A GENETIC APPROACH TO AXIOMATICS

Manfred KRONFELLNER
Vienna University of Technology
A-1040 Vienna, Wiedner Hauptstr. 8-10
e-mail: m.kronfellner@tuwien.ac.at

ABSTRACT

The genetic method is often regarded as a counter-current to the New Math and its exaggeration of formal and axiomatic mathematics. As a consequence axiomatics has been considerably reduced (nearly deleted) in school mathematics, whereas university mathematics is mostly still presented in a rigid deductive way. This discrepancy leads many freshmen to considerable difficulties, as we all know.

In this paper I will propose a synthesis between genetic and axiomatic method. In particular the axiomatic method is not only a method but also an interesting and very important subject of teaching and research itself: a milestone in the development of mathematics (Euclid), its philosophical background (Aristoteles), its purpose (Zenon), its consequences (construction with compass and ruler). Axiomatics as a model for representation of topics of mathematics (and other sciences) up to now, axiomatics as a destination of a process, not a starting point.

Examples of how to cope with axiomatics at school and at university will be discussed in this paper.

1. Preliminary Remarks

In the fifties/ sixties of the last century one of the main goal of the New Math was to reduce the gap between mathematics at school and at universities. School mathematics tried to imitate university mathematics, which was dominated in those days by the Bourbaki style. This led

- to an overemphasis of the axiomatic method
- to a high level of abstraction and formalization
- to the disdain of heuristic approaches.

Soon educators had to learn that the New Math was doomed to failure. As a counter-current to the New Math the genetic method was rediscovered. As a consequence, intuitive approaches and heuristic methods were esteemed again. Another consequence was the elimination of contents introduced by the New Math a few years ago. With some of these eliminations I agree; other eliminations I regret. In my opinion, the proponents of the counter-current failed to notice the fact that not necessarily the contents or goals were wrong, but the result of these contents together with the traditional methods of teaching and assessment. Furthermore, I am convinced that most of the abstract concepts have been placed too early in the curriculum (e.g. the concept of group in 7^{th} grade!) and caused therefore difficulties. If these concepts would have been introduced in higher classes the students had have a better chance to get a grip on them.

Abstract concepts and axiomatics usually are starting points at the university level (in books, papers and lectures), but even at this level students have difficulties to cope with them. The history of mathematics shows that these are rather final goals resp. final steps of a (sometimes long) scientific development.

Whenever axiomatics is (or was) used (at the university or at school in the New Math period resp.) I miss(ed) a discussion ABOUT this method: What is the advantage of this method? What was the reason to invent (develop) this method? What are the problems which can be treated better with this method than without it?

2. The invention of axiomatics

In ancient Greece, among others the Pythagoreans made no mean contribution to mathematics, not only their famous theorem, but also their philosophy and their theory of music, which supported their conviction that all in the universe is ordered by ratios of natural numbers. When they became aware of the problem of incommensurability they tried to apply their methods to infinity, too. Zeno showed with his famous paradoxes that these temptations may cause difficulties: Does a line consist of (indivisible) points (atoms)? Do we get these points when we bisect the line infinitely often? Can we make up a line out of points? (For more details see Boyer 1959, 23f, Kirk et al 1983, section 327-329, Struik 1967, 44; see also: Aristotle: From the Metaphysics and Physics, in Calinger 1995, 85-90.) Mathematicians of this period encountered difficulties in answering these questions. This showed the incompleteness of mathematical argumentation and produced a deep crisis of mathematics. Such inabilities led the Greek mathematicians to look for a consolidated basis of mathematics.

The answer to this question was (based on the philosophy of Plato and Aristotle) the method of axiomatics, which we can find in Euclid´s Elements: As long as the postulates and the axioms are accepted and the deductions are correct, noboby can contradict the result. This gave a feeling of certainty to the mathematicians in the discussion with critics like Zeno.

These postulates became famous centuries ago by the slogan, "construction by compasses and straight-edge". All one can construct by these "Euclidean instruments" is also deducible from the postulates of Euclid, and leads therefore to undeniable results.

The construction with compasses and straight-edge became famous especially in connection with the three classical problems of the antiquity: the duplication of the cube, the trisection of an angle, and the quadrature of the circle. (Kronfellner 2000)

Although we can find Zeno´s paradoxes in textbooks, I did not see in the student´s material their role in the history of mathematics, especially in connection with the invention of the method of axiomatics, until now.

A similar role like Zeno played Bishop Berkely with his criticism of the faulty foundation of the early calculus. (Eves 1976, 446) This can be regarded as the motivation to look for solid foundation of the calculus, a problem which needed more than one century to be solved.

3. The axiomatic characterization of the real numbers

Most of the theorems in real analysis (such as the intermediate value theorem and many others) can easily be illustrated and confirmed. For an exact proof one needs an axiomatic basis of the real numbers, which guarantees the completeness of R.

In my opinion, it is not necessary to teach the exact proofs in school. But the students should know that arguments based on graphical illustrations do not fulfil the demand on exactness which is usual (and necessary) in higher mathematics. The example of the ancient Greeks mentioned above should underline this necessity. This fact can also be illustrated by an anecdote of the german mathematician Richard Dedekind (1831-1916). When he had to prepare a lecture for freshmen at the Zurich Polytechnikum he wanted to facilitate the conclusions by avoiding arguments based on illustrations. So he came to the insight that he is missing an axiomatic basis of the real numbers. To this end, he developed the famous Dedekind cut.

4. Minimizing the axiom system

Already in ancient Greece the mathematician tried to minimize Euclid´s axiom system. The famous fifth postulate – the parallel postulate – seems not to be a proper postulate, but rather looks like a theorem. For many centuries mathematicians tried to prove this "theorem", that is, to deduce it from the other postulates. The solution that there cannot be found such a proof led to the invention of Noneuclidean geometry by Janos Bolyai.

5. Linear equations and the concept of group

For a simple genetic (but not historic) reconstruction of the development of the concept of group suitable for classroom teaching one may pose the questions:

- What do we need to be able to solve an equation like $x+a=b$?
- In which sets (structures) of mathematical objects is it possible to solve such an equation (with solutions within this set)?

The analysis of the solution ($a, b \in M$)

$x + a = b$ $\qquad$ $\exists e \in M\ \forall a \in M\quad a + e = e + a = a$ and

$\forall a \in M\ \exists a^* \in M: a + a^* = e$

$(x + a) + a^* = b + a^*$

$$x + (a + a^*) = b + a^*$$
$$x + e = b + a^*$$
$$x = b + a^* \quad (\in M)$$

shows that one needs exactly the axioms of a group. On the other hand, in every group it is possible to solve such an equation.

In some sense, these usual axioms of a group are also a counterexample to the usual temptation to minimize (or generalize) a system of axioms. In particular there exists the possibility of using the more general demands only of the existence of a left unit and left inverse elements (or right-... respectively) and to prove that these elements fulfil the conditions of right units and right inverse elements, too. In spite of this possibility, most authors demand only (for the sake of simplicity) a neutral element and inverse elements (the same elements for both sides). On the other hand they usually do not demand uniqueness in the axioms; this is proved as a theorem.

6. Once more: axiom or theorem?

What is the difference between a theorem and an axiom? Can an axiom be proved?

From my students I have to learn that such questions are not trivial! They are usually unfamiliar with these concepts. To explain the difference I use the following example:

I start teaching linear inequalities based on the axioms:

$a<b \Rightarrow a+c<b+c$

$a<b$ and $c>0 \Rightarrow ac<bc$

$a<b$ and $b<c \Rightarrow a<c$

prove further rules and apply these axioms and rules to problems. At the end of the chapter I ask the students whether we can prove $a<b \Rightarrow a+c<b+c$. I repeat that - by definition! - we cannot prove an axiom. But we can build up a new "theory" (equivalent to the previous one) based on other axioms:

$a, b > 0 \Rightarrow a + b > 0$

$a, b > 0 \Rightarrow ab > 0$

$a<b$ and $b<c \Rightarrow a<c$

Within this new system, it is possible to prove these laws, which we used as axioms in the previous system, easily as theorems. (Kronfellner/Peschek 1995, 66, problem 19137; for an extended version see Kronfellner/Peschek 1981, 139-141.)

7. Final remark

I do not completely agree with G. H. Hardy´s words, "Greek mathematics is ´permanent´,more permanent even than Greek literature. Archimedes will be remembered when Aeschylus is forgotten, because languages die and mathematical ideas do not." (Calinger 1995, 1) But these words should underline once more the importance of ancient Greek mathematics and mathematicians; and it should be our duty to use every opportunity to teach our students about their cognition, in particular the axiomatic method, which influenced not only the mathematics but also other sciences up to now.

REFERENCES

-Boyer, C.B., 1959, "The History of the Calculus and its Conceptual Development". New York: Dover
-Calinger, R. (Ed.), 1995, "Classics of Mathematics". New Jersey: Prentice Hall

-Eves, H., 1976, "An Introduction to the History of Mathematics", 4th ed., New York: Holt, Rinehart and Winston
-Fauvel, J., van Maanen, J., 2000, "History in Mathematics Education. The ICMI Study". Dordrecht: Kluwer
-Kirk, G.S., Raven, J.E., Schofield, M., 1983, "The Presocratic Philosophers. A Critical History with a Selection of Texts", 2nd edn, Cambridge: University Press
-Kronfellner, M., 2000, "Duplication of the Cube", in: Fauvel, J., van Maanen, J., 2000, 265-269
-Kronfellner, M., Peschek, W., 1981, "Angewandte Mathematik 1", 1st edition, Wien: Hoelder, Pichler, Tempsky
-Kronfellner, M., Peschek, W., 1995, "Angewandte Mathematik 1", 3rd edition, Wien: Hoelder, Pichler, Tempsky
-Struik, D.J., 1967, "A Concise History of Mathematics", revd edition, New York: Dover