SOLVING PROCESS OF A NARRATIVE COMBINATORIAL PROBLEM: AN EXPLORATORY STUDY

Elisa PETRONE, Marta MASSA, Elena LLONCH, Patricia SÁNCHEZ
Facultad de Ciencias Exactas, Ingeniería y Agrimensura – Universidad Nacional de Rosario - Avda Pellegrini 250 – 2000 Rosario - Argentina.
e-mail: epetrone@fceia.unr.edu.ar - mmassa@fceia.unr.edu.ar - ellonch@fceia.unr.edu.ar - psanchez@fceia.unr.edu.ar

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ABSTRACT

Problem solving is one of the main goals of the learning process as it concerns knowledge in action. It is regarded as the search of possibilities, evidences and goals, involving the production of inferences, arguments and strategies to validate or refute a statement. It is related to the way in which the student models the situation and applies or creates solving strategies. Formal and informal reasoning are activated in this process.

Formal reasoning is generally associated to well-defined situations, where all the relevant data are given. It is based on logical inferences where the initial premises imply implicitly a conclusion. Informal reasoning, generally associated to "open" situations, is not restricted by logical operations as it may include inferential processes (developed, sustained and evaluated by a system of beliefs or by common sense).

In the present article, we discuss problem-solving processes that are involved when students solve a combinatorial situation, written in a narrative style, and they have to recognise the relevant data and to decide possible solving strategies. These aspects were analysed in a sample of 70 students, with ages between 15 and 53 years old, that attended different Mathematics courses.

Fifteen aspects were considered as variables, among them: data comprehension, combinatorial focus, representations as support, searching criteria, solving features, formalization level, formalization type, answer type and answer content.

Information was obtained through the application of multivariate statistical techniques. This study leads to the construction of a typology attending to solving process features. The classification analysis of the protocols allowed the identification of four classes, with almost the same number of constituents, which were given the following names: *bewildered*, *rough and ordered*, *heuristic* and *formal and tidy*. They are discussed in terms of the two main bias that emerged through the study: order and formalization.

1. Introduction

Problem solving is one of the main goals of the learning process as it concerns knowledge in action. It can be seen as involving two co-operating sub-processes: the *comprehension* through which the subject organises a mental model of the situation, and the *searching* of possibilities, evidences and goals in order to generate a strategy for its solution (VanLehn, 1998; Johnson-Laird, 1983). These two processes often alternate each other, even if they are incomplete, because comprehension may still be running when the searching process starts.

If the subject deals with the same type of problems many times, she/he may learn how to solve them and she/he may cease to labour through the comprehension and searching processes. As she/he seems to recognise the stimulus of a familiar problem she/he will follow some kind of solution procedure already proved. The collection of knowledge surrounding a familiar problem is called a *problem schema*. It is usually assumed that experts know a large variety of schemas, which may have two parts that match the whole problem unambiguously: one to describe the problem and one to fit the solutions. Teachers, as experts, usually act to provide their students the schemas concerning instructional problems. This fact and the numerous repetitions of the same kind of process make these problems become prototype ones. Though initially these prototype problems could require a hard demand of comprehension and searching of solution to the students, the repetition of a schema finally transforms them in a routine or an exercise.

Frequently the subject who has shown an expert behavior finds obstacles when dealing with non-routine problems. Difficulties arise from different sources: ambiguity in selecting a schema, need of schema combination, the detection of an impossible action during the execution of a solving procedure that produces a halt, a repair of the strategy during the execution due to a failure (VanLehn, op. cit.)

Combinatorial problems provide interesting opportunities to face non-routine problems, partly because of the narrative style in which they are mostly stated. Narrative text has a closer correspondence to everyday experience than expository text does. It involves dynamic events that imply characters, goals and intentions; expositive ones include static contents such as concepts, descriptions and arguments. Many knowledge-based inferences are generated during the comprehension of narrative text, requiring the activation of knowledge structures, schemas and their integration to conform a meaningful representation of the text (Kuhn, 1991). In addition combinatorial problem solving involves the production of inferences to derive progressively typical logical laws, argumentative skills, and strategies to complete the systematic analysis of different possibilities and the exploration of the whole structure of the problem. Formal and informal reasoning are activated in the solving process.

In the present article, we discuss problem-solving processes involved in the resolution of one specific combinatorial situation written in a narrative style. Within an exploratory research, we analyse how a sample of students solved the assigned problem in order to identify indicators of their comprehension, possible bias in the interpretation of premises, the features that oriented their reasoning and the way in which they communicated the solution.

2. Method

Subjects: The participants were 70 students, with ages between 15 and 53 years old, that attended different Mathematics courses. A sample of 20 students (aged 15 to 17) proceeded from a high school, and they performed the test as a requirement to integrate its Mathematics Olympic

Team. The rest attended a first year course at the National University of Rosario, Argentina, 30 of them studying to be a high school Mathematics professor and 20 to be a Mathematics bachelor.

The participants had to read and solve individually a problem, written in a narrative style that referred to a realistic everyday situation including verbal and numerical data. They had 30 minutes to execute the task. The activity was performed before developing the specific contents.

Research Design: The present study is best described as exploratory and interpretative. The aim is to explore patterns of reasoning and strategies that students develop to solve the following narrative non-routine combinatorial problem:

> *Mary tells her friends Bob, Peter and John that she is a "psychic" and to prove it she puts 24 similar chips on a table. Then she covers her eyes and asks one of the boys to take one chip, another to take two and the last one to take three chips. Without having seen who has taken each amount of chips she promises to guess it. But she says that in order to do so, she needs Bob to take as many chips as he has taken before, Peter to take twice the number of chips he has and John to take four times the number of chips that he has taken. Once they have done so, she asks her friends to put away their chips and then she uncovers her eyes. Suppose each boy has done exactly what Mary asked. Will Mary be able to guess how many chips took each boy in the first place? How can she do that?*

In order to perform the study three analysis dimensions were defined: *personal features, combinatorial comprehension* and *solving process*. Fifteen variables were selected as indicators of different aspects involved in these dimensions, as follows:

- *Personal features*: this dimension refers to characteristics of the subjects such as gender, age, previous knowledge and current studies.
- *Combinatorial comprehension*: it searches information about the level of assurance and the ways in which the student processes the data, focuses the problem as a combinatorial task and starts to generate an effective strategy. The variables are:
    1. <u>data comprehension</u>: identifies the level of assurance in which the subject understands the information
    2. <u>combinatorial focus</u>: analyses if the subject realises the need of checking all the possible cases
    3. <u>representations as support</u>: searches for evidences of its existence as a guide for the comprehension process
    4. <u>recognised cases</u>: measures the percentage of analysed cases
    5. <u>searching criteria</u>: defines the way in which the subject organises an strategy to look over all the cases
- *Solving process*: this dimension characterizes solving features depicted by the student to arrive to the goal and the way in which she/he arguments to provide an answer. The variables are:
    1. <u>solving features</u>: characterises the process developed to solve the problem
    2. <u>formalization level</u>: takes into account correctness and order of the solving process
    3. <u>formalization type</u>: describes the procedure selected
    4. <u>content of the solving process</u>: refers to completeness, clearness and coherence of the solving process
    5. <u>answer type</u>: describes the tools used to provide an answer
    6. <u>answer content</u>: refers to completeness, clearness and coherence of the argumentation given as an answer.

The specific modalities for each variable, shown in Table I, resulted from a previous analysis of the tasks performed by a subset of 20 students, randomly selected from the sample under study. Based on them, three of the authors performed an individual analysis of the solving activities done by the whole set of subjects. A triangulation process of their different registrations followed this activity.

A data matrix of 70 files (individuals) $\times$ 15 columns (variables), which enclosed a set of 65 modalities, was obtained. A multivariate statystical analysis, applying multiple correspondence analysis and mixed cluster processing (Lebart, Morineau & Fenelon, 1985), was selected and it was used the SPAD software (C.I.S.I.A., 1988) to process the data. The matrix is represented as a cloud of points in the 15-dimensional space of the variables or in the 70-dimensional space of the individuals. The software solves an eigenvalues problem to obtain the principal directions, called factorial axes, of the topological configuration. The first factorial axis is related to the direction of maximal dispersion of the data, and its percentage of inertia measures the contribution of this axis to the interpretation of the initial data matrix. The second principal axis, orthogonal to the first, is oriented in the next greatest dispersion direction. They define the most relevant plane for the interpretation of the projected data, called factorial plane. The meaning of each axis is determined considering the neighboring and oppositions of the modalities within the projection of the cloud on this plane.

The cluster processing provides a classification of the individuals based on their similarities in a reduced number of classes. These are as much homogeneous as possible, and their centers of gravity are related to the individuals, called the *paragons,* that best represent the class because of their characteristics. The classes obtained in the present study, projected on the factorial plane, are shown in Fig. 1.

3. Results

The classification analysis of the protocols allowed the identification of four classes which were interpreted as: 1 - *bewildered,* 2 - *rough and ordered,* 3 - *formal and tidy* and 4 - *heuristic,* constituted by 20 %, 28 %, 26 % and 26 % of the sample, respectively.

Class 1 is constituted by subjects that seem to be bewildered about the task proposed. They do not register any work concerning the problem. In a few cases, after an exclusively mental work - neither supporting representations nor explanations revealing their reasoning were explicitly pointed out - they only provide isolated answers, none of them complete and clear at the same time.

Students of Class 2 partially interpret the data, they do not seem to have realized that it was necessary to check the totality of the cases and their solving process modalities involve mostly features such as incomplete, incorrect and incoherent but ordered. This is the only class in which genre appears as a determining variable: women constitute 84 % of the class.

Students of Class 3 basically check the totality of the involved cases, most of them in a systematic way, using explicit representations as a support of their reasoning. The formalization level is high, their solving processes and answers are correct, clear and complete. The whole class corresponds to students of university level.

Class 4 is completely determined by the age of the students and, consequently, by their scholar level of studies. All the students in this class, with ages between 15 and 17 years old, attend high school. They understand the problem and they basically employ graphical representations to organize their reasoning. The searching is mostly heuristic and disordered but, anyway, they complete the task in a satisfactory way.

Figs. 2a and b show, by symbols and labels projected on the most relevant plane 1-2, the distribution of the modalities that indicate the *combinatorial comprehension* and *solving process* dimensions, respectively. The factorial axis 2 divides the factorial plane in two semi planes: the modalities associated with an adequate interpretation of data lie on the left, and those that indicate misunderstanding, doubts or incapacity to solve the problem lie on the right. The factorial axis 1 (% inertia: 15.43) is conformed, at the negative extreme, basically by the proximity of the modalities: *correct and disordered* as a formalization level, *graphic solving*, *complete content* of solving process, *mixed answer* and a *graphic* solving feature (see Fig. 2a), joining to *yes* for the variable combinatorial focus, a *complete* data comprehension and a *systematic searching* criteria (see Fig. 2b). Opposite to them, on the right, appear the modalities that indicate *omission* to detect the combinatorial focus, *absence* of data comprehension, *no solving* process to recognize the formalization level and *no content* in the solving process. Therefore, the factorial axis 1 is interpreted as that defining the level of combinatorial comprehension achieved which sustains the formalization process.

The factorial axis 2 (% inertia: 8.83) is defined, at the positive extreme, by the modalities: *incorrect and ordered* as a formalization level, *mixed searching* as a criteria to organize a solving strategy, *partial* for the data comprehension and *doubtful* to detect the combinatorial focus, and, on the opposite side, by *correct and disordered* as a formalization level, *omission* of the combinatorial focus and *absence* of data comprehension. Representations as support, recognised cases, solving features and answer type are variables that do not contribute to characterize this axis as their modalities have their projections very close to zero. Therefore, axis 2 reflects the order of the solving process written on the protocols.

4. Discussion

Nature of the combinatorial problem used as instrument in the research

Problems like the “psychic” one have not been typical in the Mathematics classes in Argentina, neither in content nor in style. As regards the content, they refer to an everyday situation that acts as a *challenge to think* (Munby, 1982). Therefore, it is the type of ingenious problem that teachers sometimes offer to their students to solve on their own, assuming that only some of them, the creative and/or clever fellows, will be successful. As problems of this type are not seen as instructional ones, teachers do not work systematically on them and, consequently, students do not construct any associated schema of resolution. It is interesting to point out that this lack of a solving schema, both in students and teachers, conversely provokes that the latter disregard them as instructional, closing a circuit that seems to be hard to interrupt, though the new curricula in primary and secondary school specifically include combinatorial problems.

The resolution of this kind of problems demands a combinatorial reasoning (Piaget & Inhelder, 1951; Halpern, 1996) that may be characterized by two facts:

a) *completeness*, that is, the recognition of all the possibilities relative to a certain event (e.g., in the “psychic” problem all the alternatives in which Bob, Peter and John may pick the chips). This feature may be considered as equivalent to that introduced by Perkins (cited in Garnham and Oakhill, 1996) in the analysis of informal or everyday reasoning, where decisions and conclusions are based on a set of plausible arguments derived from evidences, and from which the subject organizes a situational modeling

b) *organization of an approach to check the possibilities*, as the order of the procedure is relevant for a successful performance of the task (e. g., the "psychic" problem requires also to design a methodical plan in order to explore the possibilities in a systematic manner).

As regards the narrative style of the problem, a previous study (Llonch et al., 2001) has shown that it demands from the student a further transformation of the explicit information into numeric or at least symbolic data in order to initiate the solving process. Relevant implicit inferences may be omitted or the student may fail to "translate" the narrative text into a scientific one (in this case, into a combinatorial language). Therefore, pure narrative problems offer additional difficulties. They involve people performing actions in pursuit of goals and include additional information acting as an obstacle or interference that has to be avoided by the reader. We may also conclude that purely narrative statements activate informal reasoning patterns that lead students:

(a) to the use of systems of beliefs (e.g., a student explicitly stated that *a riddle has nothing to do with Mathematics*) that bias the solution,
(b) to the demand of unnecessary data that adds difficulties to the situation (e.g., some students pointed that *there were insufficient data to solve the problem*).

The demand stated in the "psychic" problem introduces an interesting perspective for future research: the disturbance that may be produced by an unfamiliar question in a mathematical context *(Will Mary be able to guess…? How can she do that?)*, where habitually the student is required for a quantity (*How many*…?), to detect an existence (*Is there*…?) or to find an optimum (*Which is the greatest*…?, …*the shortest*?,… *the cheapest*?).

About problem schemes produced by the different subject classes

To summarize, the categorization schemes, stated as *bewildered*, *rough and ordered*, *heuristic*[1] and *formal and tidy*, tend to focus on the solving tendencies of the subjects. Transitions among their features are clearly depicted by the trajectories followed by the modalities of the variables *comprehension focus* and *formalization type*, as shown in Figs. 2a and b, respectively.

Basically, the whole set of secondary students understood the task in its combinatorial essence and their spontaneous solving attitude was to organize a graphic array of possibilities – arrows, tables or lines – as an heuristic. The lack of a certain conceptual base oriented them to the representation of the different situations, in coherence with the data and the proposal. The protocols showed a satisfactory searching of possibilities, although the organization to look over the different cases was varied in level of order and systematization.

About the third part of the university students tried to perform elaborate actions, with a marked tendency to introduce analytical or, at least, numerical procedures, with the occasional use of various conceptual labels or symbols to give their answers in a formal academic style. However, the informality of the spontaneous reasoning related to this new type of problem produced some disturbance in their facultylike procedures. This fact was solved by the development of more tidy graphic organizations than those seen in the previous group, which allowed the student to reach successful solutions.

The university group basically characterised by genre (Class 2) failed to detect the demand of completeness required by the task. Although they intended to produce ordered organisation of

[1] Heuristic (from the Greek word *heuriskin* that means *serve to discover*) refers to the procedure that a subject believes as a reasonable possibility to arrive to the solution or, at least, to be close to it. It is an alternative to an algorithmic procedure, that is, a detailed prescription, step by step, to get the goal.

solving strategies, they were incoherent and rough. Basically, they failed to detect the combinatorial nature of the problem.

Finally, it is an interesting fact the presence of a significant group of students that looked like bewildered when faced with a new and unknown situation. We interpret that they lack self-confidence and only act when a known schema guides their work, that is to say, when they feel like “moving on a known land”. This negative attitude is still present at university level, to which the majority of the students in this class belong.

High school students of the sample showed a different attitude – they tried to do something –, probably because they were spontaneously interested in the competition and the will to participate gave them the strength needed to persist.

6. Final Remarks

The analysis of the students’ performance in solving this type of combinatorial problem provides some evidence about two relevant facts that seem to accompany successful solvers. One of them is related to an attitude of persistency on the searching, when faced with the absence of a known schema. The second deals with the tendency of using a graphic design as an heuristic when neither a numerical nor an algebraic strategy suited properly. The latter provides an interesting framework to introduce in our secondary schools the use of graphs as an alternative and powerful topic, whose methods develop the combinatorial reasoning and constitute the nucleus of Discrete Mathematics. In Argentina teachers still should be trained in these topics, not only in their contents and didactics but also in the knowledge of the strength of their treatment. They should teach combinatorial and graph problems mainly based in the knowledge of the fact that combinatorial ability is one of the basic conditions for logical reasoning (Fischbein, 1994).

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1- Gender ▪ Male ▪ Female	**2- Age: a (in years old)** ▪ a < 15 ▪ 15 ≤ a < 18 ▪ 18 ≤ a < 21 ▪ 21 ≤ a < 30 ▪ 30 ≤ a	**3- Previous knowledge** ▪ Incomplete high school ▪ Commercial high school ▪ Technical high school ▪ Bachelor high school ▪ Other university studies
4- Current Studies ▪ High school ▪ Mathematics high school professor ▪ Mathematics Bachelor	**5- Data Comprehension** ▪ Complete ▪ Partial ▪ Absent ▪ Incorrect understanding	**6- Combinatorial focus** ▪ Yes ▪ No ▪ Doubtful ▪ Omission
7- Representations as support ▪ Explicit support ▪ Implicit support ▪ Absent support	**8- Recognised cases** ▪ 100% ▪ Between 70 % and 100 % ▪ Between 40 % and 70 % ▪ Less than 40 %	**9- Searching criteria** ▪ Systematic path ▪ Random path ▪ Mixed path ▪ Uncertain criteria
10- Solving features ▪ Mental ▪ Numerical ▪ Mixed ▪ Graphic	**11- Formalization level** ▪ Incorrect and disordered ▪ Correct and ordered ▪ Correct and disordered ▪ Incorrect and ordered ▪ Non existent	**12- Formalization type** ▪ Literal solving ▪ Symbolic solving ▪ Graphic solving ▪ Numerical solving ▪ Mixed solving ▪ No solving
13- Content of the solving process ▪ Complete, clear content ▪ Incomplete but clear content ▪ Complete content ▪ Incomplete, incoherent content ▪ No content	**14- Answer type** ▪ Literal answer ▪ Symbolic answer ▪ Graphic answer ▪ Numerical answer ▪ Mixed answer ▪ No answer	**15- Answer content** ▪ Complete and clear ▪ Incomplete but clear ▪ Complete ▪ Incomplete and incoherent ▪ No content

Table I: Variables and modalities employed to perform the analysis

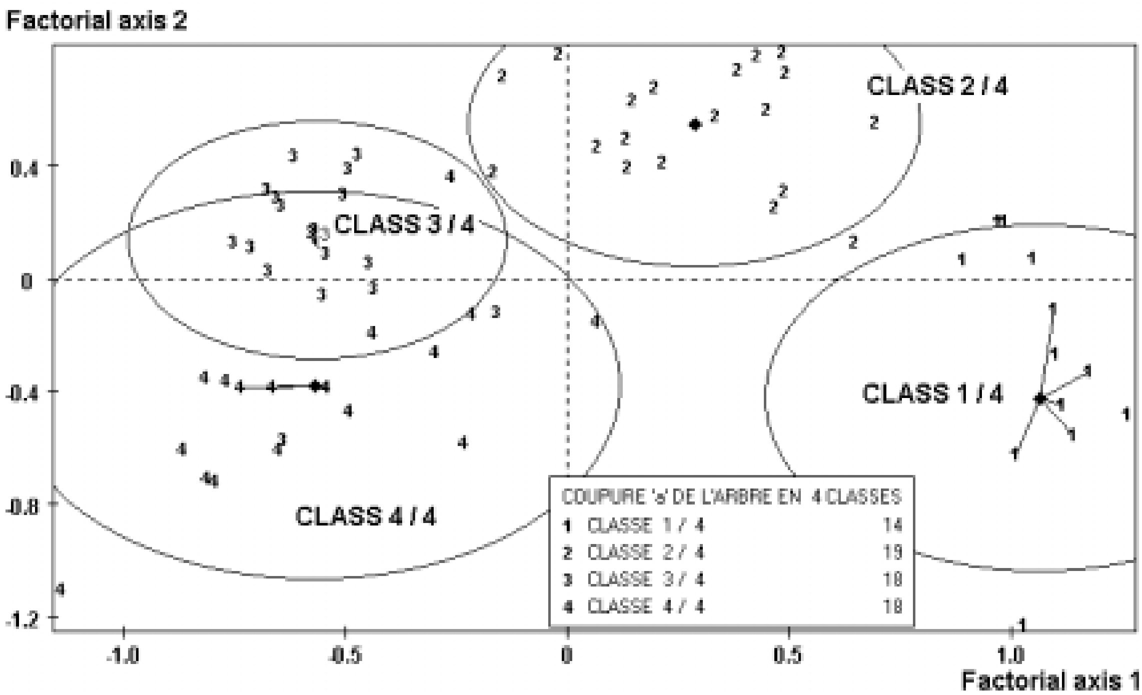
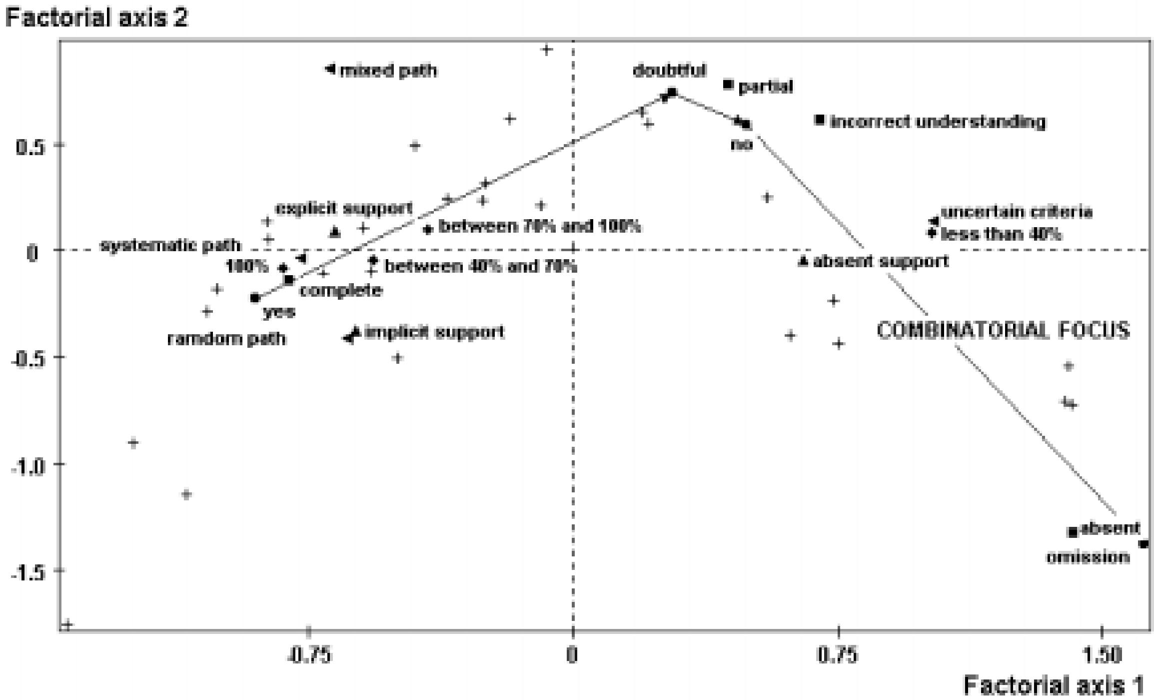
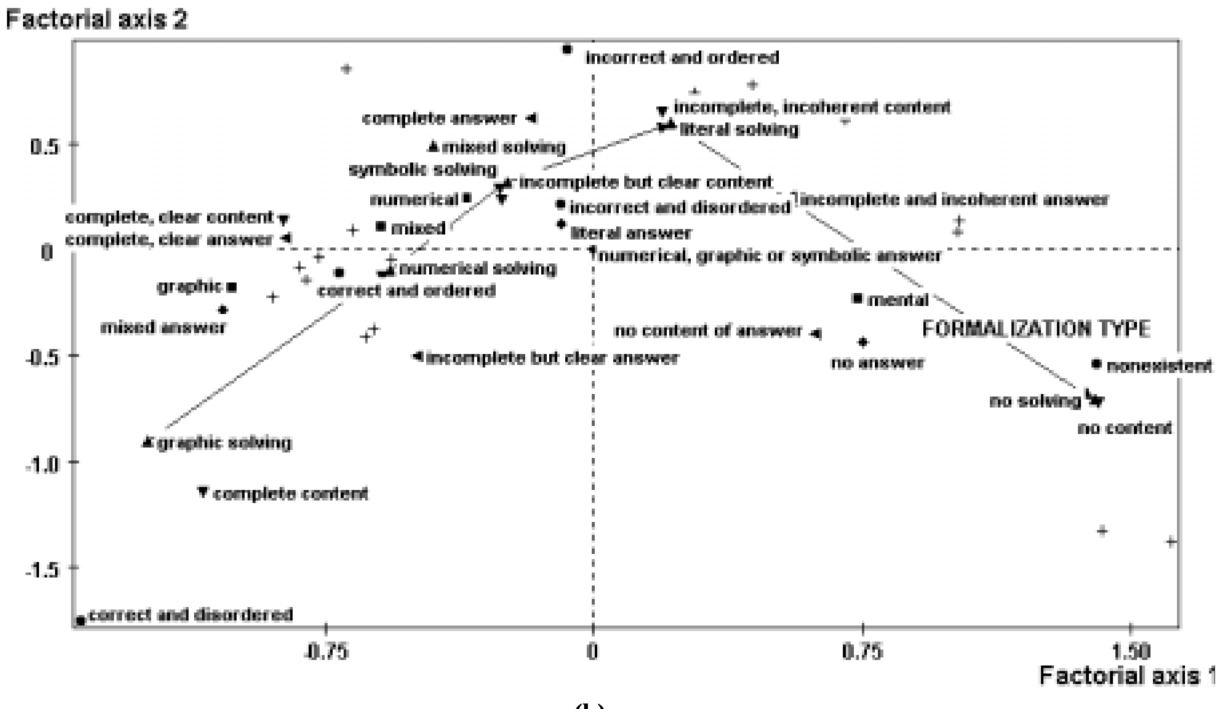


Figure 1: Classification of the students attending their affinities in the problem-solving task



(a)



(b)

Figure 2: Factorial plane defined by the first and second principal axes, with the projections of the modalities of the variables *combinatorial comprehension* (a) and *solving process* (b). The symbols (+) represent the modalities whose labels are indicated in the other figure.