A LIMIT–FREE APPROACH TO DERIVATIVES:
Report on a Classroom Project

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ABSTRACT

Led by the idea that "…using a graphing calculator to zoom in functions is one of the best ways of seeing local linearity" (Hughes-Hallet, Gleason et al. 1994), we discuss here the role of new technology in teaching and understanding mathematics and present a complete "route to derivatives", which is particularly suitable for undergraduate teaching. Starting with a formal definition of the tangent to a graph we are led to the fundamental theorem of calculus. The basic definition (the slope of the tangent) relies on graphic concepts, like screen resolution and pixels' dimensions, so, essentially, it's "limit–free". The fundamental theorem of calculus is presented in a discretized version, showing that the trapezoidal rule for areas and the central difference formula for the tangents' slopes are, in a sense, inverse one to each other. At an higher level, we can note that the central difference rule can be applied to solve numerically differential equations. Everything has been implemented on an intermediate graphic calculator (actually the TI–82, TI–83 and TI–83 Plus), and carried out in the last three years on a total population of about 750 students of life sciences, pharmacy and chemistry, aged 18–19, from two different universities in Italy. As a result we noticed a substantial improvement of low-ranked and middle-ranked students. We present also the project a "teacher bank", consisting of TI–GraphLink compatible files containing related programs and data, and more advanced teaching tools as Mathematica notebooks and QuickTime movies, especially generated by TI–8∗ screenshots.

Keywords: ICT (Information and Communication Technology), tangent line, derivative, antiderivative.

1 Does technology change mathematics?

In recent years, technology has changed and will continue to change our ideas about two central pillars of mathematics: definitions and proofs. Recent advances, like the celebrated and controversial proof by Haken and Appel of the Four Color Theorem, made us a little bit confused on these very basic fundamentals of mathematics. Actually, we do not believe that there exists a possible *objective* verification of the mathematical rigor. However, in our opinion, these remarks on the rigor of a mathematical proof should concern only professionals in mathematics. Especially at the undergraduate level, we often have to ask ourselves: it is necessary to proof everything? In our opinion, the italian mathematician Bruno de Finetti gave the answer some 60 years ago: *"I don't see why the beginner should check whether there is a mistake, which eluded everyone before him"* (de Finetti 1944). In many situations, the direct verification in few cases, or even better, a random check that we can make with a simple pocket calculator will prove more efficient than a series of lectures. If teachers in mathematics share such an opinion, the way we teach will change radically, especially if our role is to teach to students that will not pursue studies in mathematics.

In this respect, we are in a position common to many math teachers; we have to decide whether it is preferable to abandon a formal rigorous proof that has no chance of convincing or instead to present convincing arguments that are not a legitimate proof. Consider Rolle's theorem for instance. If we provide the students with a rigorous proof, our best students can probably repeat every single step in front of us, while other students may forget a single step and be stumped. Will our students bet a reasonable amount of their money on the validity of the theorems proven this way? To bet requires some conviction, most probably a simple picture will rise the stakes.

2 Can we use technology to convince?

New technology has made available to everyone high quality *graphics*. We are not referring here to graphics in mathematics, but to the kind of graphics that we meet in everyday life: DVD's, Videogames, electronic encyclopedias, the Web, advertising on TV and so on. People's minds are used to "scan" images and "record" sounds, more than to read. Images convey ideas on the spot. Obviously, images are not to be confounded with the full message, but they are extremely helpful and fast. In mathematics, pictures on the blackboard have always accompanied formal proofs for illustrative purposes to make the students get the idea. Good teacher also needed to be good drawers. With the help of suitable software, we have now the chance to present beautiful images in two or in three dimensions; and as a bonus we can create and present graphics in motion (i.e. animations). Also, the use of colors gives us an additional dimension. If we use graphics, we are using a language that students capture easily because is their own language and more important, if images are well conceived, the students' understanding really improves, and their attention span is longer. Moreover, when we need to present some subject that may already be familiar to most of the students (this is a quite typical situation), we are able to do this in a non-traditional way that is either more fascinating or more funny. We also have to remember that "mathematical images" can be surprising even if we are perfectly familiar with the subject. Mandelbrot himself, while seeing the first pictures of his famous set, thought of a software problem. So the Mandelbrot's set on screen was different from Mandelbrot's set in the mind of the "creator". To persuade a student of a theorem using a graphic calculator is the most recent possibility of a long educational path, where often the figure has been of some support to the formal proof.

There are also other situations where new technology helps to create understanding, without resorting to graphics. Think for instance to all simulation processes like the ones which appears in Montecarlo methods: a simple graphic calculator can "throw" a pair of dice thousands of times and count how many times the dice add to 4. Similarly, we can "throw" with a calculator thousands of random points in a rectangle and estimate the area of a complicated plane figure, thus unveiling a mathematical world hidden if we just use pen and paper. In such cases, experimentation is necessary and the opportunity of having a pocket laboratory opens a new world of mathematical investigations to the students.

Finally, we should not forget the possibility of using dedicated software like CABRI to teach geometry, and to use sensors, interfaces and graphical calculators to realize a classroom physics laboratory.

3 The re–definition of the derivative

Coming to the main subject, we will show that the technology available on a simple graphic calculator, via the `Zoom In` function, allows a complete *re–styling of the definition of derivative*. In order to define the derivatives as the limit of the Newton quotient, we are forced to introduce the concept of limit. The definition of the derivative as the limit of the Newton quotient introduces some additional problem: students are led to think of the tangent line as something which requires as an intermediate step a secant line "in motion" (rotation) around the fixed tangency point and with the other near intersection moving always on the same side of the tangency point; students thus lose the "bilateral" character of the derivative.

On the other hand, graphic calculators put in the hand of every student the `Zoom In` function which allows the student to view the graph of a function under a microscope: "...using a graphing calculator to zoom in functions is one of the best ways of seeing local linearity" (Hughes-Hallet, Gleason et al. 1994, p. viii).

From these ideas, we can see the need for a definition of the derivative which

- do not require the secant line as an intermediate step
- is consequently "limit–free".

The idea of teaching derivatives without worrying too much about limits appeared first with a convincing support in the Seventies in *"A First Course in Calculus"* by Serge Lang (Lang 1973). In this textbook, Lang defined the derivative in a classical way but restricted himself to an introduction of the limit theory in a substantially axiomatic way, based on algebraic rules, on the squeezing theorem, and on the direct assignment of the limits of the constant and the identity function. The first true limit–free definition of the derivative appears in 1981 in the *"Calculus Unlimited"* book by Jerrold E. Marsden and Alan Weinstein (Marsden & Weinstein 1981). The authors used the order relationship to formalize the idea that the tangent line is the unique line not *transversal* to the graph of the function (in other words any line with bigger or smaller slope "cut" in a definite way the graph of the function). In the above-mentioned "Calculus", the idea is still to get rid of the limit and the use of the zoom is proposed at an intuitive level. With respect to Marsden and Weinstein's idea, we have today the technological advantage to be able to realize everything on a pocket calculator.

We want now to formalize the fact that, with a sufficiently good "microscope", the graph of a derivable function, appears to be straight. On an initial intuitive level, we may say that *the tangent line to the graph of $y = f(x)$ at the point $P = (x_0, f(x_0))$ is the straight line which is graphically the same as the graph of the function $y = f(x)$ if we zoom in the graph enough around the point P, provided that this line remains the same even by taking a series of zooms in sub-windows, and this for every possible resolution of the actual display.*

For instance, suppose we use a calculator whose display corresponds to a grid of 200×100 pixels. Let us center the window at the point P and select the window parameters:

$$\begin{aligned} X_{\texttt{min}} &= x_0 - d \\ X_{\texttt{max}} &= x_0 + d \\ Y_{\texttt{min}} &= f(x_0) - d \\ Y_{\texttt{max}} &= f(x_0) + d \end{aligned}$$

The vertical dimension of the pixel is therefore $d/50$, and therefore, if the graph of $y = f(x)$ is graphically identified with the graph of $y = r(x) = f(x_0) + m(x - x_0)$, we will have:

$$|\, f(x) - \{f(x_0) + m(x - x_0)\}\,| \leq (1/50)\, d$$

At the leftmost ($x = X_{\texttt{min}} = x_0 - d$) and rightmost ($x = X_{\texttt{max}} = x_0 + d$) points on the screen we will have

$$|\, f(x) - \{f(x_0) + m\,h\}\,| \leq (1/50)\,|h| \tag{3.1}$$

for $|h| = d$. Requiring this to happen for every sub-window, (3.1) is the same as to ask that it is true for every h with $|h| \leq d$. Finally, if we want this to be hardware independent and to be true for every possible resolution of the display, we will have to replace $1/50$ with a positive ϵ "arbitrarily small". The derivative is then naturally the slope of the tangent line.

In a formal way, a function $y = f(x)$ is *derivable* at the point x_0 if there exists a number m (the slope to the tangent line to the graph of f at the point $P = (x_0, f(x_0))$) such that, for all $\epsilon > 0$ (no matter what the resolution of the display is) there exists a $d > 0$ (we find a graphic window P with half-width d) such that for every h with $|h| \leq d$ (in that window and in every sub-window), we have

$$|\, f(x) - \{f(x_0) + m\,h\}\,| \leq \epsilon\,|h|$$

(the graph of $y = f(x)$ become blurred with that of a straight line, namely the tangent line at P). Experts can recognize here the Fréchet notion of differential.

Starting from such a definition, it is possible to construct a complete and very "friendly" expository theory of derivatives, which uses a graphic calculator (see Invernizzi, Rinaldi et al. 2000) as a main resource. Our proposal has various important feature:

- Any graphic calculator allows viewing the definition of the derivative by simply zooming a function. If we assume a standard zoom factor 4×, by taking a sequence of four successive zooms we have a magnification factor 256. It is like having a microscope 256× which is enough to graphically straighten a derivable function with no pathology. Naturally it is impossible to straighten in this graphical way functions with corners: corners are in fact invariant under the zoom operation. See Fig. 1 and Fig. 2 (screenshots from the TI–83 Plus display).

- We can get a numerical estimate of the slope of the tangent line; it is in fact reasonable to approximate the tangent line in $(x, f(x))$ with a line which intersect the graph of the function slightly on the left and on the right of the given point, that is with the line through the points $(x-h, f(x-h))$ and $(x+h, f(x+h))$. See Fig. 3. In other words we can set

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

The last formula is the *central difference formula*, which despite its simplicity has a very good performance. The numerical value of a derivative is usually approximated on a pocket calculator by such a formula (command `nDeriv` on TI, or `d/dx` on CASIO, etc.). But we will see in next section that the central difference formula (on which essentially we develop the idea of tangent) is also very useful to discuss ordinary differential equations, the simplest of which is $y'(x) = f(x)$, linked to the fundamental theorem of calculus. The same ideas applied to an ODE like $y'(x) = f(x, y(x))$ would lead to a 2–step implicit integration scheme, which is especially suitable to study on a graphic calculator typical 2D systems (where the closed orbits tend early to break under Euler method based on the Newton quotient).

4 What's about $y'(x) = f(x)$?

To complete (even at an elementary educational level) the exposition of the differential calculus in a limit–free way we have to find a way to present the fundamental theorem of calculus. While it is difficult to trace a graph of the integral function on which to execute a sequence of zooms, in order to justify the theorem, we can resort to the programming capabilities of the calculator, to provide a *discrete version* of the theorem which will receive an unconditional approval in the classroom.

To "find" the antiderivative $y(x)$ of a function $f(x)$ (forgetting the existence problem) let us write the discrete analog

$$\begin{array}{ccc} y'(x_0) & \approx & f(x_0) \\ y'(x_1) & \approx & f(x_1) \\ y'(x_2) & \approx & f(x_2) \\ y'(x_3) & \approx & f(x_3) \\ \cdots & \cdots & \cdots \\ y'(x_n) & \approx & f(x_n) \end{array}$$

where the x_k's $(k = 0, 1, \ldots, n)$ represent a tabulation with step size h of the interval $[a, b]$ on which we wish to solve the problem at hand, and $x_0 = a, x_n = b$.

Let us approximate the derivative $y'(x)$ through the central difference formula (and with the left and right Newton formula at the endpoints):

$$\begin{array}{rcl} y(x_1) - y(x_0) & \approx & h \cdot f(x_0) \\ y(x_2) - y(x_0) & \approx & 2h \cdot f(x_1) \\ y(x_3) - y(x_1) & \approx & 2h \cdot f(x_2) \\ y(x_4) - y(x_2) & \approx & 2h \cdot f(x_3) \\ \dots & \dots & \dots \\ y(x_n) - y(x_{n-2}) & \approx & 2h \cdot f(x_{n-1}) \\ y(x_n) - y(x_{n-1}) & \approx & h \cdot f(x_n) \end{array}$$

After some obvious simplifications, we discover that

$$y(b) \approx y(a) + \frac{h}{2} \cdot \{f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)\}$$

i.e. that having estimated the derivative of $y(x)$ via the central difference formula, once these values are known, the function $y(x)$ can be reconstructed using the *trapezoidal rule*: at an intuitive level, *the central difference formula and the trapezoidal rule are inverse operations to each other.* We think that such an approach is at least equally acceptable at the undergraduate level than the standard "proof" of the fundamental theorem. Our students will work more often with tables of experimental data, like time series, than with pre-cooked formal expressions defining integrable functions. So to prove something once in a while in a discrete way is also more instructive to them and provides them with instruments that they can possibly use in their professional life.

5 Conclusions

Everything has been implemented on an intermediate graphic calculator (actually the TI–82, TI–83 and TI–83 Plus), and carried out for last three years on a total population of about 750 students of life sciences, pharmacy and chemistry, aged 18-19, from two different universities in Italy. We observed a substantial improvement of low-ranked and middle-ranked students (with respect to previous non technology-based teaching), meaning that a significative percentage of E's and D's moved to C. We're referring here to the European Credit Transfer System classification. In other words, we observed a substantial rise in the basic mathematical skills of the less gifted students, not accompanied by a simultaneous drop of the level of the best students. In practice, the number of students who fail the exam reduced in a quite remarkable way. We also noticed a strong improvement in the ability of the students to recognize mathematical aspects in other disciplines.

As a final remark, we think that many resources should be shared among teachers. In this direction, we strongly endorse the project of a "teacher bank", consisting of TI–GraphLink compatible files containing related programs and data, and more advanced teaching tools as Mathematica notebooks and QuickTime movies, especially generated by TI–8∗ screenshots. For a concrete example see:
http://www.dsm.univ.trieste.it/∼inverniz/quicktime/zoom.mov.

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6 Figures

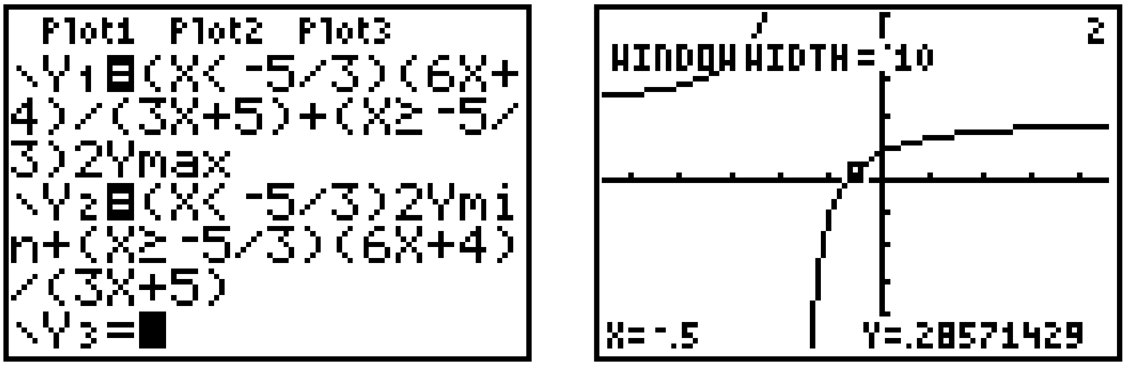

Figure 1: Definition and graph of the function $f(x) = (6x + 4)/(3x + 5)$. Notice that the definition avoids the drawing of the vertical asymptote.

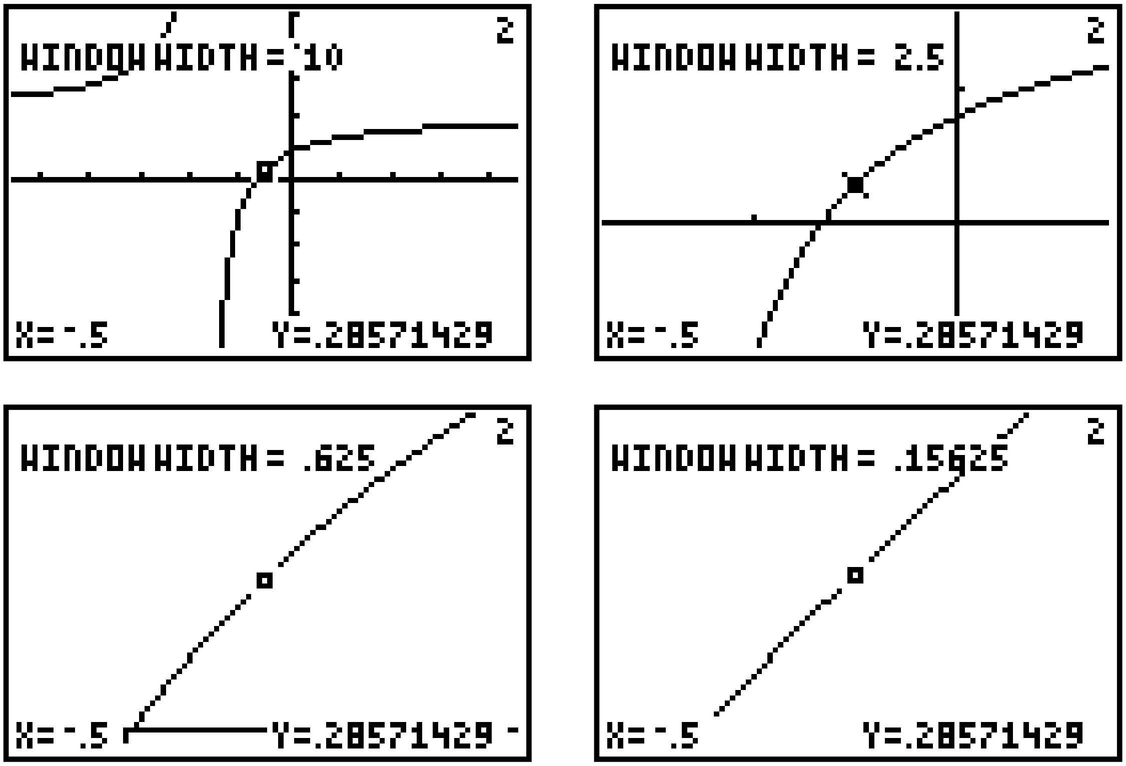

Figure 2: Graph of the function $f(x) = (6x + 4)/(3x + 5)$ and a sequence of zoom $4\times$ realized on the TI–83 Plus near $x_0 = -0.5$. The number in the right upper corner refers to which branch of the function the cursor is on.

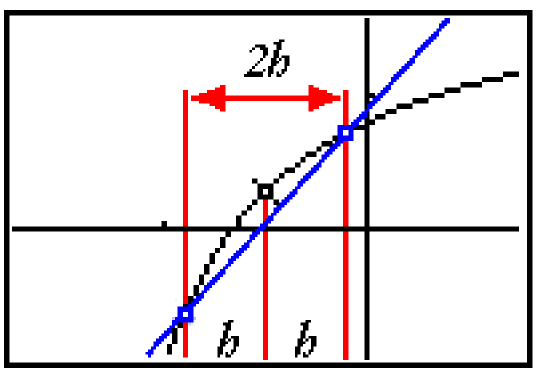

Figure 3: Secant to the function $f(x) = (6x + 4)/(3x + 5)$ near $x_0 = -0.5$ illustrating the central difference formula.