

DYNAMIC GEOMETRY SOFTWARE NOT ONLY FOR SIMPLE DRAGGING

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ABSTRACT

Dragging is an integral part of Dynamic Geometry Software (DGS), but one runs into danger to determine only invariants without asking for the reasons and looking for arguments. It is shown, how this danger can be reduced by the additional mode of rearranging that makes possible a DGS with "overlay-technology". Together with a geometry based on translations, rotations and reflections, students are instructed to discover unaided ideas for visual proofs that can be extended to pure mathematical proofs.

Keywords: DGS, Experimental Mathematics, Proofs

1. Introduction

The National Council of Teacher of Mathematics and the Consortium for Mathematics recommended among others the following main-goals for the future of geometry (Mayes 2001):

- (a) **Geometry as an Experimental Science:** Geometric objects and concepts should be studied more from an experimental and inductive point of view.
- (b) **Geometry as a Formal Deductive System:** Local axiomatic systems which allow the student to explore, conjecture, then prove their conjectures, should replace the long sets of pre-formatted theorems.

These goals of experimenting, conjecturing and proving are higher level cognitive skills and require active student learning. Dynamic Geometry Softwares (DGS) provide interactive and dynamic learning envions. These tools reduce the computation, construction and measurement burdens so that the student can focus on the higher cognitive skills. Together with the celebrated drag mode the student can discover invariants by himself. DGS supports so the first goal mentioned above , together with functional thinking students can discover most of the traditional theorems of undergraduate geometry (Kautschitsch 1998, 2001):

Conjecturing = Finding of invariant properties.

But this kind of DGS contains the danger of restricting only to experiments and of looking only for invariants. There is no time and there are not adequate technological possibilities for answering the question: Why is there the invariant? What are the reasons? So usual DGS is a highly efficient tool for the process of conjecturing but geometry in general is not experienced as a deductive system.

2. The rearrange-mode: DGS with "overlay - technology"

Usual DGS, a mixture of dragging, measuring and calculating, misleads to a reduction in the focus on proofs. But proofs are still the corner stone of mathematics and especially geometry was and is the main field for teaching and learning proofs. Such a (direct) proof is (only) a deduction of a statement A from other statements A_i using some logical rules:

$$A_1 \wedge A_2 \wedge \dots \wedge A_n \rightarrow A.$$

The correctness of the proof depends not on the fact, if the statements A_i are already proved. Therefore a strong axiomatic-deduction foundation of geometry is not necessary for learning techniques of proving in school-mathematics. The main-goal of a proof for our purpose is – beside the verification of the conjecture – to demonstrate the **logical connections** between theorems. Students should be able to answer the question, why the observed invariants are valid. By the way students learn to keep on selected rules and statements, a soft skill useful for life and business. Two central questions arise:

- a) How do students get the conclusion A?
- b) Which statements A_i are useful for deduction?

Dragging and measuring support the first question, while the rearrange-mode, described below, supports the finding of appropriate A's. Such kind of DGS (for example realized in the package THALES developed at the Department of Mathematics in Klagenfurt/Austria) (Kadunz/ Kautschitsch 1993), has the following additional feature:

It allows a **breaking off constructive relations** and also a **re-establishing** of them.

This feature of breaking off constructive relations permits

- to act with the broken out objects, especially to rearrange them
- to look for "beautiful" figures by changing of position of some partial figures
- to carry out also transformations of congruence such as translation, rotation and reflection of partial figures

- to duplicate objects and moving one for a simultaneously comparing of the initial- and final configurations ("**overlay - technology**").

By this rearrange-mode a dynamic sampling of decomposition, fitting together, complementing and matching is realized. The cooperation of the modes of measuring, calculating and rearranging offers a micro-world, that simulate a plane with movable parts that makes synthetic geometry possible. That is the main reason why with use of DGS with rearrange-mode one does not run into danger to determine only invariants by dragging without asking for arguments, but it needs a reorientation of geometry and teaching.

3. "Reorientation" of Geometry

The main-feature of DGS with rearrange-mode is the possibility to carry out congruence-transformations with partial figures. In the packages THALES there are buttons for translations, rotations and reflections on lines for interactively chosen parts of construction. Knowledge of properties of these congruence-transformations are assumed as already known. Students have a lot of experience with motions, so it is natural- especially when handling with DGS - to use properties of congruence-transformations as **visual evidences**, above all:

- (V) Measure of **lengths**, **areas** and **angles**, parallelism and incidence are preserved under congruence-transformations.

Beside these visual evidences only two visual logical rules are used:

- (L1) If two figures are congruent then corresponding parts are equal.
- (L2) Removing equal parts of an equal figure - it remain figures, which must be equal.

By testing this programme with pupils it turned out that the following strategies were very useful:

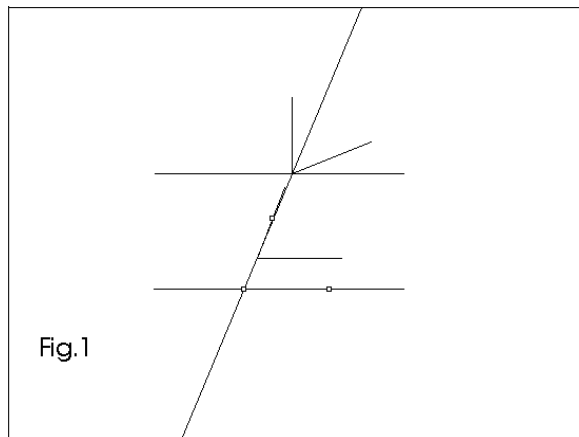
- (ST1) Complete to "beautiful" larger figures.
- (ST2) Decompose in and match suitable partial figures
- (ST3) Carry out the transformations consecutively several times
- (ST4) Inscribe suitable subsidiary lines

All these should be done to

- (ST5) Search known constellations such as congruent or similar triangles, the Side-Splitter-Theorem, the Screen Angle Theorem and so on.

The collection of (V), (L1)-(L2),(ST1)-(ST5) and the theorems, listed at the end of this section we call the "**visual encyclopaedia**". For learning the technics of proving this encyclopedia should be developed by using only the visual evidences (V), (L) and already proved theorems to prove the following ones. The development was tested twice with 15-16 years old students. Most of the suggestions are well-known in the literature, but they require pencil, paper and (dangerous) scissors and a plenty of time. We did it in one week (!) and most of the theorems were discovered unaided, certainly a merit of DGS.

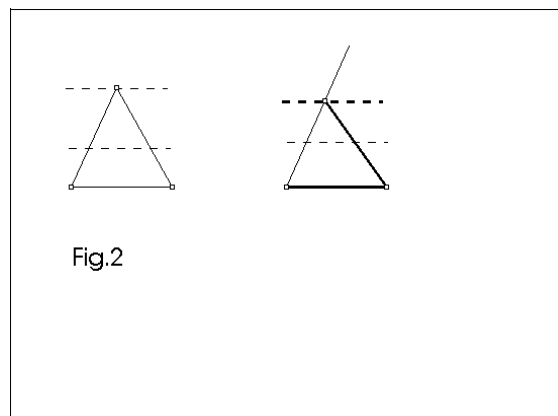
The theorems of angles on parallel lines (Equality of corresponding angles, interior angles, vertical and alternative angles) and the Congruence Theorems on triangles play a leading role in the development. The theorems on angles can also be used as visual evidences, but they are direct conclusions of the properties (V) of our transformations (we use the visual evidence: Translations preserve parallelism).



Strategy: Construct a second angle and move it by the overlay-technology.
From the very beginning on it is essential to give the reasons for the matching processes (e.g. in Fig. 1 the parallel lines).

It should be mentioned that a single figure can not replace the acting with DGS.

With the strategy of subsidiary line (in order to generate the above equal angles) one gets the theorems of the sum of angles in triangles and quadrilaterals and about the exterior angles.



Strategy: Look for the known constellation "angles on parallel lines". It turned out that pupils do not discover the well known line by their own, the motion (!) of one side in the translation-mode after breaking off the relation leads pupils to the all proving subsidiary line. They were not able to imagine this line.

For the further development the Congruence Theorems on triangles are essential, for example the A.S.A. Congruence Theorem:

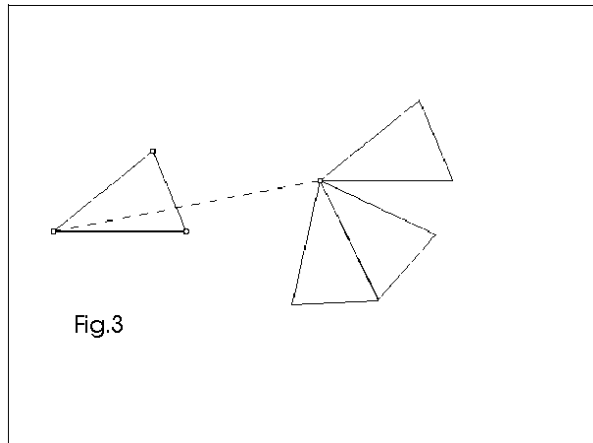


Fig.3

Strategy: Composition of translation, rotation, reflection on a line.

With the help of the S.A.S-Theorem one gets the Isosceles Triangle Theorem:

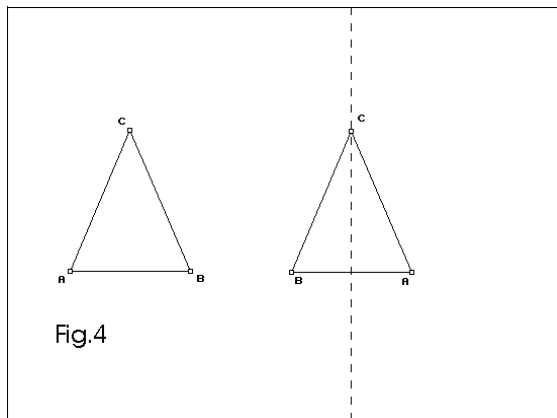


Fig.4

Strategy: Subsidiary line, reflection on the bisector of $\angle C$.

Essential: Explain the matching with the S.A.S-Theorem and use the geometric logical rule (L2).

Helpful: The inscription is moved in the same way.

This theorem is used essentially in the section "angles in a circle", for example for the Cyclic Quadrilaterals Theorem and the Screen Angle Theorem (Inscribed angles that intercept the same are equal):

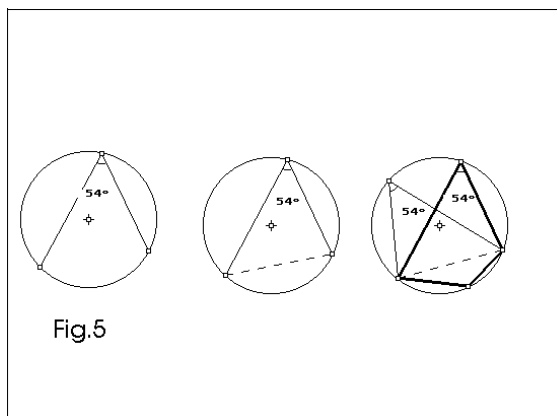
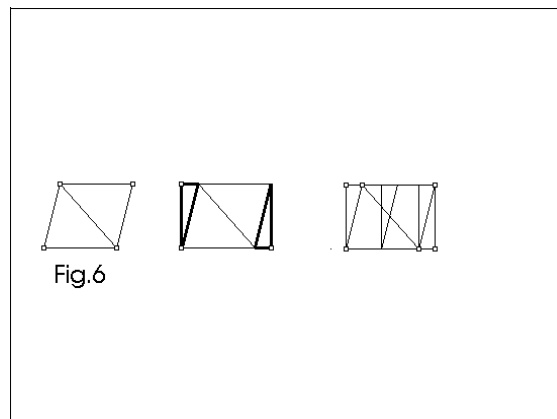


Fig.5

Strategy: Subsidiary line, completion to a cyclic quadrilateral (because it was shown before: In a cyclic quadrilateral opposite angles are supplementary).

Completion to larger "nice" figures together with the "overlay-technology" and the geometrical logical rule (L) allow self-discovery of the usual formulas for areas and the theorems in right triangles **before** similarity.



Strategy: Complete to a larger well-known figure such as rectangle, parallelogram and so on.

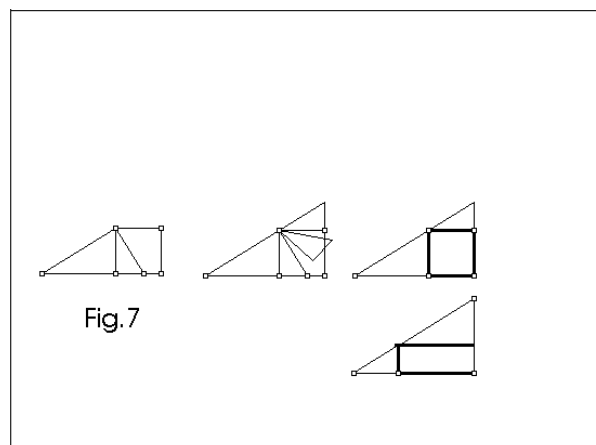
Change the position of the additional right triangles with the overlay-technology and use screen-splitting for comparing the initial and the final state:

Remove equal pieces from the equal figure and use (L2).

Such deductions of the formulas for areas without calculations are very instructive for students.

The next example concerns the theorem about the altitude to the hypotenuse.

Before knowledge of similarity it is a difficult didactical problem to discover the quadratic relationships of the sides in a right triangle. In Kautschitsch 1998 I have shown how this problem can be mastered by the drag-mode of a DGS together with the "**dependence-graph- technology**". Once the quadratic relationship is discovered one can proceed as follows:



Strategy: Rotate the small right triangle after duplicating it (overlay- technology).

Duplicate the whole figure and change the position. Remove equal pieces and use (L2).

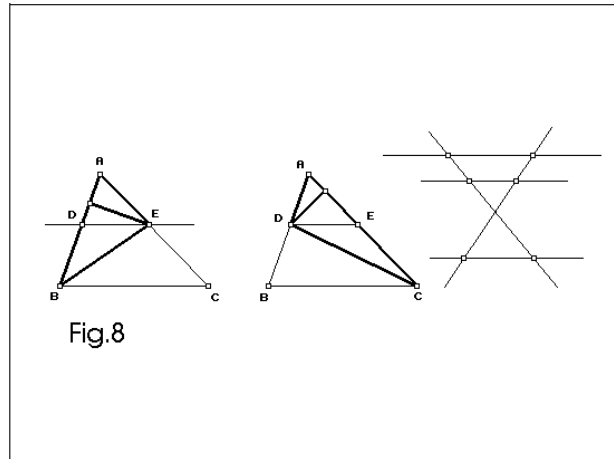
Again it is very important to give reasons why a right triangle is attained (vertical angles are equal, the sides of a stretched angle form a line). The resulting rectangle consists of those segments into which the altitude divides the hypotenuse.

It is well-known that the theorems on right triangles can easily be derived from similarity. In order to use only our mentioned visual evidences and already proved theorems for the developing of the theory of similarity, we use the fact about the areas of triangles that is a direct corollary of the known formula for the area of a triangle, namely:

If two triangles have equal altitudes, then the ratio of their areas is equal to the ratio of the lengths of their basis.

Naturally an excursion about ratios and proportions is necessary. Then we get easily the Side-Splitter Theorem:

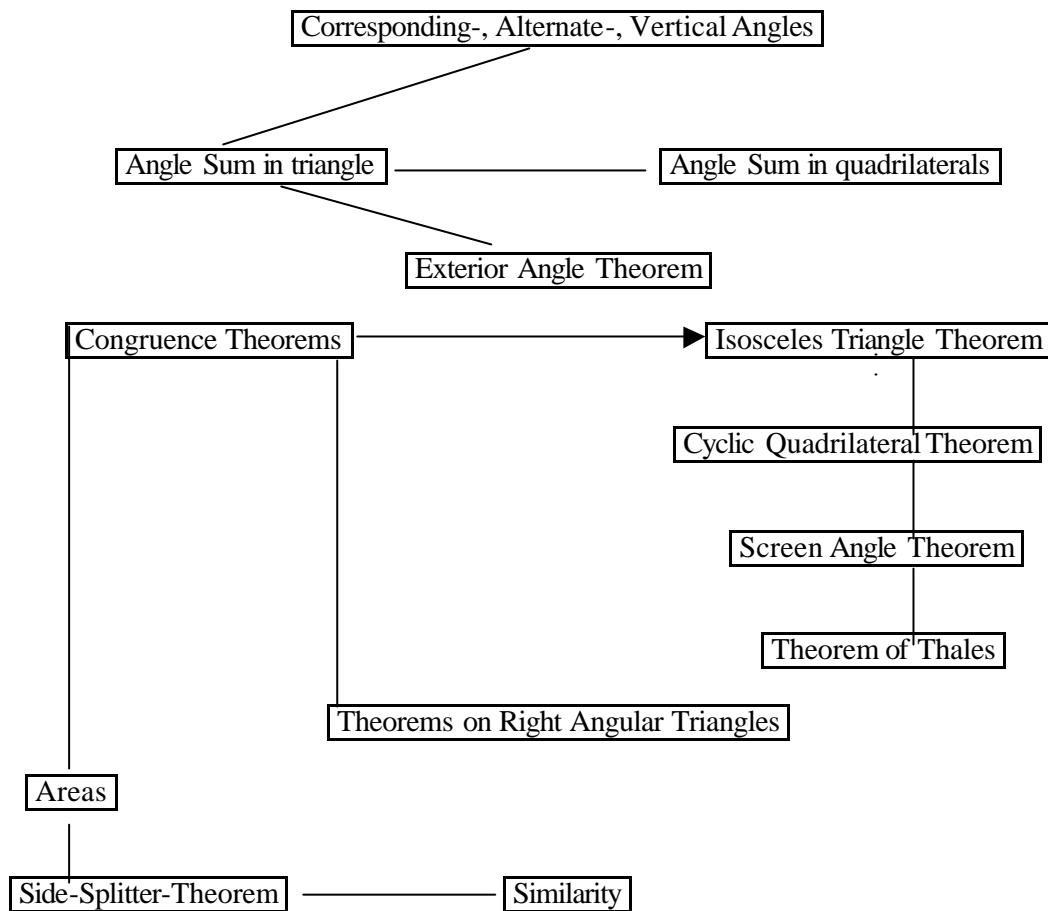
If a line parallel to one side of a triangle intersects the other two sides in different points, it divides the sides in the same ratio.



$$\frac{AD}{DB} = \frac{\Delta AED}{\Delta DEB} = \frac{\Delta AED}{\Delta DEC} = \frac{AE}{EC}$$

A generalization with the help of rotations generates the general "Side-Splitter-Constellation". By computation we get the usual statements about proportions in similar triangles.

Summing up we get the following connections between the theorems, which form together with (V), (L1), (L2), (ST1)-(ST5) the above mentioned "visual encyclopaedia":



4. Report on a Course

At the University of Klagenfurt/Austria we held two one-week courses with 15-16 years old students to test the DGS with overlaying-technology and the above described “visual encyclopaedia” concerning the following topics:

- Are the students able to make conjectures without any help?
- Are the students able to find proofs?

It turned out that they could discover many traditional theorems as well as exotic one's. For conjecturing especially the drag-mode with "dependence-graph"-technology was very efficient. But in fact this method led away from ideas for proving the discovered conjectures. This disadvantage could be reduced by the rearrange-mode with "overlay-technology", since this method is nearer to synthetic geometry and also rearranging can be a source for conjectures.

Example: Theorems in right angle triangles.

By measurements students “see” that the altitude to the hypotenuse of a right triangle forms two triangles that are similar to each other and to the original one.

But it is well-known that similar triangles can easily be recognized, but students have difficulties to find the corresponding sides. Next the students tried to get the Side-Splitter-Constellation (they knew that this had something to do with similar triangles). By overlay-technology they could move the triangle $\triangle ACH$. Since also inscriptions move with the triangle, they could read off the right proportion.

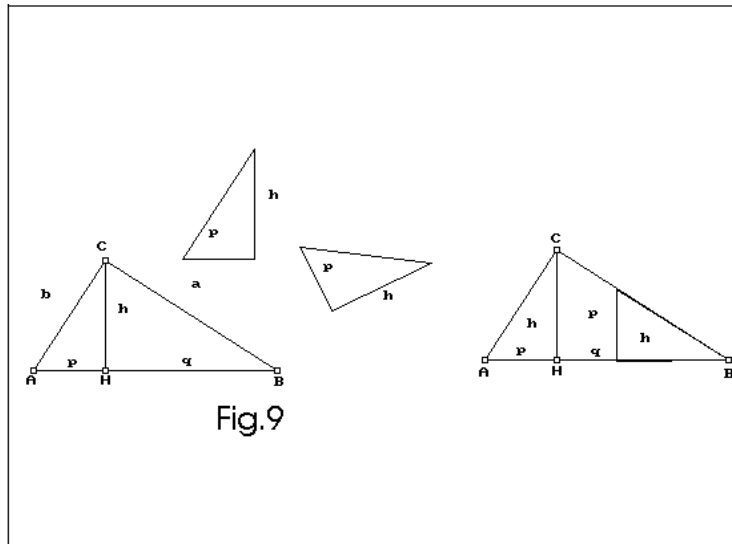


Fig.9

All students could explain the matching of the moved triangle by theorems about angles.

Example: Van Schooten's Theorem

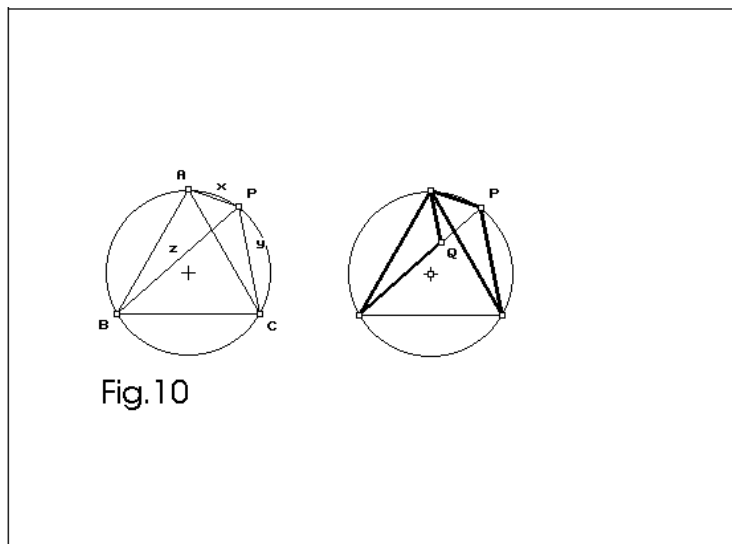


Fig.10

Given an equilateral triangle ΔABC and a point P on the circumscribed circle. What can one say about the lengths of $x=PA$, $y=PC$, $z=PB$?

By measuring and the drag-mode (students begin always to measure and to drag) they discovered the relationship $z=x+y$. They were convinced of the validity, but they wanted to know, what the reasons were for the validity.

First strategy: Since z should be $x+y$, x was marked off on the line z and the subsidiary line AQ was drawn.

Second strategy: Looking for congruent triangles. By measuring angles and lengths they saw, that the triangles ΔABQ and ΔAPC were congruent. Movement of a duplicated triangle showed this by matching of corresponding lines and therefore they claimed: $BQ=y$.

Now the difficult part remained: What are the reasons for this congruence?

Since there are only few theorems in the "visual encyclopaedia", some students concluded first by the Screen Angle Theorem the equality of the angles in B and C . By measuring they saw the equilateral triangle PQA . It took a long time, that some could conclude again by the Screen Angle Theorem that the angle $\angle QPA$ measured 60° , and by the Isosceles Triangle Theorem the other angles

must then be also 60° . Therefore $QA=x$ and again by the Screen Angle Theorem the angles $\angle BQA$ and $\angle APC$ measure 120° , so by the A.A.S-Theorem the triangles must be congruent and by (L1) $BQ=y$.

5. Conclusions

The success of this programme is based on the following facts:

- a) **DGS with overlay technology** offers escape routes for students in hopeless situations into the familiar domains of transformations of figures by translations, rotations and reflections.
- b) **DGS with dependence-graph-technology** shows relations that even can not be seen.
- c) The “visual **encyclopaedia**” consists only of **few** theorems, two visual logical rules and five strategies. It is so a “minimal” generating set for other theorems.
- d) The contents of this encyclopaedia were learned by doing on their own. This procedure helps to discover or recognize well known constellations in unfamiliar situations.

Most of this programme can be done with paper, pencil and scissors. But the use of DGS has many advantages:

(a) The constructions are precise and can be repeated quickly. Many relations can be seen directly and guide so the process of thinking. Subsidiary lines can be discovered by motions of parts of the figure. The imaginative faculty in general is too weak. DGS offers constellations that students can hardly imagine. So DGS offers imaginations outside of the head but of the same or better quality (moveable, precise, only correct imaginations).

(b) Measurements and calculations facilitate the finding of congruent or similar triangles or other equal parts.

(c) The overlay-technology permits comparison of the initial with the final state by screen-splitting and Congruence Transformation of partial figures. This is a source for conjectures that even includes ideas for proving.

(d) Congruence Transformations covers many properties, so the usual long sets of pre-formulated and pre-sequenced theorems can be replaced.

The development of this programme shows the student how a mathematical proof does work. If each matching process is explained by the chosen visual evidences and already proved theorems then no dragging for getting more examples is necessary. So pure mathematical proofs are obtained, expressed only in actions with pictures.

Visual Proving = Finding of always practicable actions with pictures.

We got the experience that students understand the proof, if they could describe the actions and gave reasons for matching. Writing down the arguments was a problem and did not increase the understanding.

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