

**HOW TO FIND THE INTERNAL ANGLE OF A REGULAR POLYGON:
STRATEGIES OF PRE-SERVICE TEACHERS**

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ABSTRACT

The task of finding a regular polygon internal angle can be explored by students from middle school to college and beyond. This task can be investigated in many different ways from which it is possible to learn more about various properties of basic geometrical shapes such as triangles, quadrangles, and regular polygons. The study presented here is aimed at characterizing the solution strategies of pre-service teachers that were asked to find the internal angle of a given regular polygon. In this paper we describe the different solution strategies given by the pre-service teachers and discuss the contribution of the whole class discussion to the learning process. Since the different solution strategies all shared some common features, we suggest that this task could promote mathematical generalization.

Introduction

Finding a regular polygon internal angle is an issue to explore for students from middle school to college and beyond. Various studies investigated different aspects of regular polygons such as: connections between the number of polygon sides, angles and area (Battista, M., 1985; Waters, W. M., Jr., 1987; Killgrove, R. B. and Koster, D.W., 1991); Connections between a regular polygon sides' length and the length of its diagonals (Tzamir, Tirosh and Stavi, 1997); Construction of regular polygons, and their internal angles aided by a ruler and caliper (Austin and Austin, 1979; Benson and Borrkovitz, 1982); Construction of different regular polygons by joining squares, corner to corner (Muscat, 1992); Connection between regular polygon and its central angle (Happs and Mansfield, 1992); Connection between the n-sided regular polygon area inscribed in a circle and the circle' area, as n approaches to infinity (Kich, 1979), and so on.

Researchers examined also possible connections between the number of polygon sides and the value of its related internal angle. Troccolo (1987) presented a method for accurately constructing regular polygons with a given specified side length. This method is based on the idea of inscribing the regular polygon in a circle, dividing it into triangles and finding the base angle of each central triangle.

While engaging in activities connected to mathematical definitions, Borasi (1992) used the task of finding the internal angle of a regular pentagon inscribed in a circle. The students were given two hints related to circle properties (i.e., equal radii, central angle of 360°). Using these given hints, the students divided the pentagon into five congruent isosceles triangles, first found the central angle, then the pentagon internal angle. In another activity related to the connection between polygon sides and its internal angles, Borasi (ibid.) asked students to define a polygon. Their definitions were based on the theorem "In an n-sided polygon, the sum of the interior angles measures $(n-2)*180^{\circ}$ " (p. 45).

The present study focuses on a variety of strategies given by pre-service teachers while trying to find the internal angle of a regular pentagon, and to generalize it to n-sided regular polygon.

The study

The aim of the current study was to characterize strategies of pre-service teachers that were asked to find the internal angle of a given regular polygon.

Forty-two pre-service teachers participated in the study. The participants took part in a two-hour workshop dealing with regular polygons. Each student was given two tasks:

- (i) What is the internal angle of a regular pentagon?
- (ii) What is the internal angle of an n-sided regular polygon?

During the first part of the workshop, the participants were asked to reply to the above questions individually. In the second part of the workshop there was a full class discussion, based on the written reports.

Findings

First task – individual work

In order to find the internal angle of a regular pentagon, nine different strategies were raised by the participants.

Strategy (a): using the fact that a regular pentagon can be inscribed in a circle whose center is the center of gravity of the pentagon, many participants divided the regular pentagon into five isosceles

triangles (as shown in Fig. 1 below). They found β ($\beta = 360^\circ/5$), and calculated the regular pentagon angle α based on the isosceles triangle ($\beta + \alpha/2 + \alpha/2 = 180^\circ \Rightarrow \alpha = 108^\circ$).

Strategy (b): the regular pentagon was divided into three triangles by drawing two diagonals from one of the pentagon vertex (Fig. 2). The pentagon angle α was found by calculating the sum of the three triangles ($5 \cdot \alpha = 3 \cdot 180^\circ \Rightarrow \alpha = 108^\circ$).

Strategy (c): the internal angle α was found by using the fact that the external angle of a regular pentagon is $360^\circ/5$, and hence the internal angle α equals $180^\circ - 360^\circ/5$.

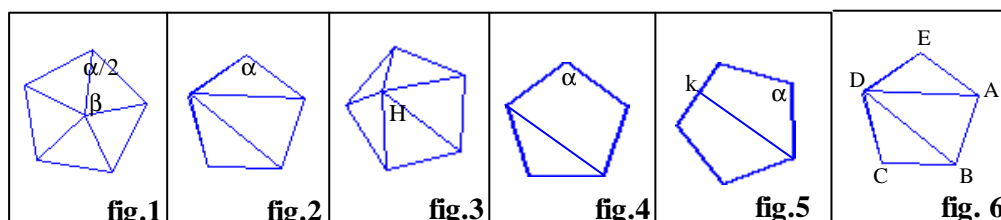
Strategy (d): the internal angle α was found by using the formula $\alpha = [180^\circ(n-2)]/n$ and by assigning $n = 5$, α was found.

Strategy (e): the internal angle α was found by dividing the regular pentagon into five triangles and connecting the pentagon vertices to an added reference point (H) inside the pentagon (Fig. 3). The pentagon angle α was found by calculating the difference between the sum of the five triangles and the inner angle H ($5 \cdot 180^\circ - 360^\circ = 5 \cdot \alpha \Rightarrow \alpha = 108^\circ$).

Strategy (f): dividing the regular pentagon into a quadrangle and a triangle by drawing one diagonal from one of the pentagon vertex as shown in Fig. 4. The pentagon angle α was found by calculating the sum of the quadrangle and triangle angles ($5 \cdot \alpha = 360^\circ + 180^\circ \Rightarrow \alpha = 108^\circ$).

Strategy (g): this strategy involved dividing the pentagon into two quadrangles (as shown in Fig. 5). The pentagon angle α by calculating the difference between the sum of the two quadrangle's angles and the straight angle k ($2 \cdot 360^\circ - 180^\circ = 5 \cdot \alpha \Rightarrow \alpha = 108^\circ$).

Strategy (h): division of the pentagon to two overlapping trapezes (ABCD and DEAB as shown in Fig. 6). The pentagon angle α was found by calculating the difference between the sum of the two trapeze angles and overlapped triangle (ADB) ($2 \cdot 360^\circ - 180^\circ = 5 \cdot \alpha \Rightarrow \alpha = 108^\circ$).



Strategy (i): By listing the known factors (as shown in Table 1), and using the arithmetical series properties' the internal angle α was found.

The regular shape	sum of angles	Inner angle
Equilateral triangle	180°	60°
Square	360°	90°
Pentagon	540°	108°

Table 1: strategy (i)

Table 2 illustrates the distribution of the strategies used by the pre-service teachers while solving the first task.

strategy	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)	Total
Num. of students	17	12	1	5	1	6	1	1	1	45*

Table 2: Distribution of strategies for the first task

Second task – individual work

The participants used four different strategies to find the internal angle of an n-sided regular polygon.

Strategy (1): using the fact that a regular polygon can be inscribed in a circle whose center is the polygon's center of gravity, the internal angle of n-sided regular polygon was expressed by finding the central angle β ($\beta = 360^\circ/n$), and calculating the polygon's angle α using the isosceles triangle ($\beta + \alpha/2 + \alpha/2 = 180^\circ \Rightarrow \alpha = 180^\circ - 360^\circ/n$) (Figure 7).

Strategy (2): the regular polygon was divided into n-2 triangles by drawing n-3 diagonals from one of the polygon vertex (Fig. 8). The polygon angle α was expressed by the equation $n \cdot \alpha = (n-2) \cdot 180^\circ$ (meaning, $\alpha = 180^\circ(n-2)/n$).

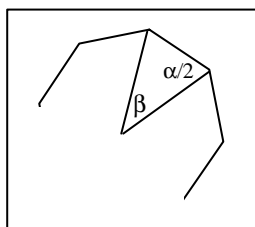


Figure 7

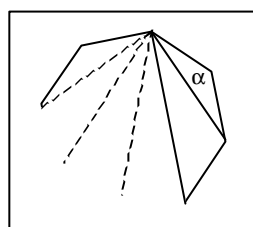


Figure 8

Strategy (3): using the fact that the external angle of a regular polygon is $360^\circ/n$, and hence the internal angle α equals $180^\circ - 360^\circ/n$.

Strategy (4): stating the formula $\alpha = [180^\circ(n-2)]/n$ in order to express α .

Table 3 illustrates the distribution of the strategies used by the pre-service teachers while solving the second task.

Strategy	(1)	(2)	(3)	(4)	Total
Num. of students	25	7	1	9	42

Table 3: distribution of strategies for the second task

Classroom discussion

At the beginning of the classroom discussion each participant presented the strategies he used to find the pentagon internal angle. When introducing strategy (e), in which a reference point H was constructed inside the pentagon, one of the participants proposed checking a case in which H falls on one of the pentagon sides (Figure 9). The participants were asked to explore this particular case and they arrived at the solution that the pentagon angle α can be found by calculating the difference between the sum of the angles of four triangles and the straight angle H ($4 \cdot 180^\circ - 180^\circ = 5 \cdot \alpha \Rightarrow \alpha = 108^\circ$).

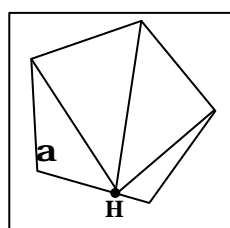


Figure 9

While discussing the former case, two questions were raised: could the internal angle of a regular pentagon be found if the reference point H falls outside the pentagon? Would it give the same solution regardless of the location of H, or would there be a different solution for each location? At this stage the participants were asked to work with the 'Geometry Inventor' computerized program in order to explore the different cases.

Investigations within the environment led to the conclusion that this case includes five different sub-cases (Fig. 10):

- (1) H falls in area A.
- (2) H falls in area B.
- (3) H falls in area C.
- (4) H falls on one of the segments DI or EI.
- (5) H falls on one of the rays IG or IF.
- (6) H falls exactly on I.

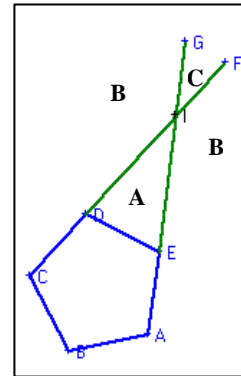


Figure 10

Figure 10.1 below describes the first sub-case. In this case the internal angle α of a regular pentagon can be found by calculating the difference between the sum of the angles of the four triangles (HDC, HCB, HBA, and HAE) and the angles of the triangle that partially overlaps them (HDE) as follows: $4 \cdot 180^\circ - 180^\circ = 5 \cdot \alpha \Rightarrow \alpha = 108^\circ$.

Figure 10.2 describes the second sub-case. In this sub-case the internal angle α of a regular pentagon can be found by calculating the sum of the two triangles angles (HDC and HCB) and the angles of the concave quadrangle (HBAE) minus the angles of the triangle that partially overlaps them (HDE) as follows: $2 \cdot 180^\circ + 360^\circ - 180^\circ = 5 \cdot \alpha \Rightarrow \alpha = 108^\circ$.

Figure 10.3 describes the third sub-case. When the internal angle α of a regular pentagon can be found by calculating the sum of the two concave quadrangle angles (HDCB and HEAB) minus the angles of the triangle that partially overlaps them (HDE) as follows: $2 \cdot 360^\circ - 180^\circ = 5 \cdot \alpha \Rightarrow \alpha = 108^\circ$.

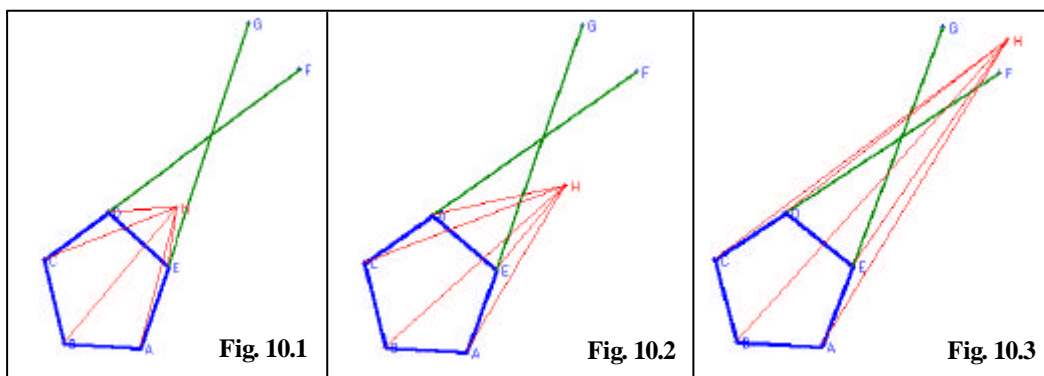
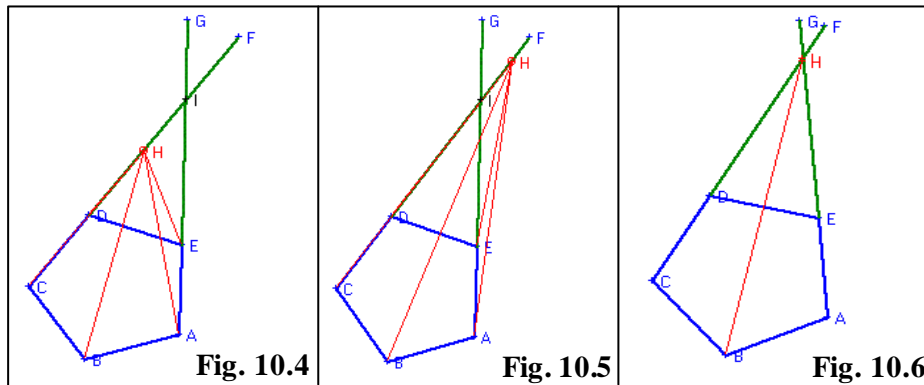


Figure 10.4 describes the fourth sub-case. In this instance the internal angle α of a regular pentagon can be found by calculating the sum of the angles of the three triangle's (HCB, HBA and HAE). One can see that $3 \cdot 180^\circ = 4\alpha + \angle DHE + \angle HED$, but since $\alpha = \angle CDE = \angle DHE + \angle HED$ ($\angle CDE$ is an external angle to triangle DHE), $3 \cdot 180^\circ = 5 \cdot \alpha \Rightarrow \alpha = 108^\circ$.

The fifth sub-case is described in Figure 10.5 where the internal angle α of a regular pentagon can be found by calculating the sum of the triangle and the concave quadrangle angles (HCB and HBAE).

Then, $180^{\circ} + 360^{\circ} = 4\alpha + \angle DHE + \angle HED$, yet again $\alpha = \angle CDE = \angle DHE + \angle HED$ and hence $180^{\circ} + 360^{\circ} = 5\alpha \Rightarrow \alpha = 108^{\circ}$.

In case H falls on I (the intersection of DF and EG as shown in Fig. 10.6) two triangles are formed (HAB, HCB) $\angle A + \angle B + \angle C + \angle DHE = 360^{\circ}$. Assigning $\angle DHE = 2\alpha - 180^{\circ}$ in the above equation will result in $5\alpha - 180^{\circ} = 360^{\circ}$ from which $\alpha = 108^{\circ}$.



As was mentioned earlier, in the second task, (finding the internal angle of an n-sided regular polygon) most of the participants used strategy (a) to solve the general case, while the others used strategies (b), (c) or (d). Classroom discussion yielded generalizations for the other strategies.

None of the participants used strategy (e) in the general case. In the classroom discussion the participants concluded that this strategy could also be generalized. They discovered that the n-sided regular polygon could be divided into n triangles by adding a reference point (H) inside the polygon (Fig. 11). The internal angle α can be found by calculating the difference between the sum of the n triangles and the inner angle H as follows: $n \cdot 180^{\circ} = n \cdot \alpha + 360^{\circ} \Rightarrow \alpha = 180^{\circ} - 360^{\circ}/n$.

The attempts to generalize strategy (f) caused some confusion. Some of the participants thought that the polygon should be divided into one quadrangle and the remainder into triangles (option A). Others argued that for $n > 5$ partition of the polygon could be into as many quadrangles as possible and the remainder into triangles (option B). In Figure 12 we can see an example of a hexagon divided into one quadrangle and two triangles (option A), or into two quadrangles (option B).

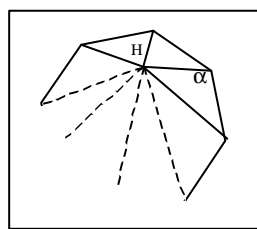


Figure 11

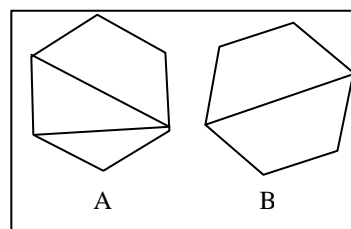


Figure 12

Generalization of option A to an n-sided regular polygon yielded a division into one quadrangle and $n-4$ triangles. In this case, the sum of the internal angles of the polygon will be $360^{\circ} + (n-4) \cdot 180^{\circ}$ and hence $\alpha = 180^{\circ}(n-2)/n$.

Generalization of option B to an n-sided regular polygon resulted in two instances: (1) the number of the polygon vertices (i.e., n) is even; (2) the number of the polygon vertices is odd. In the first instance the polygon can be divided into $(n-2)/2$ quadrangles, which yielded $360^{\circ} \cdot (n-2)/2 = n\alpha \Rightarrow \alpha = 180^{\circ}(n-2)/n$. The second instance was difficult to investigate. After checking a few examples, the conclusion was that in this instance the partition of the polygon will include one triangle and $(n-3)/2$

quadrangles. The suitable equation is $360^{\circ} \cdot (n-3)/2 + 180^{\circ} = n\alpha \Rightarrow \alpha = 180^{\circ}(n-2)/n$ (notice that since every quadrangle can be divided into two triangles, there could be many other possible polygon partitions).

As shown in table 1, when calculating the internal angle of the regular pentagon, most of the participants did not use the known formula: $180^{\circ}(n-2)/n$ but divided the pentagon into different simple geometric shapes (quadrangle, triangle, etc) and used their properties in order to solve the task. Although the formula for the sum of a polygon's internal angles ($180^{\circ} \cdot (n-2)$) is known and relatively not complex, yet, the participants preferred to use other strategies rather than those based on it. One possible reason is that since they had to find the angle of a regular polygon, they were looking for strategies connected to symmetrical shapes. Most of the participants divided the pentagon into three triangles or divided the regular pentagon into five isosceles triangles, using the property that a regular pentagon can be inscribed in a circle, the center of which is the center of gravity of the pentagon. The tendency to divide a polygon into triangles is one of the common heuristics among students solving geometry problems (Borasi, 1992), since one of the first geometric shapes they are introduced in school is the triangle. A major part of geometry lessons is dedicated to the learning about the properties of triangles and quadrangles and most of the proofs presented use triangles' congruence. This might be the reason they try to use the triangle's properties, when available, for solving geometrical problems. Another possible reason for the surprising outcome is that there was a minor use of the formula for the sum of a polygon's angles is the loose connection between algebra and geometry. The participants preferred to use concepts within geometry to solve a geometric problem rather than to use algebraic formulae.

Discussion

The problem presented in this study is an example of a simple task with various solution strategies. Analyzing the different solutions can prompt mathematical generalization as well as many other desirable learning situations in the spirit of the NCTM Standards (2000). Analysis of the emerging solution strategies shows that there are two different kinds of generalizations in this task, the combination of which provides a powerful tool for the learning process. The first kind of generalization is to infer from a specific regular polygon (pentagon) to an n-sided regular polygon. The second kind of generalization is embedded in the participants' emerging strategies.

The regular pentagon divides the surface into three parts: the area inside the pentagon, the pentagon side and the area outside the pentagon. Looking carefully at the different emerging solution strategies shows that most strategies include the addition of a reference point H and its connection to the pentagon's vertices (Figure 13).

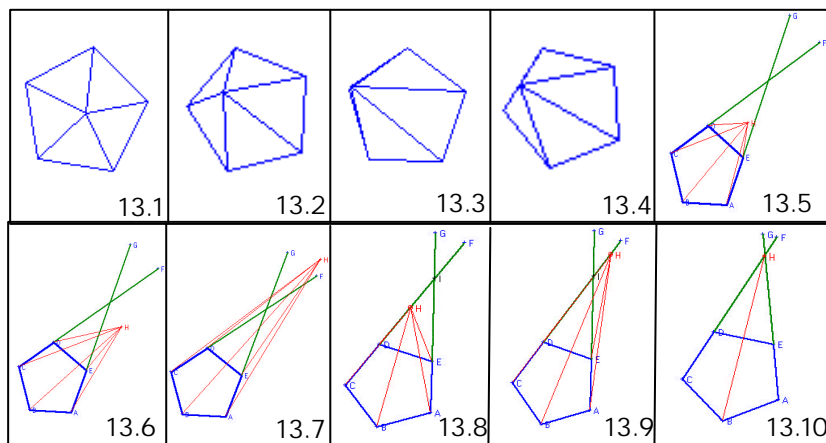


Figure 13

Analysis of the emerging solution strategies led to two main categories:

- (1) The addition of a reference point.
- (2) The partition of the polygon to simple geometric shapes (triangle, quadrangle)

The first category can be divided into three sub-categories by referring to the location of the reference point: (1.1) adding a reference point inside the pentagon; (1.2) adding a reference point on the pentagon edge; (1.3) adding a reference point inside the pentagon (Diagram1).

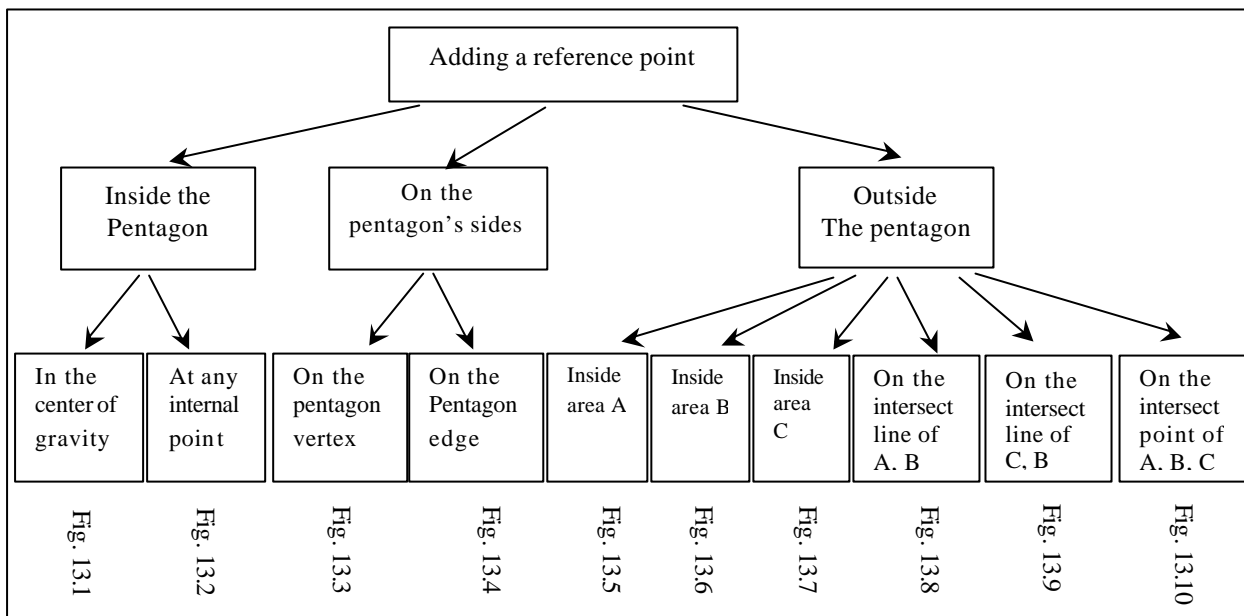


Diagram 1

The classroom discussion resulted in many more strategies. The exchange of ideas between the participants during the classroom discussion and their interaction enabled them to think of new directions for possible solution strategies. Within the classroom discussion a “mutual entity” evolved, and eventually yielded new interesting strategies. According to the socio- cultural approach the group plays an important role in an individual’s learning process (Cobb, Wood & Yackel, 1993). The group encourages an individual to reflect on his thinking process and as a result he can develop and deepen his understanding (Cobb et al., 1997). Within a group, the individual can achieve more than he could if he were working by himself (Voigt, 1994).

While looking for new solution strategies, pre-service teachers felt that they had to use previous knowledge like: the sum of angles in a triangle; the connection between the triangle’s external angle and its inner angles; etc. The engagement of pre-service teachers in this kind of activity could promote their awareness to the educational importance of problems with various strategy solutions. Much research was carried out regarding the educational value of making connections between different mathematical issues or concepts (NCTM, 2000; Even, 1990; Hiebert & Carpenter, 1992; Cornu & Dubinsky, 1989; Coxford, 1995; Reimer & Reimer, 1995). According to Nöss, Healy and Hoyles (1997), “Mathematical meanings derive from connections: intra-mathematical connections, which link new mathematical knowledge with old, shaping it into a part of the mathematical system” (p.203).

The forgoing discussion has demonstrated main point of the activity: creating the need to apply previous knowledge in performing a new task. We hope that the pre-service teachers will take this point with them and use it in their own teaching.

This paper describes an example of a simple task that can be solved in many different ways. Individual work, reflection on its outcomes, and classroom discussion can lead to more sophisticated ways of solution strategies. Discussing those strategies could promote mathematical generalization.

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