-factors of powers of a base in numeral expressions, i.e. the role of digits with respect to place value.

Clearly the direct impact of such field research upon the preparation of teachers is appropriate. But, there are usually mathematical prerequisites to such teacher preparation courses. Traditionally, congruence-modulo structures are included in algebra and elementary number theory based courses, which may be part of prerequisite mathematical courses. When it is known that teacher preparation students are participants of such courses, professors might consider somewhat the possibility of some emphasis upon applications of such structures and perhaps include some ideas associated with applications to the cast out idea.

The authors appreciate the constructive remarks and technological advice of a colleague, Dr. David Snyder, with respect to this paper.

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Examples of Applying Cast Outs to Checking and Graphs

Example 1. The cast out technique for checking arithmetic of counting numbers expressed in verbal numeral form is independent of any specific base. In the following examples the following will represent what number is being cast out, $\overrightarrow{number}$.

$$\begin{array}{lll} thirteen & \overrightarrow{five} & three \\ \underline{+nine} & \overrightarrow{five} & \underline{+four} \\ twenty-two & \overrightarrow{five}\ two & seven\ \overrightarrow{five}\ two\ CK \end{array} \qquad \begin{array}{lll} seven & \overrightarrow{six} & one \\ \underline{\times five} & \overrightarrow{six} & \underline{\times five} \\ thirty-five & \overrightarrow{six}\ five & five\ \overrightarrow{six}\ five\ CK \end{array}$$

Example 2. The cast out technique for checking arithmetic of counting numbers expressed in tally numeral form is independent of any specific base. In the following examples a slash, /, will represent one (a tally mark). To check subtraction or division one merely applies the definition of each in terms of addition or multiplication and checks such.

$$\begin{array}{rrl} ///////////////// & \overrightarrow{seven}\ /// & \\ \underline{-/////////} & \overrightarrow{seven}\ // & \\ //////// & \overrightarrow{seven} & \underline{+/} \\ & & ///\ \overrightarrow{seven}\ ///\ CK \end{array} \qquad \begin{array}{l} \quad\ \ \, // \\ ///////\overline{)//////////////} \end{array} \quad \begin{array}{rcl} // & \overrightarrow{three} & // \\ \underline{\times///////} & \overrightarrow{three} & \underline{\times /} \\ //////////////\ \overrightarrow{three}\ // & & //\ \overrightarrow{three}\ //\ CK \end{array}$$

The following examples will all be in base five and consequently the arrow will simply indicate the cast out of fours.

Example 3. This example is in terms of a simple grouping numeral structure where / represents one, f represents five, t represents twenty-five, etc. (the powers of base five), and 0 represents zero.

$$\begin{array}{lll} ff\,/// & \to & / \\ \underline{+f\,//} & \to & \underline{+///} \\ ffff & \to 0 & ////\to 0\,CK \end{array}$$

$$\begin{array}{ll} fff\,/// & \to // \\ \underline{-f\,//} & \to /// \\ ff\,/ & \to \underline{+///} \\ & f\,/ \to //\,CK \end{array}$$

$$\begin{array}{lll} f\,// & \to & /// \\ \underline{\times f\,/} & \to & \underline{\times //} \\ tfff\,// \to // & & f\,/ \to //\,CK \end{array}$$

$$\begin{array}{l} \qquad\;\; //// \\ f\,/\,\big)\overline{ffff\;////} \end{array} \qquad \begin{array}{ll} //// & \to 0 \\ \underline{\times f\,/} & \to \underline{\times //} \\ ffff\,//// \to 0 & 0 \to 0\,CK \end{array}$$

Example 4. This example is in terms of a multiplicative grouping numeral structure where 0, 1, 2, 3, 4 are the digital co-factors of a term and / represents one, f represents five, t represents twenty-five, h represents one hundred twenty-five, etc, which are the exponential co-factors of a term.

$$\begin{array}{lll} 3f2/ & \to & 1/ \\ \underline{+4f1/} & \to & \underline{+1/} \\ 1t2f3/ \to 2/ & & 2/ \to 2/CK \end{array}$$

$$\begin{array}{ll} 1t2f2/ & \to 1/ \\ \underline{-3f3/} & \to 2/ \\ 3f4/ & \to \underline{+3/} \\ & 1f0/ \to 1/CK \end{array}$$

$$\begin{array}{lll} 3f2/ & \to & 1/ \\ \underline{\times 2f3/} & \to & \underline{\times 1/} \\ 1h3t4f1/ \to 1/ & & 1/ \to 1/CK \end{array}$$

$$\begin{array}{l} \qquad\;\; 3/ \\ 3f2/\,\big)\overline{2t1/} \end{array} \qquad \begin{array}{lll} 3/ & \to & 3/ \\ \underline{\times 3f2/} & \to & \underline{\times 1/} \\ 2t1/ \to 3/ & & 3/ \to 3/CK \end{array}$$

Example 5. This example is in terms of integers expressed in base five positional numeration.

$$\begin{array}{lll} {}^{-}342 & \to & 3 \\ \underline{+{}^{-}413} & \to & \underline{+0} \\ {}^{-}1310 \to 3 & & 3 \to 3CK \end{array}$$

$$\begin{array}{ll} 142 & \to 3 \\ \underline{-{}^{-}334} & \to 2 \\ 1031 & \to \underline{+1} \\ & 3 \to 3CK \end{array}$$

$$\begin{array}{lll} {}^{-}41 & \to & 3 \\ \underline{\times 34} & \to & \underline{\times 3} \\ {}^{-}3044 \to 1 & & 14 \to 1CK \end{array}$$

$$\begin{array}{l} \qquad\qquad 211 \\ {}^{-}102\,\big)\overline{{}^{-}22022} \end{array} \qquad \begin{array}{ll} 211 & \to 0 \\ \underline{\times {}^{-}102} & \to \underline{\times 1} \\ {}^{-}22022 \to 0 & 0 \to 0CK \end{array}$$

Example 6. This example is in terms of rational numbers expressed in base five.

$$\begin{array}{lll} {}^{-}3.301 & \to & 1 \\ \underline{+4.232} & \to & \underline{+3} \\ 0.431 \to 0 & & 4 \to 0CK \end{array}$$

$$\begin{array}{ll} {}^{-}12.03 & \to 2 \\ \underline{-{}^{-}31.41} & \to 3 \\ 14.33 & \to \underline{+3} \\ & 11 \to 2CK \end{array}$$

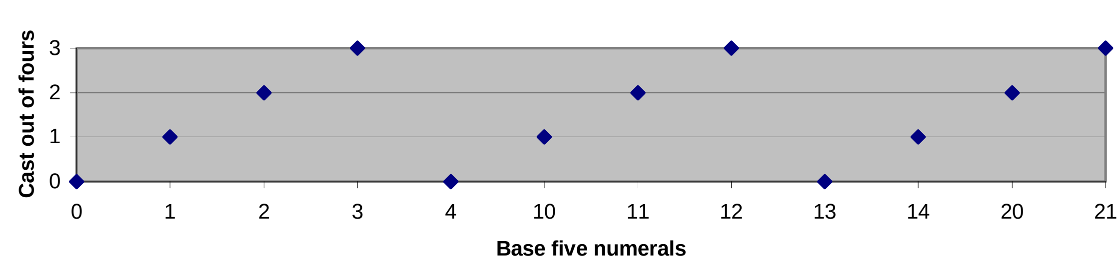
$$\begin{array}{lll} 1.23 & \to & 2 \\ \underline{\times {}^{-}23.4} & \to & \underline{\times 3} \\ {}^{-}40.442 \to 2 & & 11 \to 2CK \end{array}$$

$$\begin{array}{l} \qquad\qquad {}^{-}2.03 \\ {}^{-}31.4\,\big)\overline{120.302} \end{array} \qquad \begin{array}{ll} {}^{-}2.03 & \to 3 \\ \underline{\times {}^{-}31.4} & \to \underline{\times 0} \\ 120.302 \to 0 & 0 \to 0CK \end{array}$$

Example7. This example is in base seven where the arrow represents the cast out of six.

$$
\begin{array}{rllll}
 & \overline{{}^{-}14.65} & & & \\
0.34 \big) & {}^{-}6.0432 & {}^{-}14.65 \rightarrow 2 & & \\
 & \underline{-{}^{-}6.0466} & \underline{\times 0.34} \rightarrow \underline{\times 1} & & \\
 & 0.0034 & {}^{-}6.0466 & 2 \rightarrow 2 & \\
 & & \underline{+0.0034} & \rightarrow & \underline{+1} \\
 & & {}^{-}6.0432 \rightarrow 3 & & 3 \rightarrow 3CK
\end{array}
$$

Graph 1. Graph of the cast out of fours of base five counting numbers.



Graph 2. Graph of the cast out of fours of integers in base five.