# AN APPROACH FOR THE EFFECTIVE INTEGRATION OF COMPUTER ALGEBRA IN AN UNDERGRADUATE CALCULUS AND LINEAR ALGEBRA COURSE

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## ABSTRACT

In this paper we describe an approach for an effective integration of computer algebra systems in an elementary calculus and linear algebra course.

In our mathematics courses at Wageningen University, an education and research centre for the biological, environmental, agrotechnical and social sciences, we have noted that students often show a lack of conceptual understanding while using computer algebra systems. A reason for this seems to be that the students do not establish a right link between the computer algebra techniques and their mental approach of mathematics. We have composed a framework that aims at establishing such a link. Because the students have developed their mathematical way of thinking in close relation with paper-and-pencil methods, this framework is based on an integration of computer algebra and paper-and-pencil techniques. We have used this framework for the set-up of an elementary calculus and linear algebra course for first year students in social sciences.

We first describe this framework, which is made up of several steps. In these steps the use of paper-and-pencil and computer algebra alternate and reinforce each other. Next we show how we worked out this approach for an example from calculus: the determination of the stationary points and extremes of functions of two variables. In this example also the graphic facilities of the computer algebra system are exploited. The last part of the example is an application on maximising the profit of a production process, both without and with constraints.

**Keywords** computer algebra, integration of technology, teaching scenarios, functions of two variables

# 1. Introduction

In our mathematics courses at Wageningen University, an education and research centre for the biological, environmental, agrotechnical and social sciences, we have noted that students often show a lack of conceptual understanding while using computer algebra systems. A reason for this seems to be that the students do not establish a right link between the computer algebra techniques and their mental approach of mathematics. In this study we describe the set-up of an elementary calculus and linear algebra course for first year university students in social sciences, in which we attempt to establish such a link in a systematic way. In this course we have integrated the use of a computer algebra environment into a more traditional course, but with special attention for the connection between both approaches. In particular, we have composed a framework that aims at an effective integration of paper-and-pencil work and computer algebra techniques. This framework is made up of several steps in which the use of paper-and-pencil and computer algebra alternate and reinforce each other.

In section 2 we describe the educational setting and the aim of the use of a computer algebra environment in this course. In section 3 we describe our framework for the integration of paper-and-pencil and computer algebra techniques. We continue with an illustration of our framework in section 4, describe some results in section 5, and complete the paper with a discussion in section 6.

# 2. Educational setting and aim of the use of computer algebra in the course

The course had been set up for first year university students in social sciences. Before they entered university, most of these students had taken a curriculum in upper secondary education preparing for a study in social sciences at university level. That curriculum contained mathematics courses in which the mathematics was dealt with in a realistic context, but algebraic skills such as formal manipulation were not highly developed. We note that the students had not made use of a graphing or symbolic calculator in that curriculum.

Our university course covered subjects from calculus and linear algebra. In the course applicability of the mathematics received more emphasis than its theoretical finesses. Applications relevant for the social sciences were included. Also the course aimed at conceptual insight rather than at far reaching technical skills. During a period of six weeks the students had to attend four 2-hour lessons each week. Three of these weekly lessons were given in a more traditional classroom setting without computer facilities, whereas in the other weekly lesson a computer algebra environment was available. In the more traditional lessons, alternately the teacher explained the mathematics and the students were studying the subject, for instance by making assignments. During these lessons the students just had a hand held calculator at their disposal without graphing or symbolic facilities. In the other weekly lesson computer algebra was used in combination with paper-and-pencil techniques.

In this course we did not aim at developing large skills in the use of computer algebra, or at acquiring a thorough knowledge of it. The amount of time available for the use of computer algebra (all together only six lessons of two hours) was not sufficient to achieve such goals. Instead, the aim of the use of computer algebra in this course was to support the mathematical learning process of the students. For this reason we selected an easily accessible computer algebra program (we chose Derive 5.0). Besides, the use of computer algebra gave us the opportunity to treat applications that would demand too much technical skill or too much time if dealt with without a computer.

# 3. A framework for the integration of computer algebra and paper-and-pencil techniques

Computer algebra can be expected to facilitate the process of gaining conceptual insight, see e.g. Heid (1988). An efficient use of computer algebra in the teaching of mathematics is not self-evident, though, see e.g. Artigue (1997), Lagrange (1999), Drijvers (2000), Drijvers and Van Herwaarden (2000). Students often show a lack of conceptual understanding while using computer algebra systems. As mentioned above, a reason for this seems to be that they do not establish a right link between the computer techniques and their mathematical way of thinking. The students have learned mathematics using paper-and-pencil methods, and their mental approach of mathematics has developed in close relation with these methods. Therefore, one can suppose that a successful internalisation of computer algebra techniques can be reached by an appropriate link with paper-and-pencil methods.

In the course we have tried to establish such a link in a systematic way. In the lessons without computer facilities we treated the subjects of the course in a more traditional (paper-and-pencil) way. In the weekly lessons with computer algebra facilities we have tried to integrate the use of computer algebra and paper-and-pencil work. We have taken the following approach, in which four steps are distinguished. First the students make an exercise with paper-and-pencil. This exercise is of a type they have already dealt with in one of the more traditional lessons, but not too elaborate or requiring too much technical skill. In the second step the students have to solve this exercise using computer algebra. To let them not get stuck in the computer manipulations at this stage, we have provided sufficient details on the required computer algebra commands; in some cases the expected computer algebra output has also been added. In the third step the students have to make some similar exercises to obtain more practice. Finally (step 4), they have to make some more difficult assignments, for example extensions of previous exercises that are too elaborate to handle without computer algebra. These assignments may also be applications from the social sciences. But, when useful, also in these assignments paper-and-pencil questions are included to achieve an appropriate link with the computer algebra work.

# 4. An illustration

The framework of section 3 will now be applied to an example: the determination of the stationary points and extremes of a function of two variables. In one of the more traditional lessons the students have already learned how to determine partial derivatives, stationary points, and extremes by hand. Then in the next lesson with computer algebra the function

$$F(x,y) = x^2 y - 2x^2 - 2y^2 + 4y + 1$$

is considered.

*Stationary points.* First the students have to determine the stationary points of this function using paper-and-pencil. After calculating the partial derivatives, they are expected to factor these derivatives, if possible, and to make appropriate combinations to determine the stationary points. In this case we expect them to obtain the equations

$$\begin{cases} F_x(x,y) = 2xy - 4x = 2x(y-2) = 0 \\ F_y(x,y) = x^2 - 4y + 4 = 0, \end{cases}$$

from which they should deduce the combinations

$$\begin{cases} x = 0 \\ x^2 - 4y + 4 = 0 \end{cases} \quad \text{and} \quad \begin{cases} y - 2 = 0 \\ x^2 - 4y + 4 = 0. \end{cases}$$

From these combinations they obtain the stationary points (0, 1), (2, 2) and (–2, 2). For our students one of the difficult steps is to make those right combinations. In particular, a common error is the wrong combination of $x = 0$ with $y - 2 = 0$, leading to an incorrect solution (0, 2). Another common error is that students do not obtain the third stationary point, because they forget that the equation $x^2 = 4$, which results from solving the second combination, has two solutions.

Next the students determine the stationary points with computer algebra. First they have to check the partial derivatives already obtained by hand. Next they have to determine the stationary points with Derive in two different ways: graphically and algebraically. For both approaches we have described in rather much detail the actions and commands they have to carry out. We thus hope to prevent that at this stage difficulties with computer manipulations would distract the students' attention from the mathematics. In the graphical approach the students make a plot of the equations $2x(y - 2) = 0$ (in red) and $x^2 - 4y + 4 = 0$ (in blue). Then each point of intersection of a red graph and a blue graph represents a stationary point. In this way the students can see that the wrong combination mentioned above does not correspond with the intersection of two differently coloured graphs. Also those students that had missed the third stationary point should now become aware of this and correct their paper-and-pencil work. To determine the stationary points with computer algebra in an algebraic way we let the students solve the system of both equations with Derive's Solve > System command. As a check the expected screen output

$$[x = 0 \wedge y = 1, x = 2 \wedge y = 2, x = -2 \wedge y = 2]$$

is included in the accompanying text. Some of the students have difficulties in interpreting this notation. In fact, in this case the paper-and-pencil result appears to be helpful in explaining the computer algebra notation. Thus the paper-and-pencil work and the computer algebra method are used to reinforce each other.

*Extremes*. Next the students have to investigate if the stationary points are extremes. In one of the more traditional lessons they have already learned how to classify stationary points (using the determinant of the Hessian matrix). Now in this computer algebra lesson they first have to classify the stationary points of $F(x,y)$ with paper-and-pencil. They obtain that (0, 1) is a maximum, whereas (2, 2) and (–2, 2) are saddle points. Then they have to investigate the stationary points with computer algebra, again both algebraically and graphically. In the algebraic approach they have to check their paper-and-pencil work using Derive, and to correct it, if necessary. In the graphical approach the students have to plot level curves of the function $F(x,y)$. Such a plot yields a very instructive picture of the behaviour of the function, in particular near the maximum and the saddle points. In the accompanying text we have again provided the computer algebra command that the students can use to produce the plot:

$$\text{vector}(F(x,y) = z, z, -2, 4, 0.05),$$

which yields 121 level curves. By providing this command and the screen settings we hope that the students do not get stuck in the computer manipulations. In this case the aim of the plot is to enlarge the students' understanding of the subject and not to master this vector-command. Finally, the students check their results by plotting 3D-graphics of $F(x,y)$ in the neighbourhood of the stationary points, together with the (horizontal) tangent planes in these points. Thus the paper-and-pencil and computer algebra methods complement each other and improve the students' understanding of the subject.

*An application.* We note that up to here steps 1 and 2 of the framework introduced in section 3 have been applied twice, both for the determination and the classification of the stationary points. We now let the students continue with a similar exercise to obtain some more practice (step 3). Then in the final step the students have to turn their attention to an application from economics: the maximisation of the profit of a production process. We consider a firm producing a single product that is sold in two different markets. Say, $x$ and $y$ are the outputs in the two markets. Certain simple assumptions for the demand curves and the total cost function lead to the following profit function in the variables $x$ and $y$:

$$P(x, y) = -5x^2 - 2xy - 8y^2 + 4200x + 10200\,y\,.$$

First we let the students determine the marginal profits (first order partial derivatives) for some specified values of $x$ and $y$. Next they have to determine the stationary point and to investigate if it is an extreme, in particular if it is a maximum. They may answer these questions, which are of a type they have become familiar with by now, with paper-and-pencil or computer algebra, as they prefer. It turns out that most of them solve these questions using Derive. The application ends with a question that is new for the students. Suppose that the firm's production capacity is constrained by $x + y = 801$. In that case the (unconstrained) maximum, $P(300, 600)$, is not attainable. The students have to plot the constraint and level curves of the profit function in one figure, making use of Derive's vector-command (see above). Of course, the next question is to find the maximum of the profit function subject to the constraint. To obtain this maximum, the variable $y$ (or $x$) can be isolated from the constraint and substituted in the profit function. Maximising the resulting function of one variable then yields the solution.

# 5. Results

At the end of each computer algebra lesson we asked individual students for their opinion. We also interviewed a sub-population of the students at the end of the course. In general the students were positive on the set-up of the course. They remarked that

- the use of computer algebra created the possibility of checking their paper-and-pencil results; it enabled them to discover their mistakes, and it clarified the methods
- the alternation of paper-and-pencil and computer work helped them to 'keep awake'; they had to work very intensively during the computer lessons
- in some cases too much repetition of paper-and-pencil work already carried out in the lessons without computer algebra had been included in the computer lessons
- the computer commands had been described in sufficient detail; the computer work had not raised too many obstacles
- because of the link between the paper-and-pencil and computer algebra work, they had the feeling that they knew what they were doing when using the computer ('not just pushing buttons')
- for many of them the computer algebra lessons, and in particular the integration with the paper-and-pencil work, played an important and useful part in preparing for the final written exam, even though it had to be made without computer algebra.
- Specifically on the computer lesson on functions of two variables the students remarked the benefit of the computer graphics for their understanding of the subject.

We also observed the students' reactions in the weekly computer algebra lessons. We noticed that in general the students worked very intensively. In the lesson on functions of two variables many students had questions on the computer algebra approach to determine the stationary points graphically. They needed explication why the points of intersection of the red and blue graphs represented the stationary points. This created an opportunity to show, when necessary, that the wrong combination of factors leading to the incorrect solution $(0,2)$ is mistaken, indeed. It appeared that some students confused the red and blue graphs with level curves of $F(x,y)$. Specifically the plot with 121 level curves (in colour, and being drawn on the screen gradually) caught the students' attention. From their reactions and our enquiries we deduced that most of them could tell the behaviour of the stationary points (extreme, saddle points) from these level curves. The students also produced 3D-plots of the function in the neighbourhood of the stationary points, together with the horizontal tangent planes. Many of them pointed out, though, that they considered the plot with level curves to be more informative. The question to determine the maximum of the profit function subject to the constrained production level appeared to be difficult. Many students who reached this question, managed to plot the constraint and level curves of the profit function in one figure. They could also point out where in this plot the constrained maximum is attained (the point where the 'constraint line' is tangent to one of the 'iso-profit curves'). So it seemed that they had obtained a good understanding of the problem. Most of them needed a clue, though, to calculate the constrained maximum algebraically: only after the teacher's suggestion to isolate one of the variables from the constraint and to substitute the result in the profit function, they succeeded in obtaining the correct solution.

The results on the final written exam were good, but it is hard to assess the influence of our integration of paper-and-pencil and computer algebra methods. One of the assignments was to determine the stationary points of a function of two variables, similar to the exercise in section 4, and to classify them. The results on this assignment were rather good, slightly better than the results of a comparable group of students in a course before the introduction of computer algebra. But the results on a question to determine a constrained maximum were not very encouraging. Many of the students just checked some points; only 10-15% substituted the constraint into the object function, and only half of them knew how to continue. Without guidance by the teacher this assignment is apparently too difficult for the students.

Finally, the teachers, who could compare the course with the former traditional course, were unanimously positive on the set-up. It was their impression that the carefully staged interaction with computer algebra caused the students to be more conscious of their paper-and-pencil work. Comparing with computer lessons of other courses where computer algebra and paper-and-pencil work had not been integrated that systematically, the teachers remarked that the students seemed to know better what they were doing when using the computer for their calculations.

# 6. Discussion

We now reflect on the framework that we adopted for the integration of paper-and-pencil and computer algebra techniques. We expect that this integration is helpful for the conceptual understanding of the mathematics involved. One obvious reason is that in this set-up the students work out assignments in two different ways: with paper-and-pencil and with computer algebra. Not only that repetition, but especially the interaction between both approaches may be expected to support the mathematical insight. As an example we mention the interaction between the graphical computer algebra approach for the determination of the stationary points (as the points of intersection of red and blue graphs) and the algebraic paper-and-pencil approach. We recall that the students thus obtain insight why the wrong combination of factors (pointed out above as a common error) is incorrect, indeed. Another aspect is the checking of paper-and-pencil results with computer algebra, in combination with the correction of mistakes in the paper-and-pencil work. We note that the teacher also plays a part in this by focussing the student's attention again on his/her paper-and-pencil results. We emphasise that in steps 1 and 2 of the framework the paper-and-pencil and the computer algebra work should not raise obstacles that divert the attention from the mathematics. At this stage of the process attention should not be focussed on technical problems, but on basic concepts and techniques.

We remark that within our framework the graphic facilities of computer algebra programs can be successfully exploited. A good example is the plot of level curves, which yields an excellent picture of the behaviour of a function of two variables near an extreme or a saddle point and thus enlarges the insight in these concepts. These plots may be helpful for the students to create a 'mental picture' of these concepts. In turn, these 'mental pictures' can reinforce the paper-and-pencil approach. Also 3D-plots may be helpful, though in our example the behaviour of the function can be deduced better from the level curves, as noted by the students.

The framework for the integration of paper-and-pencil and computer algebra techniques described in this paper appears to be a good and efficient approach in our educational setting. We think, though, that also in other educational settings this systematic framework might be useful, because it aims at developing conceptual insight and concurs with the way students have learned mathematics, i.e. using paper-and-pencil. Also for more advanced subjects it may be useful. For example, it can be utilised for an extension of the example in section 4 with Lagrange's method for the determination of extremes subject to constraints.

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