LINEAR TRANSFORMATIONS AND EIGENVECTORS WITH CABRI II VIA MAPLE V.

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ABSTRACT

Teaching of Linear Algebra to beginners raises many cognitive problems related to the three thinking modes intertwined: geometric, computational (with matrices) and algebraic (Symbolic). We first study linear transformations in $\mathbf{R}^2$ and $\mathbf{R}^3$ in the Maple V environment. Here the mode is only symbolic and computational. To bring the geometric mode, students can be shown animations programmed with Maple; this will improve their geometric understanding, but during such animations Maple takes the role of a moviemaker and prevents students from participating as actors. Then we use the Cabri microworld where Maple animations can be rendered with its two functions "Locus" and "Animation". However in this micro-world, students can produce easily their own movies, change transformations, vectors and run their own explorations. We claim that students performing with Cabri will enhance their geometric and as well conceptual understanding and will link better the three thinking modes in Linear Algebra.

Topics of Linear Algebra chosen in this presentation are: linearity of transformations and the search of eigenvalues and eigenvectors in $\mathbf{R}^2$ and $\mathbf{R}^3$.

Keywords: Linear Transformations, Eigen-vectors and values, Pedagogical scenarios.

1. Introduction.

What is a linear transformation? What are their eigen-vectors and values?

The usual definitions found in any textbook of Linear Algebra are:

A linear transformation T defined on a vector space V has to satisfy the following conditions:

(i) $T(cv) = cT(v)$ for any scalar c and vector v

(ii) $T(v+w) = T(v) + T(w)$ for any pair v,w of vectors.

An eigenvector of the linear transformation T is a non-zero vector v that satisfies to the property: $T(v) = cv$ and the scalar c is then called the eigenvalue associated with the eigenvector v of T.

Which mode of interpretation did our students choose to grasp these definitions? First, the word 'transformation' has a metaphorical meaning; it indicates a change and even a movement in the space. Then how are understood the three conditions in the definitions above? Geometrically?, arithmetically? or algebraically?

These three modes of interpretation were observed and analyzed by several authors (cf. A. Defence, T. Dreyfus, J. Hillel, A. Sierpinska & S. Khatcherian).

2. Experience with Maple V

Our pedagogical scenarios for teaching the concept of linear transformations that we have been using since 10 years will be presented first. Our students being first exposed heavily to the algebra of vectors and matrices, we use matrices as prototypes of linear transformations. The two conditions *(i) & (ii)* are easily accepted as coming from the properties of the algebra of matrices:

(i') $A . (cu) = c (A.u)$

(ii') $A . (u+v) = A . u + A . v$

During a 2-hour workshop, ten years ago with grid papers and pencils, now in a computer lab with a CAS such as Derive first and then Maple, students are given a set of 2x2-matrices together with a 2xn- matrix representing a closed polygon with n vertices, one of which is the origin (0,0); this polygon has few right angles and pairs of parallel sides.

Students are asked to plot first the initial polygon, evaluate its area, look at its orientation when following the order in the matrix; then for each given matrix they repeat the same task: plot, area, orientation of the new polygon equal to the image under the transformation studied; they are to collect all observations into a big tableau with initial entries equal to the given 2x2-matrices; those observations are about the preservation of parallelism and the origin for each linear transformation, preservation of right angles only for symmetries, homotheties and rotations, change in areas and orientation depending on the determinant of the matrix of the transformation. Finally they are requested to write their own matrix with determinant = 0 and find out what happens in such a case to the image of a closed polygon. Students can easily see that the 2 column vectors of their matrix span the line of projection onto which the image has collapsed.

After such a workshop, during regular class time we can discuss with more comprehension on the geometric role of the conditions*: (i) & (ii):*

- The image of any line through the origin is again such a line.
- The image of any pair of parallel lines is again a pair of parallel lines.
- The image of any closed polygon is again a closed polygon or a closed interval in the case of a singular matrix.

Finally the role of the matrix as a code for a geometric transformation is clarified by exhibiting the initial basis (u1, u2) (standard basis is commonly used here) and the image pair (v1, v2) = (T(u1),T(v1)) that is also a basis if the transformation is not singular, i.e. is invertible or equivalently of determinant not equal to 0.

Then during a second class, a new lesson (created in fall 2001) constructed with Maple on eigenvectors and eigenvalues was presented. Using a CAS like Derive or Maple helped me to diminish the ambiguity in the students' minds between the arithmetic and geometric modes of representation of a linear transformation. The confusion for some students between the matrix multiplication and the scalar multiplication was alleviated thanks to different commands used in Maple.

We went back to all transformations exhibited during the first class, searching for eigenvectors and then their eigenvalues. In a geometrical context, it was easy to make the observations:

- The directions of lines preserved by the transformation give eigenvectors.
- Eigenvalues measure the ratios of the of the two collinear vectors lengths.

The situation of rotations with no eigenvectors was geometrically clear for students but then we had to stress the relation with the irreducible (over **R**) characteristics polynomial $x^2 + 1$.

A new matrix K=[[1,1],[4,1]] encoding a linear transformation was introduced. The directions of the eigenvectors are rather easy: the two symmetric lines y= -2x and y = 2x associated with the eigenvalues: 3 and –1. Then we run the experience with a symmetric matrix L= [[2,1],[1,2]] that gives the two orthogonal lines of eigenvectors: y = -x (eigenvalue=1) and y = x (eigenvalue=3). Finally we dealt with a regular symmetric markovian matrix [[0.3, 0.7],[0.7,0.3]].

Using Maple we obtain easily the eigen-vectors and -values by running an animation. The initial variable vector w = [cos(t),sin(t)] is chosen on the unit circle, then the algebraic calculation of T(w) is performed and at the same time the image vector T(w) is plotted. The animation will be executed with the parameter t running over the circle and can be interrupted as soon as w and T(w) appear collinear. In the case of a markovian matrix T, we ask Maple to evaluate the sequence of iterates T^n of T and its limit. The limit matrix consisting of two identical column vectors is to be compared with the eigenvector of T associated with its dominant eigenvalue 1.

Pictures with Maple

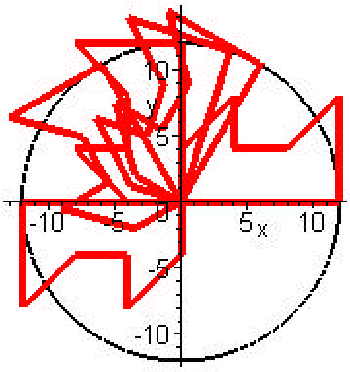


Fig. 1

Different Rotations of the same Polygon

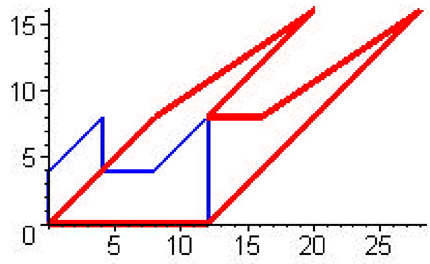


Fig. 2

Shear Transformation encoded

More Pictures with Maple

Case of $\quad K = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$ with the eigenvalues 3 and – 1.

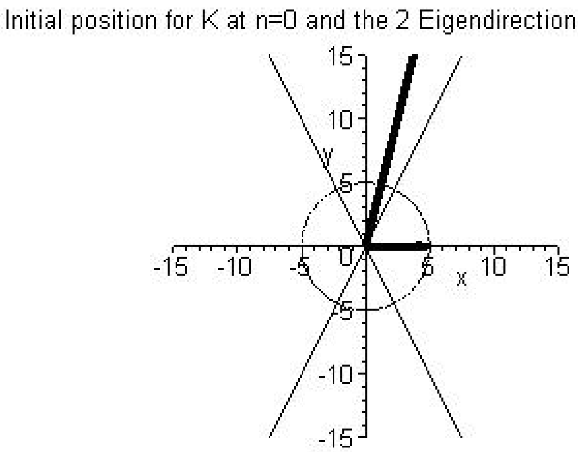


Fig. 3

Case where $L = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$, with the eigenvalues 6 and 2.

Initial position for L at n=0 and the 2 Eigendirections y=x & -x

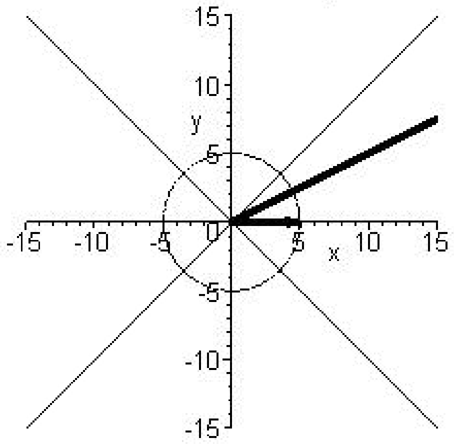


Fig. 4

Scenarios with Cabri II

Since the Fall semester 2001 we have been able to use CABRI II at Dawson College, during my Linear Algebra classes. We adapted to the Cabri environment, the pedagogical scenario described in the previous paragraph.

During the computer lab the macro-construction of a linear transformation depending on the origin O, an initial basis (u1, u2), the image pair (v1, v2) is given for help but not shown to students; only later during regular class time the macro construction will be explained and the importance of the conditions*: (i) & (ii)* will be stressed out.

We give again to students, as during the Maple exercise, the same set of 2x2-matrices encoding well-known linear transformations. Using the standard set of axes and associated grid of Cabri, students may draw their own closed polygon having O as one vertex, with parallel sides and few right angles. Then they point one vector w onto this polygon. To be able to use the Cabri "macro" of linear transformation, they will choose the standard basis for (u1, u2) and then draw two vectors v1 and v2 originating from O. They should be instructed that the pair (v1, v2) represents the image pair (T (u1), T(u2)) that happens to be the two column vectors of the 2x2-matrix representing T. Now the superb " locus" function of Cabri will trace the whole image of the polygon under the transformation T. How to change the transformation T? With Cabri it is a very easy task, as the student just needs to move the 2 vectors v1 and v2. For each matrix given during the Maple task, here we just need to put the vectors v1 and v2 into the positions of the column vectors of the matrix. Then as in a dream, the previous locus changes simultaneously to the new image polygon. We ask the students to use the animation function for two reasons:

- To observe the simultaneous moves of w onto the initial polygon and of T(w) on the image. Parallelism, right angles and orientation could be analyzed during this animation. This could not be done with Maple.

- Given a parametric family of linear transformations, as rotations in the last Maple experiment, we can animate the pair (v1, v2) together with the family of images of the polygon.

It seems that with this new scenario with Cabri, we should gain in clarity for the geometric relationships to observe.

Now we are we going to explain our second lesson on eigen-vectors and –values in this new environment of Cabri. Are we going to gain for the students, more clarity with Cabri than with Maple?

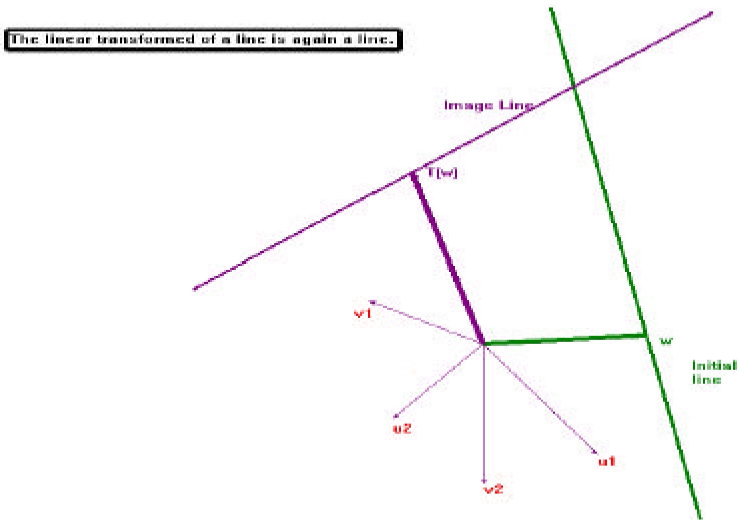
During the animation of the vector w running onto the initial polygon P while its image T(w) traces the image of P, students can be asked to be attentive at the instant when vector and image are collinear. We may also drop the polygon scenario and instead start with a circle as initial object;

We shall fix the vector w pointing to that initial circle and explore the movement of T(w); during the animation, we shall stop whenever the 2 vectors are collinear, ask Cabri to give their coordinates and to calculate their ratios in order to obtain the associated eigenvalue. Later we analyze with students the shape of the locus of T(w) by posing the following questions: Are the two axes of symmetry of the locus given by the lines of eigenvectors? If it is not always the case, for which type of matrix does it happen? To meet this instance we added to our study the symmetric matrix Q associated to the quadratic form $x^2 + y^2 + xy$.

Finally we looked at a markovian matrix T; Cabri can plot the orbit of iterates T^n (w) for any initial markovian vector w; The sequence will converge to the eigenvector associated with the dominant eigenvalue 1 and will be found on one of the line of eigenvectors.

We may conclude now that in this Cabri micro-world, students can change transformations, vectors and run their own explorations. We claim that students performing with Cabri will enhance both geometric and conceptual understanding and will be able to link the three thinking modes of Linear Algebra.

Cabri Picture   Fig. 5



All following Illustrations are Cabri pictures:

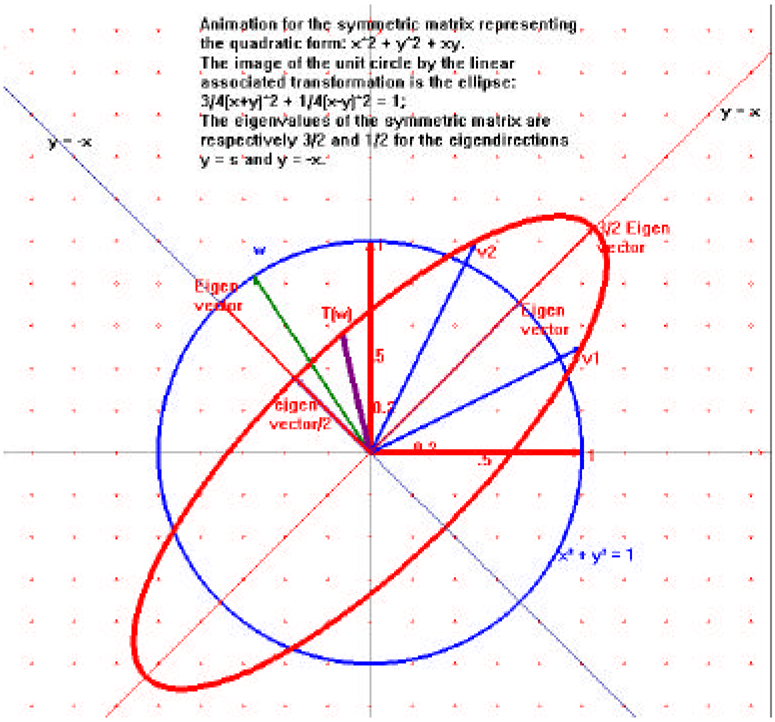


Fig. 6

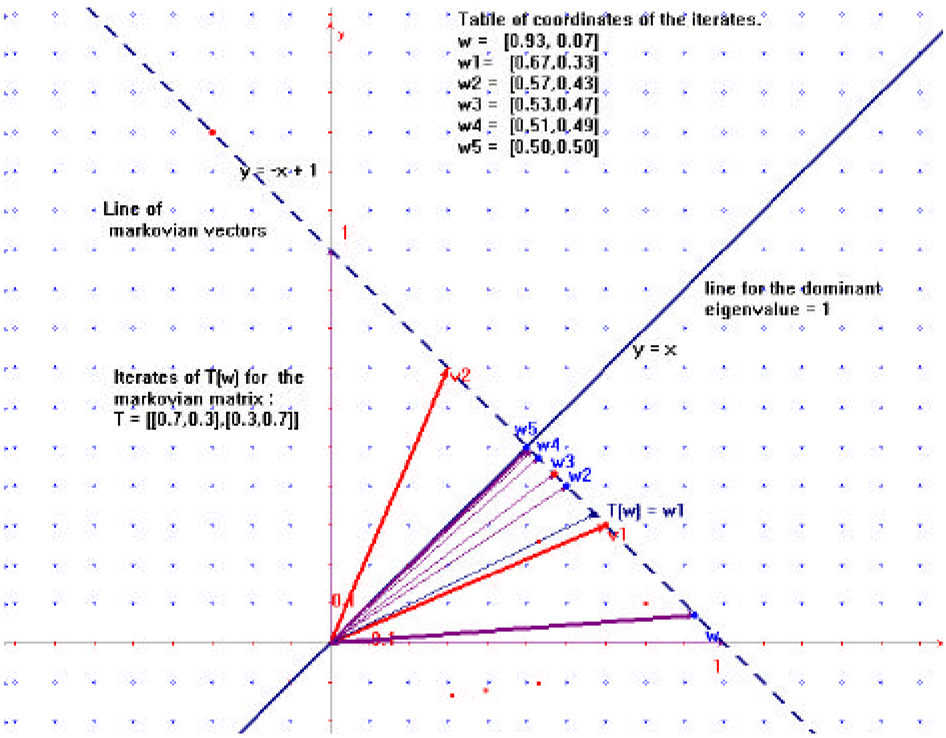


Fig. 7 : Iterates of w for a markovian transformation T converge to the line y=x.

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