THE USE OF THE HISTORY OF MATHEMATICS IN THE LEARNING AND TEACHING OF ALGEBRA

The solution of algebraic equations: a historical approach

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ABSTRACT

The Ministry of Education (MPI) and the Italian Mathematical Union (UMI) have produced a teaching equipment (CD + videotapes) for the teaching of algebra. The present work reports the historical part of that teaching equipment.

From a general point of view it is realised the presence of a "fil rouge" that follow all the history of algebra: the method of analysis and synthesis.

Moreover many historical forms have been arranged to illustrate the main points of algebra development. These forms should help the secondary school student to get over the great difficulty in learning how to construct and solve equations and also the cognitive gap in the transition from arithmetic to algebra.

All this work is in accordance with the recent research on the advantages and possibilities of using and implementing history of mathematics in the classroom that has led to a growing interest in the role of history of mathematics in the learning and teaching of mathematics.

1. Introduction

We want to turn the attention to the subject of *solution of algebraic equations* through a historical approach: an example of the way the introduction of a historical view can change the practice of mathematical education. Such a subject is worked out through the explanation of meaningful problems, in the firm belief that there would never be a construction of mathematical knowledge, if there had been no problems to solve.

Gaston Bachelard (1967, p. 14) has written: "*It is precisely this notion of problem that is the stamp of the true scientific mind, all knowledge is a response to a question.*" That is, the concepts and theories of mathematics exist as tools for solving problems.

Also Evelyne Barbin (1996) has pointed out that. "*There are two ways of thinking about mathematical knowledge: either as product or as process. Thinking about mathematical as product means being concerned with the results and the structure of that knowledge, that is to say, with mathematical discourse. Thinking about mathematical as process means being concerned with mathematical activity. A history of mathematics centred on problems brings to the fore the process of the construction and rectification of knowledge arising out of the activity of problem solving.*"

Algebra (mostly that part relative to the so-called "literal calculus") is the more suitable branch of mathematics for the use of the method of analysis. Such a method is very old and still today one of the best definitions [together with that of synthesis] is that given by Pappus in his *Collection*: "*Now, analysis is the path from what one is seeking, as if it were established, by way of its consequences, to something that is established by synthesis. That is to say, in analysis we assume what is sought as if it has been achieved, and look for the thing from which it follows, and again what comes before that, until by regressing in this way we come upon some one of the things that are already known, or that occupy the rank of a first principle. We call this kind of method "analysis" as if to say* ***anapalin lysis*** *(reduction backward). In synthesis, by reversal we assume what was obtained last in the analysis to have been achieved already, and, setting now in natural order, as precedents, what before were following, and fitting them to each other, we attain the end of the construction of what was sought. This is what we call "synthesis.".*" [Pappus, *Book 7 of the Collection* (tr., comm. A. Jones), 2 vols., New York, Springer, 1986]

Simply with reference to an educational point of view, we can say that analysis is a "backward reasoning". In (Rojano & Sutherland, 2001) this method is used for explaining the solutions of word problems.

2. The method of analysis in the construction of an equation

Before considering some algebraic problems drawn from the history, let us consider a typical problem that students deal during the first year of high school as an example of the method of *analysis*.

Problem. *In a rectangle the difference between its sides is 12 m and the perimeter is 224 m; find its area.*

We suppose that such a rectangle exist; to find its area we have to know the base and the height of rectangle; but we know the difference between the base and the height. Therefore

base = height + 12

or

height = base – 12.

However we know the semi-perimeter, that is 112 m. Then, if we take away the base from the semiperimeter, we can get the height. At this point, starting from what the problem asks, we have reached what we are given: the difference between the sides and the semiperimeter.

We suppose that the height is known and call it x and then the base is $x + 12$. From such an analysis we get

$x = 112 - \text{base}$

$x = 112 - (x + 12)$.

From that it is easy to obtain $x = 50$, the height; then we get the base, 62, and consequently we can compute the area, which is 3100 m^2. Geometrically we realise the symmetry of problem (that is, the base can be exchanged with the height).

We can note, from a didactical point of view, that the backward reasoning comes, step by step, from the question: what do I need to compute…? We go on putting these questions until we find, by splitting the problem, something known (given by the text of problem).

3. The Egyptian Rule of False Position

The next problem is one of 85 problems in the Rhind Papyrus, now housed in the British Museum.

Problem. *A quantity whose seventh part is added to it becomes 19.*

In modern notation the problem is equivalent to the solution of the equation $x + (1/7)x = 19$. The Egyptian method of solution, called the *Rule of False Position*, consist of giving to the unknown quantity x at the left side the beginning value 7, so that the resulting value at the right side is $7 + (1/7)\cdot 7 = 8$.

The argument goes on supposing that, if some "multiple" of 8 gives 19, than the same "multiple" will produce the sought number.

Therefore we can solve the problem by the proportion

$8 : 19 = 7 : x$ that is $x = (19/8)\cdot 7$.

The "False Position" in the history of mathematical education

Till the nineteenth century the rule of "false Position" is proposed again to present to the students first-degree equations. In the handbook *Elementi di matematica* [Elements of mathematics] by V. Buonsanto, Società Filomatica, Naples 1843 (pp. 117-119) we find the following passage:

"We shall look for a number, which solve the problem: but you will find it only by means of a *false* number, which does not solve it. This is the *rule of simple false position.* You have been told: A third and a quarter of my money are 24 ducats. How much money has I? Since you don't know the true number of ducats, you suppose that who is speaking gets 12 ducats. *This number, supposed in such an arbitrary way, is called position.* But it is easy to see that such a *supposition is false*, because a third and a quarter of 12 are 4 + 3 = 7 and so your friend should have not 24 ducats for a third and a quarter, but 7. However you can argue in this way. If 7 are the result of the false position 12, what number does 24 come from? You will do 7 : 12 = 24 : 288/7 and 288/7 = 41 and 1/7. Your friend has 41 and 1/7 ducats. To solve such problems you can suppose every number, but it is better to choose it in a way to avoid fractions. It is also better to choose a small number."

The rule of false position was also taught to American students of XIX century and is present in the textbook *Daboll's Schoolmaster's Assistant*, that was, till 1850, the most popular book of arithmetic in that country.

We can find still in recent works some notes on this method. The rule of false position can be used today, besides teaching first-degree equations (Winicki, 2000, and Ofir & Arcavi, 1992), to analyse as the spreadsheet works, e.g. the hidden algorithms (Rojano & Sutherland, 2001).

4. A Babylonian problem considered also by Diophantus

Babylonian algebra consisted of a totally algorithmic method formed by a list of operating rules to solve problems (*rhetorical* algebra). The algorithms were illustrated by numerical examples, however the recurrent use of some terms gives us a first concept of *symbolism*. Instead Diophantus introduces (in his *Arithmetica*) a literal symbolism and a form of language half way between "rhetorical" and "symbolic", that is "syncopated". In particular he introduces the "*arithme*" an indeterminate quantity of units that becomes a real unknown. Diophantus accepts only exact rational positive solutions, while Babylonians accepted also approximations of irrational solutions.

Problem. *Find two numbers whose product is 96 and sum is 20.*

Using modern notation the problem becomes

$$\begin{cases} x + y = 20 \\ xy = 96 \end{cases} \quad \text{or, in general form} \quad \begin{cases} x + y = b \\ xy = a \end{cases}$$

which is equivalent to quadratic equations $z^2 - bz + a = 0$.

What follows is the rhetorical solution of scribe (*instructions*) and his modern "*translation*".

instructions	*translation*
1. Divide by two the sum of numbers 20:2 = 10	$\frac{b}{2}$
2. square $10^2 = 100$	$\left(\frac{b}{2}\right)^2$
3. subtract the given area, 96, from 100 100 – 96 = 4	$\left(\frac{b}{2}\right)^2 - a$
4. take the square root 2	$\sqrt{\left(\frac{b}{2}\right)^2 - a}$
5. the base is 10 + 2 = 12	$x = \frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 - a}$
the height is 10 – 2 = 8	$y = \frac{b}{2} - \sqrt{\left(\frac{b}{2}\right)^2 - a}$

This method of solution shows that the Babylonians knew some laws of algebraic operations, made substitutions and solved by algebraic methods quadratic equations and systems equivalent to quadratic equations (Bashmakova & Smirnova, 2000).

The following Diophantus' method of solution (also used by Babylonians) is called "*plus or minus*".

1. "The difference between two numbers is two *arithme*" $x - y = 2\varsigma$

2. “If we divide the sum into two equal parts, each part will be half the sum that is 10” $\frac{x+y}{2}=\frac{b}{2}$

3. “If we add to one part and subtract from the other one half the difference of number, that is one *arithme*, we find again that the sum of two numbers is 20 units and the difference is two *arithme*”

$$\begin{cases} x+y=b \\ x-y=2\varsigma \end{cases}$$

4. “Let us suppose the bigger number is 1 *arithme* plus 10 units that are half the sum of numbers; therefore the smaller one is 10 units minus 1 *arithme*”

$$\begin{cases} x=\frac{b}{2}+\varsigma \\ y=\frac{b}{2}-\varsigma \end{cases}$$

5. “It is necessary that the product of two numbers is 96”

$$\left(\frac{b}{2}+\varsigma\right)\left(\frac{b}{2}-\varsigma\right)=a$$

6. “Their product is 100 units minus a square of *arithme*, that is equal to 96 units”

$$\left(\frac{b}{2}\right)^2-\varsigma^2=a$$

7. “And the *arithme* becomes 2 units. Consequently, the bigger number is 12 units and the smaller one is 8 units and these numbers meet the statement ”

$$\varsigma=\sqrt{\left(\frac{b}{2}\right)^2-a} \qquad \text{from which} \qquad x=\frac{b}{2}+\sqrt{\left(\frac{b}{2}\right)^2-a} \qquad \text{and}$$

$$y=\frac{b}{2}-\sqrt{\left(\frac{b}{2}\right)^2-a}\,.$$

The description of Diophantus shows awareness in the use of unknowns that we shall find only in the works of Arabic mathematicians.

5. Algebra and geometry in Euclid and Bombelli

Traditionally Book II of the Euclid’s *Elements* (but also part of Book VI) is considered as an example of “geometrical algebra”, also if this name can be misleading because the formulation is completely geometrical. We don’t want to enter into the merits of debate concerning geometrical algebra (still far from over) that has seen engaged some famous mathematicians as Unguru, Van der Waerden, Freudenthal and Weil. We want instead to stress that the so called *problems of applications of areas*, also if explained and solved in geometrical way, can be considered equivalent to first and second-degree equations.

The first application consists of constructing a rectangle of area S on a given base a and finding its height.

This problem is equivalent to first-degree equation $a\cdot x = S$.

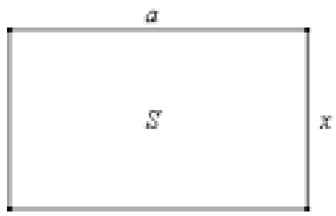

Figure1

Such a problem is solved by Euclid both in Book VI by means of proportions theory (thinking $S = b \cdot c$) and in Book I by means of that we can call a theory of equivalence of polygons.

Bombelli (*Algebra*, Book IV) proposes the same problem again in the following form: "*Find a line that is in proportion to .c. as .b. is to .a.*" Therefore we have to find the fourth proportional after three segments *a*, *b*, *c*. [$a : b = c : x$]

Bombelli in his *Algebra* gives for this problem two different constructions, both taken from Euclid.

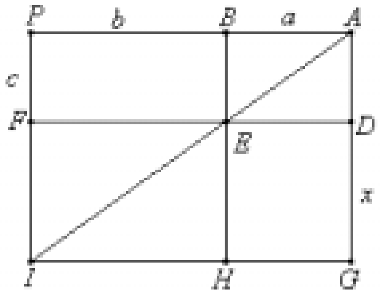

Figure 2

In the first one he considers the rectangle *FPBE* (Fig. 2), whose sides are *b* and *c*, then, having set $BA = a$, he joins points *A*, *E*, *I* and constructs the rectangle *PAGI*. The two rectangle *PBEF* and *EDGH* are equivalent for the Proposition I.43 of the Euclid's *Elements* and therefore *DG* is the solution of the equation $ax = bc$, that is

$$x = \frac{bc}{a}.$$

In the second construction (Proposition [73]) Bombelli uses the Thales' theorem and sets $AB = c$, $BC = a$ and $CD = b$ (Figure 3) and using the Proposition VI.12 of the Elements, concludes that $AB : BC = DE : CD$, so that *DE* is the required solution.

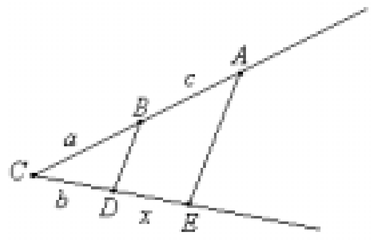

Figure 3

The method of Bombelli can be outlined in the following way:

1. *Enunciation of the problem*
2. *Geometrical construction of the solution*
3. *Solution of a numerical example via algebra.*

Always *the geometrical solution precedes the algebraic one*. Yet it is apparent from the analysis of the single cases, that *it is the equation, or better the form of its algebraic solution, which determines the subsequent steps of the construction* (Giusti, 1992).

We can note that this part of *Algebra* (Books IV and V), devoted to the application of algebra to geometry, marks almost a turning point, and sometimes a bringing forward, of the analytic geometry of Descartes (Bashmakova & Smirnova, 2000 and Giusti, 1992).

6. The Arab algebraists of the Middle Ages, the Italian algebraists of 16th century and the solution of the cubic equation

We owe to Arab algebraists, beside the introduction of word "*algebra*", the more and more aware use of substitutions to simplify the solutions of problems; Diophantus had already proposed such a method.

Moreover we find in the works of *Abu Kamil* (850-930?), more than in those of *Al-Khwarizmi* (800?-847), complicated transformations of expression with irrational numbers as the following problem shows.

Problem. *Divide 10 into two parts x and 10 − x to get*

$$\frac{x}{10-x}+\frac{10-x}{x}=\sqrt{5}\,.$$

The relative quadratic equation is

$$\left(2+\sqrt{5}\right)x^2+100=\left(20+\sqrt{500}\right)x$$

that, multiplying by $\sqrt{5}-2$, becomes

$$x^2+\sqrt{50000}-200=10x\,.$$

But Abu Kamil finds another simpler solution setting $y=\frac{10-x}{x}$. He obtains immediately the equation

$$y^2+1=\sqrt{5}y$$

which has the solution

$$y=\sqrt{1+\frac{1}{4}}-\frac{1}{2}\,.$$

In this way we arrive to the linear equation

$$\frac{10-x}{x}=\sqrt{1+\frac{1}{4}}-\frac{1}{2}$$

that could be solved as

$$\frac{10}{x}-1=\sqrt{1+\frac{1}{4}}-\frac{1}{2}$$

that allows determining the unknown *x*, but it gives a result with an irrational denominator. Abu Kamil instead finds

$$10-x=\sqrt{1+\frac{1}{4}}x-\frac{1}{2}x \quad \text{that is} \quad 10-\frac{x}{2}=\sqrt{1+\frac{1}{4}}x$$

and squaring both the sides he obtains, after some calculations, the equation

$$x^2+10\,x=100$$

of which he finds the solution $x = \sqrt{125} - 5$.

The method of making a substitution of an unknown to reduce a more difficult equation to a simpler one will become, as we shall see, quite usual.

The Italian algebraists of 16th century have used these substitutions to solve cubic equations. We know that this mathematical "discovery" is the result of the works of *Scipione del Ferro* (1456-1526), *Girolamo Cardano* (1501-1576) e *Niccolò Fontana* (1500-1557) called *Tartaglia* [the "stammerer"]. Del Ferro begins with the equation $ax^3 + bx = c$ that he immediately reduces to the form $x^3 + px = q$ (p, $q > 0$), dividing by a. Tartaglia, in his famous cryptic poem, assumes that the solution is of the form

$$x = u - v.$$

Then the equation can be reduced to the form

$$u^3 - v^3 + (u - v)(p - 3uv) = q.$$

If one imposes on u and v the additional condition $3uv = p$, then u and v can be determined from the system

$$\begin{cases} u^3 - v^3 = q \\ uv = \dfrac{p}{3} \end{cases}$$

Or also

$$\begin{cases} u^3 - v^3 = q \\ u^3v^3 = \left(\dfrac{p}{3}\right)^3 \end{cases}.$$

Putting $z = u^3$ we see that this system is equivalent to the quadratic equation

$$z^2 - qz - (p/3)^3 = 0,$$

which means that

$$x = \sqrt[3]{\sqrt{\frac{q^2}{4} + \frac{p^2}{27}} + \frac{q}{2}} - \sqrt[3]{\sqrt{\frac{q^2}{4} + \frac{p^2}{27}} - \frac{q}{2}}.$$

Let us consider, as an example of application of this method, the equation $x^3 + 6x = 20$. We set $u^3 - v^3 = 20$ and $u^3v^3 = 8$. We get $u^3 = 6\sqrt{3} + 10$ and $v^3 = 6\sqrt{3} - 10$ or $u^3 = -6\sqrt{3} + 10$ and $v^3 = -6\sqrt{3} - 10$. In both cases we get

$$x = \sqrt[3]{u} - \sqrt[3]{v} = \sqrt[3]{6\sqrt{3} + 10} - \sqrt[3]{6\sqrt{3} - 10} = 2.$$

Remark. It is possible to reduce the standard cubic equation (in modern notation)

$$ax^3 + bx^2 + cx + d = 0$$

to the form

$$y^3 + py = q,$$

used by the Italian algebraists by means of the substitution

$$x = y - \frac{b}{3a}.$$

Viète will use a similar method to obtain the quadratic formula.

7. The quadratic equation in Viète and Descartes

The method of Viète

A possible way to obtain the quadratic formula was proposed by Viète in *De aequationum recognitione et emendatione Tractatus duo* (1591). He uses a substitution quite similar to that of Italian algebraists of 16th century. Viète begins with the equation

$ax^2 + bx + c = 0$

(of course, he uses different symbols for the unknowns and the parameters). He puts $x = y + z$ and obtains

$a(y + z)^2 + b(y + z) + c = 0$

$ay^2 + (2az + b)y + az^2 + bz + c = 0.$

To eliminate the first degree term it is necessary that

$2az + b = 0,$

from which we get $z = -\frac{b}{2a}$. The substitution in the equation gives

$$ay^2 + a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) + c = 0$$

that is

$4a^2y^2 = b^2 - 4ac.$

From this he obtains

$$y = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

and lastly, using again the variable x

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

the well known quadratic formula.

The method of Descartes

In the Book I of the *Geométrie* (1637) Descartes gives detailed rules to solve quadratic equations. He uses, with a different approach, the classic Greek geometry; particularly the problems of applications of areas (Bos, 2001).

Hyperbolic application

a) Equation: $x^2 - ax - b^2 = 0$ $(a, b > 0)$.

Construction:

1. Draw a right-angled triangle AOB with $OA = \frac{1}{2}a$, $OB = b$ and $\angle AOB = 90°$.
2. Draw a circle with center A and radius $\frac{1}{2}a$.
3. Prolong AB; the prolongation intersects the circle in C.
4. $x = BC$ is the required line segment.

[Proof: BA intersects the circle in D; by *Elements* III.36 $BC{\cdot}BD = OB^2$, i.e., $x(x - a) = b^2$, so $x^2 - ax - b^2 = 0$.]

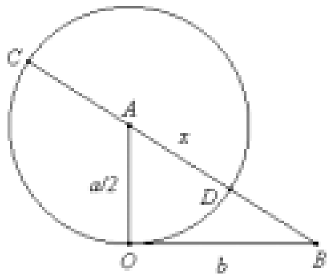

Figure 4

b) Equation: $x^2 + ax - b^2 = 0$ ($a, b > 0$).

The construction is the same of previous case: it is enough to put $x = BD$.

Elliptic applications

Equation: $x^2 - ax + b^2 = 0$ ($a/2 > b > 0$).

Construction:

1. Draw a line segment $AB = a$, with midpoint O.
2. Draw a semicircle with center O and radius $\frac{1}{2}a$.
3. Draw the line tangent at B to semicircle and mark on that line $BP = b$ in the half-plane where the semicircle is.
4. Draw a line through P parallel to AB. It intersects the semicircle in Q and S is the projection of Q into AB.
5. $x = SB$ is the required line segment, but also $x = AS$ is a solution.

[Proof: By *Elements* VI.8 $BP^2 = SB{\cdot}AS$, i.e. $b^2 = x(a - x)$, so $x^2 + ax - b^2 = 0$.]

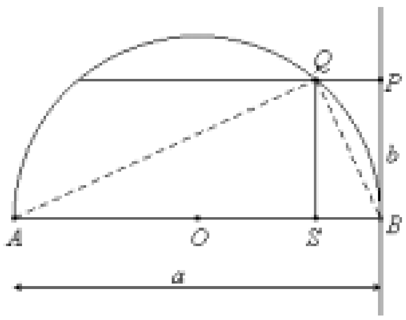

Figure5

Remark. In the cases of hyperbolic application Descartes constructs only the positive solution. Actually, also if he uses negative numbers in calculations, he doesn't give a geometrical meaning of negative numbers and therefore doesn't use negative abscissas.

7. Conclusions

The problems and the selected subject are meant to give relevance to the history and also to motivate and deepen student understanding of subject matter. Student can also see how problems were solved before the use of what to us are familiar equations and realise how a good symbolism make life easier for us in studying mathematics.

On the other hand it is difficult for student to give meaning to the "handling of symbols", when he meets the first time with algebraic equations. In this case the use of geometry, that gives a concrete meaning to symbols, can help student to overcome this epistemological obstacle.

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