“THALIS”

A REPRESENTATION SYSTEM FOR UTILIZATION IN TEACHING AND LEARNING FRACTIONS

Stavros ORFANOS
Annis Marias 32, 85100 Rhodes, Greece
Tel. 0241-23437
e-mail: orfanos@rhodes.aegean.gr

Fragiskos KALAVASSIS
University of the Aegean
Demokratias 1, 85100 Rhodes, Grece
e-mail: kalabas@rhodes.aegean.gr

ABSTRACT

The purpose of this paper is to present an attempt to promote the development of teachers’ own intuitive mathematical knowledge of rational numbers (1) with the use of a new teaching tool which is named “Thalis”. For the construction of this tool consideration has been taken in children’s difficulties to produce adequate intuitive models to represent rational numbers and operations with them.

Currently, to teach concepts of rational numbers, traditional representation systems are utilized; some of them are not self-consistent, since they are capable of producing contradictory situations, whereas, there are others, self-consistent but over-specific (2) since they are capable of producing multiple representations of a problem’s solution.

This paper recommends a new representation system for the teaching of rational numbers which has the form of a natural transformer. This system does not allow for any misconceptions since, as it will be discussed, it is a model of the field of rational numbers. Moreover, since it is a natural transformer, it permits authentic measurement activities and ratio computations in school contexts.With this new system an improvement is expected in:

1. Children’s ability to experiment
2. Teachers’ ability to plan constructivistic activities for the teaching of rational numbers

The paper presentation structure for Thalis will be the following:

1. Representation systems of rational numbers
2. Informal presentation of Thalis
3. Examples of Thalis use in representing some operations of rational numbers
4. Discussion about Thalis being a model of the field of rational numbers.
5. A teaching script with the use of Thalis

A central part of the Curricula (ages 9-13) of elementary schools in most countries focuses on the teaching of fractions. However, didactical research indicate that a significant number of students have serious difficulties in understanding the concepts of fractions and techniques of operations with fractions.

The most important proposed interpretations for these difficulties are(1)

- Rational numbers are used less than natural numbers.
- Many children find it difficult to accept a given fraction as a number and tend to view it as two whole numbers.
- Students often incorrectly attribute observed properties of operations with natural numbers to those with rational numbers.
- The many different interpretations of, and notations for, rational numbers.

Also(5),

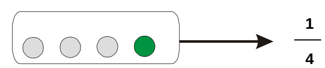
- The over-development of the instrumental versus the conceptual knowledge.
- The non-use of the students' intuitive knowledge.

On the other hand, we observe that textbooks in order to facilitate teaching of rational number concepts and operations, use a rich variety of representations with different roles. Some representations are used for a better visualization of the part-whole interpretation of the fraction, some others for a better visualization of addition and some for better visualization of multiplication, etc.

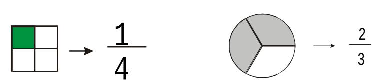
Our hypothesis is that many of the difficulties students have in understanding fractions are related to the nature and the consistency of the textbooks' representations

In Greek textbooks, most common representation systems contain:

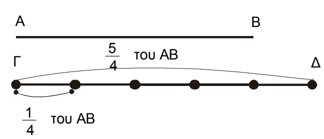
1. Representations with discrete objects E.g.



2. Representations with two or three dimensional figures

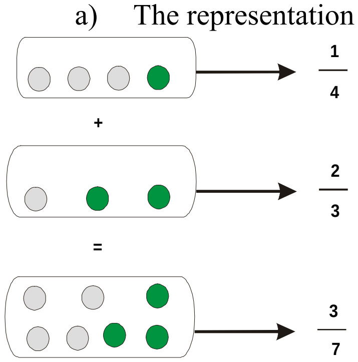


3. Representations with straight lines.

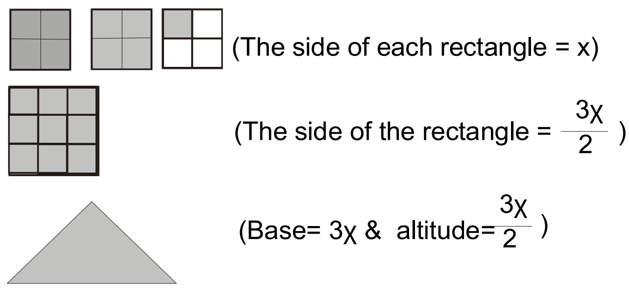


These representation systems are either non self-consistent, since they are capable of producing contradictory situations, or self-consistent but over-specific since they are capable of producing multiple representations of a problem's solution. More specifically:

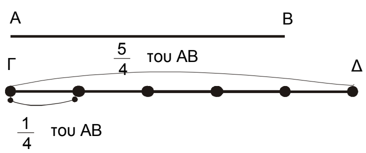
a) The representation system, which uses discrete objects, is not self-consistent since it leads to paradoxes. For instance, in order to compare $\frac{1}{4}+\frac{2}{3},\frac{1}{2}$ it is possible, by using this representation system, to elaborate the following "proof" which leads us to the paradox $\frac{1}{4}+\frac{2}{3}=\frac{3}{7}<\frac{1}{2}$



b) The representation system, which uses two or three-dimensional figures, is over-specific since it is capable of producing multiple representations of a problem's solution.



For example on the left hand figure there are some of the possible representations of the problem's "Find $\frac{9}{4}$ of the rectangle of side x" solution.



c) The left hand figure is a common representation system which shows the result $\frac{5}{4} of$ AB but it does not explain how to find ¼ of AB.

d) Textbooks indications are guiding students to construct only subdivisions of 2 of any manipulating aids. For example, textbooks show ways of finding the ¼ but they do not show ways of finding the 1/7 of a paper sheet.

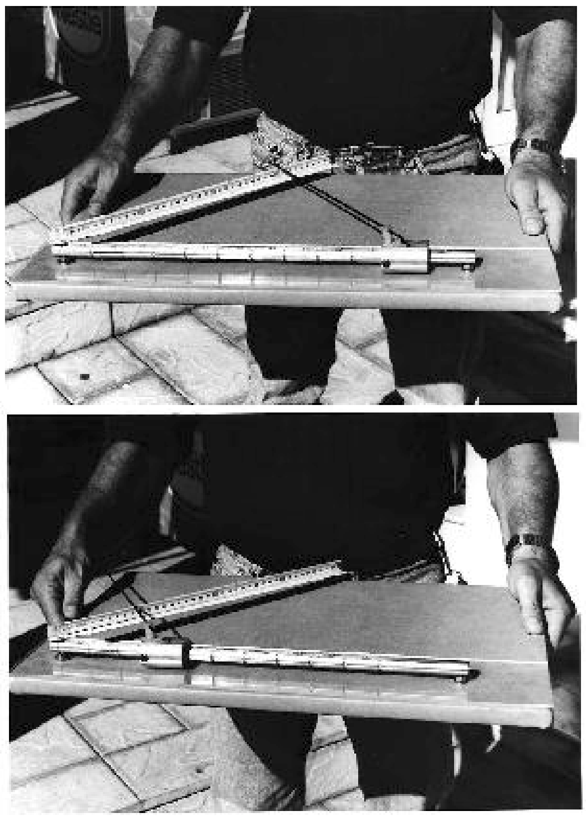
These kinds of problems, in the use of the above representation systems pose, in our opinion, a serious issue that concerns the quality of the school textbooks. It is a direct reason for the difficulties faced by the students when comprehending fractions.

2

Based on the above analysis we propose an alternative representation system named "THALIS" which has the form of a natural transformer and consists of:

1. A wooden board
2. A numbered axis (a) in the bottom of the board
3. A small wagon which can move along the axis (a)
4. A rotating needle placed on the wagon
5. A numbered axis (b) which forms an acute angle with axis (a) and has the same origin O as axis (a)

This mechanism has a function, which transforms lengths in the following way. A line segment OA is drawn on axis (b). In order to compute $\frac{p}{q}$ of OA, the wagon is placed on axis split q, the needle rotates as to indicate A, and the wagon moves to axis split p. Then the needle indicates a certain point B on axis (b). OB is the requested segment



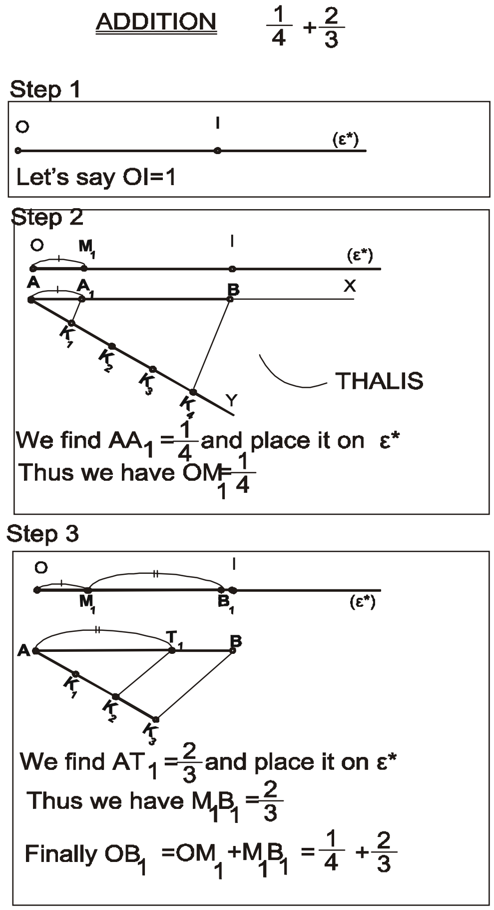
E.g.: In order to find the 3/10 of a line segment with a length of 30 cm, the wagon is placed on 10; the needle rotates until it indicates the end of the segment: i.e. 30.Then the wagon moves to 3.Now the needle shows the 3/10 of the line segment, which is a 9 cm segment.

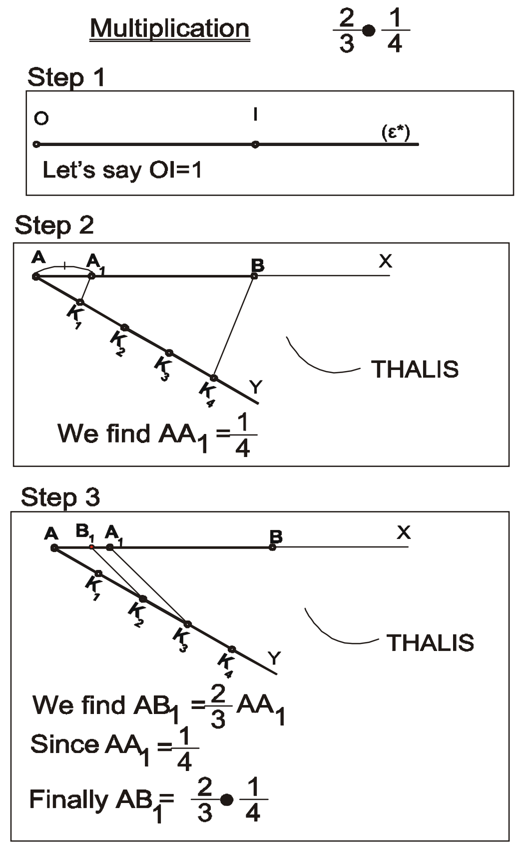
THALIS succeeds on the following:

1. It visualizes fractions as well as all of their properties and operations.
2. It is self-consistent since it is not capable of producing contradictory situations.
3. It is not over-specific since it does not produce multiple representations of a problem's solution.
4. It can easily create any subdivision of the form a/b of the usual manipulating aids (e.g.: you can easily find the 1/7 of a paper sheet)

3

Below are some examples of the use of THALIS





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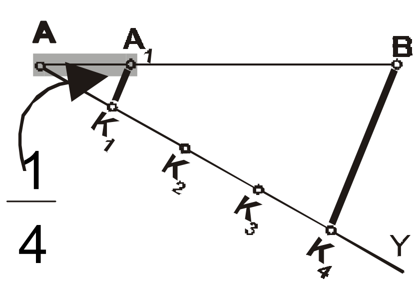
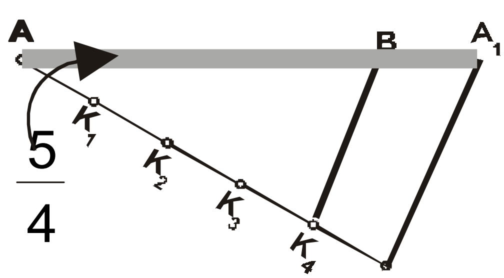
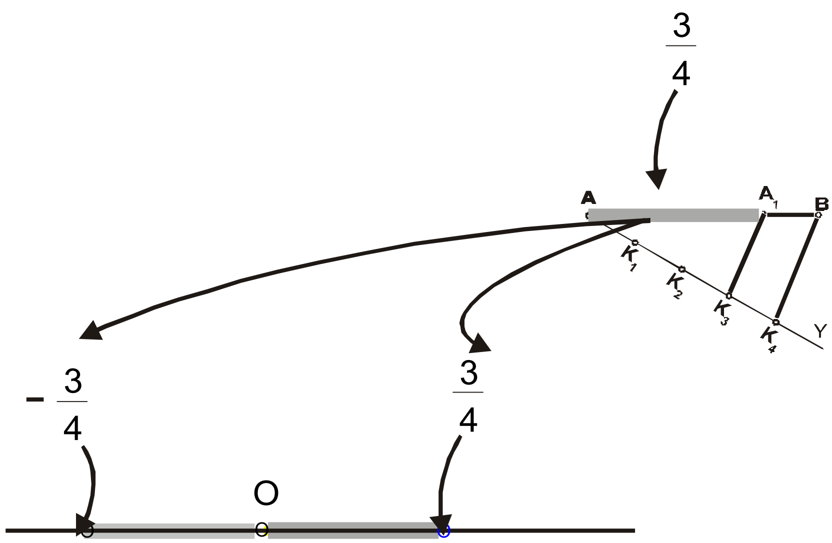
In order to support the scientific soundness of THALIS, we must prove that it is a mathematical model of the field of rational numbers. This way, the structures of THALIS and the field of rational numbers match completely. For that purpose, we follow the steps below. First, we find the set K of segments whose length is a rational number. The following classic method of finding the segments of length $\frac{a}{4}, a=1,2,3,4,5,...$ can also be applied for finding segments of length $\frac{a}{2},\frac{a}{3},\frac{a}{5},..., a=1,2,3,4,...$



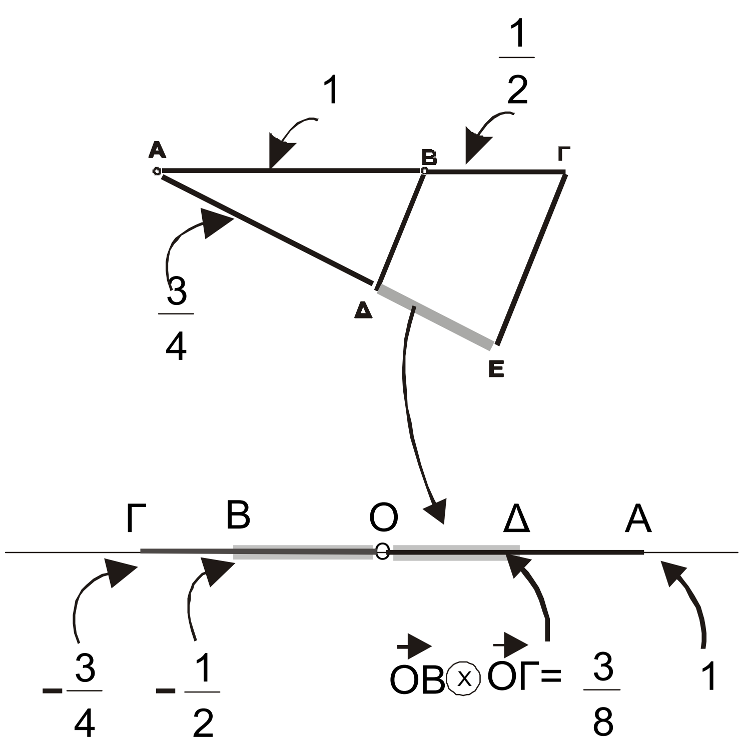


Each of the above segments creates two opposite vectors on an axis. For example, the figure below shows how the segment of length 3/4 creates two opposite vectors.



We name F, the set of all vectors of the axis that have been created in the above way. Thus, the F set contains all the vectors of an axis whose length is a rational number.

We supply the set F with the usual vector addition $\oplus$. Also, if $\vec{e}$ is the unit vector and $\vec{a}, \vec{\beta}$ are any two vectors of the F set, we define their product $\vec{a} \otimes \vec{\beta}$ as the vector $\vec{\gamma}$ with has positive direction if $\vec{a} \uparrow\uparrow \vec{\beta}$, and it has negative direction if $\vec{a} \uparrow\downarrow \vec{\beta}$ and length for which the proportion $\frac{|\vec{\gamma}|}{|\vec{\beta}|} = \frac{|\vec{a}|}{|\vec{e}|}$ is valid.



The left hand figure shows how to find the product of the vectors $\vec{OB}, \vec{O\Gamma}$ which have the same direction and $|\vec{OB}| = \frac{1}{2}$ & $|\vec{O\Gamma}| = \frac{3}{4}$

If we consider the language of the fields $L = \{+,-,\cdot,0,1\}$ and the L-structures Δ_1,Δ_2 with domain the set of rationals Q and the set F, correspondingly, then, we can easily find that all symbols of L keep their interpretation on the structures Δ_1,Δ_2. That means that there is an isomorphism $\omega : \Delta_1 \rightarrow \Delta_2$. Since the structures are isomorphic, therefore Δ_1 constitutes a model of Δ_2, which is a model of the field of rational numbers. Δ_2 is the basis of THALIS; therefore THALIS constitutes a model of the rational numbers field.

5

We propose to use THALIS to construct a composium of classroom activities aimed at promoting the students' construction of rational numbers as mathematical objects, and their construction of transformations and operations upon those objects. Put another way, the objectives of this collection are to have students first construct transformations as "**things to act with"(LEVEL 1)** and then to reconstruct them as **"things to act on"(LEVEL 2)**.(4)

More specifically:

LEVEL 1

- There is a discussion which concerns the finding of e.g.: $\frac{1}{2},\frac{1}{3},\frac{1}{4}$ of several quantities
- THALIS is introduced along with the instructions for its use
- Students experiment
- Students will be asked to think of fractions in a new way; not as a part of a whole, but as a transformation of a quantity into another.(4)

LEVEL 2

- In order to clarify that each rational number can be represented in infinite ways, the following problem may be given: "Find the $\frac{2}{3},\frac{4}{6},\frac{6}{9}$ of a thread. What do you observe?"
- Compare $\frac{1}{2},\frac{1}{3}$
- Find the products $\frac{15}{2}\cdot\frac{2}{3},\frac{2}{3}\cdot\frac{15}{2}$. What do you observe?
- Find the sum $\frac{15}{2}+\frac{2}{3}+\frac{5}{6}$

Before using THALIS in the school context, teachers must become familiarized with it, in order to comprehend the specific problems that the traditional representation systems had in the past, and subsequently feel comfortable and secure with its use.

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