ON THE RELATION BETWEEN MATHEMATICS AND PHYSICS IN UNDERGRADUATE TEACHING

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ABSTRACT

The historical development of Mathematics and Physics suggests that:

(a) Mathematics and Physics have always been closely interwoven, in the sense of a "two-ways process":

- Mathematical methods *are used* in Physics. That is, Mathematics is not only the "language"of Physics (i.e. the tool for expressing, handling and developing logically physical concepts and theories), but also, it often *determines* to a large extent the *content* and *meaning* of physical concepts and theories themselves.
- Physical concepts, arguments and modes of thinking *are used* in Mathematics. That is, Physics is, not only a *domain of application* of Mathematics, providing it with problems "ready-to-be-solved" mathematically by already *existing* mathematical *tools*. It also provides, ideas, methods and concepts that are crucial for the creation and development of *new* mathematical concepts, methods, theories, or even whole mathematical domains.

(b) Any distinction between Mathematics and Physics, seen as general attitudes towards the description and understanding of an (empirical, or mental) object, is related *more* to the point of view adopted while studying particular aspects of this object, than to the object itself.

Points (a) and (b) imply that:

(c) Any treatment of the history of Mathematics independent of the history of Physics is necessarily incomplete (and vice versa).

(d) By accepting the importance of the historical dimension in education, the relation between Mathematics and Physics should not be ignored in teaching these disciplines.

It is possible to illustrate the above points with the aid of many important examples, which can also be didactically relevant by following a historical-genetic teaching approach. In this paper, we illustrate this qualitatively by means of three examples (at the same time surveying author's work in this area in the last few years):

- The possibility to introduce and/or illustrate important geometrical and algebraic concepts on the basis of Relativity Theory.
- The complex, deep interconnections between Differential Equations, (Functional) Analysis and Quantum Mechanics.
- The physical origin of many basic concepts and theorems of the theory of Dynamical System and of Ergodic Theory.

1. Introduction

The present paper rests on the following two points:

(a) The appreciation by many mathematicians, mathematics educators and historians of the significance of the introduction of a historical dimension in Mathematics Education (ME).

(b) The well-known fact that there has been a close interrelation between Mathematics and Physics throughout their historical development.

Both points can have a lasting effect on the way Mathematics is taught and learned. In what follows, I will elaborate on (a) and (b), connecting them and illustrating them by means of 3 examples at the university level. Details can be found in the literature; hence the present paper is also a survey of author's work in this area in the last few years.

1.1. Comments on (a): At least implicitly, the way Mathematics is presented and/ or taught reflects a philosophical and epistemological point of view about the *nature* of Mathematics. In particular, that Mathematics is conventionally presented deductively reflects a point of view, according to which Mathematics is simply a collection of axioms, definitions, theorems, and proofs, that is, *only* the *results* of the mathematical activity. As a consequence, Mathematics is supposed to evolve more or less by a linear accumulation of new results (cf. Lakatos 1976, pp.1-2). Hence, what is essential is to learn these results in their final "polished" form. Such a point of view has a lasting effect on what parts of Mathematics should be taught and how this should be done (Schoenfeld 1992, p.341). This is particularly evident at the university level, given that there, it is often tacitly taken for granted that once the student has made his/her choice to study (either "pure", or "applied") Mathematics, he/she has to learn it independently of the way it is presented.

However, in this way it is not appreciated that Mathematics is a human enterprise, hence that "doing Mathematics" is an equally important aspect of Mathematics itself that should not be left out (cf. Grugnetti, Rogers et al. 2000, §§ 2.2.2, 2.3.3). On the contrary, there is an ever-increasing agreement that helping students to become aware of the evolutionary nature of Mathematics may lead them to a deeper and more solid understanding of Mathematics.

Therefore, if Mathematics is conceived, not only as a collection of logically complete finished products, but also as the *process* by which these products are conceived, formulated, developed and justified, it becomes clear that a historical dimension in teaching and learning Mathematics is helpful, or even necessary. Actually, history makes clear that the deductive organization of any mathematical domain is a posteriori (i.e. once this domain is sufficiently mature). At the same time, history provides a natural framework for helping students to become aware of Mathematics in the making. Introducing a historical dimension into ME has important advantages that cannot be analyzed here (see Tzanakis, Arcavi et al.2000, §7.2 for a comprehensive analysis), and can be done in a variety of ways depending on several factors, like the emphasis one wants to put on the subject taught, the level of education etc (see Tzanakis, Arcavi et al. 2000, §§7.3, 7.4). I will focus on two advantages only.

- History constitutes an important resource of *relevant* questions, problems and expositions, valuable both in terms of their content and their potential to *motivate***,** *interest* and *engage* the student. Thus, historically inspired exercises, problems, or small research projects, may stimulate the student's interest and contribute to enhance curriculum alongside those exercises and problems, which may seem 'artificially' designed. In this way, aspects of the historical development of a subject include "real" Mathematics, so that they become part of the student's "working knowledge". Consequently, history *in* ME no longer appears as something alien to "Mathematics proper", but forms an integral part of it.

- History reveals interrelations among different mathematical domains, or, of Mathematics with other disciplines and suggests that mathematical activities and results may be interdependent. Thus, integration of history in teaching may help to interrelate domains, which at first glance appear unrelated. It also provides the opportunity to appreciate that fruitful research in a scientific domain does not stand in isolation from similar activities in other domains. On the contrary, it is often motivated by questions and problems coming from apparently unrelated disciplines and often, having an empirical basis. This is especially true for Physics and leads us to point (b) mentioned above.

1.2. Comments on (b): What has been said above about the role of history in teaching and learning Mathematics is equally valid for Physics as well (Tzanakis & Thomaidis 2000). On the other hand, as it has already been mentioned, history shows clearly the close, interconnected development of Mathematics and Physics, which cannot be ignored in teaching and learning these disciplines, in view of what has been said in §1.1. This close interrelation can be seen in two different, but *complementary* perspectives:

(1) *From a historical point of view*, there are 3 different ways by which Mathematics and Physics are interrelated, influencing each other (Tzanakis 2000):

(a) Physical theories and the appropriate mathematical framework evolve *in parallel*, often as the result of the work of the same persons. This is the case of the foundations of infinitesimal calculus and of classical mechanics in the 17th century, mainly through the work of Newton and Leibniz; or, the parallel development of vector analysis and of electromagnetic theory in the second half of the 19th century, mainly by Maxwell, Gibbs and Heaviside (Crowe 1967).

(b) *New mathematical* theories, concepts or methods are formulated in order to solve *already existing physical problems*, or to provide a solid foundation to methods and concepts of Physics. The emergence of the basic ideas of classical Fourier analysis, through the study of heat conduction constitutes a typical example. Dirac's introduction of his delta function in quantum mechanics, and its later clarification in the context of the theory of generalized functions is a more recent example (Lützen 1982, ch.4, part 2)[1]. Finally, the introduction in the second half of the 19th century of Boltzmann's ergodic hypothesis in classical statistical mechanics led to the foundations of ergodic theory in the 1920's and 1930's through the work of G. Birkhoff, J. von Neumann and E. Hopf (Sklar 1993, ch.5; see also section 3.3. here).

(c) The formulation of a *mathematical theory precedes its physical applications*. Its use is often made *after* the corresponding physical problems naturally indicate the necessity of an appropriate mathematical framework. A famous example is Einstein's work on the foundations of the general theory of relativity in the period 1907-1916, on the basis of riemannian geometry and tensor analysis developed in the second half of the 19th and early 20th centuries, mainly by Riemann, Christoffel, Ricci and Levi-Civita (Pais 1982, ch.12). Another example is provided by the fact that on the basis of spectroscopic data, Heisenberg realized in 1925 that atomic magnitudes have the algebraic structure of (infinite dimensional) complex matrices and he was thus led to the formulation of matrix mechanics (Mehra & Rechenberg 1982 ch.3; see also section 3.2 here).

[1] Of course, it is well known that the delta function appeared much earlier, in the 19th century in the work of many mathematicians and physicists, in a number of equivalent forms (Lützen 1982, ch.4, §34).

These examples are indicative of the intimate relation between Mathematics and Physics and lead us to look at this relation from another perspective (Tzanakis 2001)

(2) *From an epistemological point of view* Mathematics and Physics are much closer to each other than it is usually thought:

(a) Mathematics and Physics have always been closely interwoven, in the sense of a "two-ways process":

•Mathematical methods are used in Physics. By this I mean that not only Mathematics is the "language" of Physics (i.e. the tool for expressing, handling and logically developing physical concepts and theories), but also it often *determines* to a large extent the *content* and *meaning* of physical concepts and theories themselves.

•Physical concepts, arguments and modes of thinking are used in Mathematics. Thus, Physics not only constitutes a reservoir of problems "ready-to-be-solved" mathematically (i.e. a *domain of application* of already *existing* mathematical *tools*), but it also provides *ideas*, *methods* and *concepts* that are crucial for the *creation* and *development* of *new* mathematical concepts, methods, theories, or even whole mathematical domains.

(b) Any distinction between Mathematics and Physics, seen as general attitudes towards the description and understanding of an object[2], is related *more* to the point of view adopted while studying particular aspects of this object, than to the object itself.

The general characteristics of the relation between Mathematics and Physics described in (1) and (2) above can be integrated into teaching in several different ways. In the next sections I will illustrate the above points in terms of 3 different examples and with the aid of what may be called a historical-genetic approach

2. A Historical-Genetic Approach

As already mentioned in the previous section, a historical dimension can be introduced into teaching in several ways that have been discussed elsewhere, depending on several factors (Fauvel & van Maanen 2000). The discussion here is confined to what may be called a historical-genetic approach, presented in more detail in the literature (Tzanakis, Arcavi et al 2000, Tzanakis 2000, Tzanakis & Thomaidis 2000).

It is an approach adopting the point of view that a subject should be taught, only after the learner has been *motivated* enough to do so by means of questions and problems, which the teaching of the subject may answer (cf. Toeplitz 1963, Edwards 1977). In other words, the subject to be taught should acquire a necessary character for the learner, so that he/she can appreciate its significance in clarifying particular issues and in answering specific problems. This character of necessity of the subject constitutes the central core of the meaning to be attributed to it by the learner. Therefore, such an approach emphasizes less the *way of using* theories, methods and concepts, and more the *reasons* for which these theories, methods and concepts *provide answers* to specific problems and questions, without however disregarding the "technical" role of mathematical knowledge.

It is clear that such a point of view is not restricted to Mathematics only. In particular it is equally applicable to Physics (Tzanakis & Coutsomitros 1988, Tzanakis & Thomaidis 2000). For both disciplines, a historical perspective offers interesting possibilities for a deep, global understanding of the subject, according to the following general scheme:

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[2] By this term I mean not only concrete, empirically conceived objects, but also mental objects like concepts, questions, problems etc.

(1) The teacher has a *basic* knowledge of the historical evolution of the subject, so that he/she is able to identify the *crucial steps* of this historical evolution and appreciate their significance. These steps consist of key ideas, questions and problems, which opened new research perspectives and enhanced the development of the subject.
(2) (Some of) these crucial steps, are *reconstructed*, by explicitly, or implicitly integrating historical elements, so that these crucial steps become didactically appropriate.
(4) Many *details* of these reconstructions are incorporated into exercises, problems, small research projects and more generally, *didactical activities* that give the opportunity to the learner to acquire technical skills and a better sense of the concepts and methods used. For instance, one may use sequences of historically motivated problems of an increasing level of difficulty, such that each one presupposes (some) of its predecessors. Their form may vary from simple exercises of a more or less "technical" character, to open questions which presumably should be tackled as parts of a particular study project to be performed by groups of students.

This general scheme forms the basis of what can be called a *historical-genetic approach* and seems to have distinct advantages that have been analysed in the references given above. Here we add only a few comments:

One may argue, that an obvious possibility to use history in the presentation of a mathematical and/or physical subject is to retrace its historical evolution. However, the formulation of the problems which led to its birth, and are presented today as part of modern Mathematics and/or Physics, would be too advanced for the learners, or may look completely foreign to them. Usually, its *strictly* historical presentation, in which all the fine details of the historical development are given, is not didactically appropriate, even at the university level. This is due to the fact that the historical evolution of a scientific domain, contrary to what is sometimes naively assumed, is almost never straightforward and cumulative. On the contrary, it is rather complicated, involving periods of stagnation and confusion, in which prejudices and misconceptions exist and it is greatly influenced by the more general cultural milieu, in which this evolution takes place. Moreover, the conceptual framework and the mathematical terminology and notation vary from one period to another. Finally, the didactical, social and cultural conditions of the students today are very different from the corresponding conditions in which mathematicians, who created and developed the subject under consideration, were living. Hence, strictly respecting the historical order makes the understanding of the subject more difficult (Thomaidis & Tzanakis preprint).

Therefore, integrating history in teaching Mathematics and/or Physics, should mean that a historically motivated thinking framework for the learner has been created, in which various aspects of the mathematical subject under consideration can be illustrated. In this respect, the *crucial* steps of the historical evolution of the subject are didactically important because whether or not a step in the historical evolution is crucial, is judged *a posteriori*. In other words, such *a step is crucial* exactly *because* it opened new research paths, it clarified the meaning of new knowledge, it suggested the most convenient and clear formulation of this knowledge and in general *it enhanced the development of the subject*. *Therefore*, such a step in the historical evolution is *in principle* didactically relevant.

It is in the above perspective that I will comment in the next section on three specific examples, which at the same time illustrate the deep, continuous and multifarious interrelation between Mathematics and Physics.

3. Examples

3.1 Algebra, Geometry and Relativity Theory

Einstein laid the foundations of the Theory of Relativity in two seminal papers. In 1905 he presented the Special Relativity Theory (SR) and after many years of intensive work and unsuccessful attempts, in 1916, he arrived at a new theory of gravitation, the General Relativity Theory (GR), in a long paper where he presented both its physical foundations and the mathematical methods to be used (both papers are reprinted in Sommerfeld 1952).[3] Although his papers were fairly complete, and full of fundamental consequences, both theories were developed further by many others in the next years.

Today, SR is a standard subject in undergraduate curricula for Physics students, whereas an introductory course in GR is usually addressed to postgraduate, or advanced graduate Physics (and occasionally, Mathematics) students. However, basic aspects of both GR and SR that played an important role in the development of new Mathematics, and enhanced the development of our understanding of physical phenomena, can be presented at a much earlier stage as an illustration of this new Mathematics and their place in the scientific edifice (both inside and outside Mathematics). This can be done by following an approach inspired by history, along the lines suggested in the previous section.

Some crucial historical elements[4]:

(a) SR is based on the so-called Lorentz transformations (LT) that gives the transformations between inertial coordinate systems. Einstein gave the derivation of these transformations in 1905 using the basic principles of SR, namely, the *Principle of Special Relativity* (the laws of Physics are invariant under a coordinate transformation between inertial systems, i.e. systems moving with constant velocity with respect to each other) and the *Principle of the constancy of the light speed* (light has the same speed in all inertial systems, whether or not the source is moving). His derivation is elementary and appeals very much on physical intuition and some tacit assumptions (about the homogeneity of space).

(b) Others (Voigt in 1887, Larmor in 1900, Lorentz in 1899 and 1904) have derived the LT earlier as a consequence of the search for the coordinate transformations that leave unaltered Maxwell's equations in electrodynamics. By the end of the 19th century, it had been realized that these were the transformations between inertial systems, as a consequence of the famous Michelson-Morley experiment (and other similar ones).

(c) Poincaré in 1904 derived the LT by following a more mathematically oriented approach. He explicitly used the group structure of the sought transformations and determined their general form, as well as, fundamental consequences of SR, like the relativistic law of velocity addition.

(d) In 1908, in a seminal lecture (reprinted in Sommerfeld 1952), Minkowski introduced the concept of spacetime and revealed the rich geometrical content implicit to Einstein's 1905 paper on SR. This was the crucial step, without which GR could not have been developed.

(e) In his conceptual analysis of the physical and mathematical foundations of GR, Weyl (in 1918) argued that its basic physical principles imply that spacetime has the structure of a conformal rather than a (pseudo)riemannian manifold (i.e. only ratios of infinitesimal

[3] Perhaps it is less known that Hilbert arrived almost simultaneously to the field equations of the theory by following a different route. I will not touch upon Hilbert's contribution here (for a detailed study see Mehra 1973, Pais 1982 §14(d)). I simply mention that Hilbert knew Einstein's struggle for a new theory of gravitation and approached the subject from a different point of view.

[4] More historical details and references to the original literature can be found in Tzanakis 1999.

spacetime distances have a meaning, not the infinitesimal distances themselves; Weyl 1918/1952, p.204). As a consequence, he considered that the physically relevant basic geometrical structure of spacetime is not its (pseudo)metric. To proceed further, he argued that the basic structure is parallelism (i.e. the existence of a connection), an important concept introduced in 1917 by Levi-Civita and Hessenberg (Weyl 1918/1952, p.202). In this way, Weyl was led to introduce and study the first example of what later became known as gauge theory and gauge transformations (Weyl 1918/1952, Weyl 1921/1952, section 16).

It is beyond the scope of this paper to give a detailed epistemological analysis of points (a)-(e), which supports the claims made in §1.2. This will be done implicitly, by commenting on the didactical relevance of (a)-(e) along the lines of section 2.

(1) It is possible to derive the LT in two dimensions (one spatial and one temporal) by following Minkowski's key ideas: (i) the introduction of the concept of spacetime as a natural idea implied by Einstein's 1905 analysis of the relative character of simultaneity of events and (ii) its immediate consequence that the constancy of the light speed trivially implies that the sought transformation leaves invariant the so-called light cone (i.e. the surface on which light signals lie).

This derivation uses elementary matrix algebra and proceeds in close analogy with the determination of the form of plane rotations in analytic geometry: Rotations conserve the Euclidean distance x^2+y^2 in the xy plane, whereas LT conserve the Minkowski (pseudo)distance $x^2- y^2$ (which is zero on the light cone, y being the time coordinate). For details see Tzanakis 1999, section 3.

(2) Strictly speaking, conservation of the light cone implies only that the transformations are conformal, a fact whose significance seems to have been appreciated first by Weyl (see (e) above). It is more advanced, especially in 4 dimensions, to show that in the context of SR these conformal transformations are indeed isometries of the Minkowski (pseudo)distance if we assume that they map straight lines to straight lines, a consequence of the validity of Newton's law of inertia in SR. This derivation may constitute a small project, which can proceed along the lines of Poincaré, using explicitly the group structure of the transformations sought. A number of fundamental consequences of SR can be obtained in this way (velocity addition, length contraction etc), at the same time illustrating important abstract concepts, like group, commutativity, pseudo-Euclidean structure, conformal transformations etc. (for details and references to the original literature see Tzanakis 1999).

(3) Conformal transformations in the special case of similarities can be also introduced naturally by looking for the symmetry group of Maxwell's equations. It is a nice example to consider this problem for both the Laplace and the wave equation and to arrive at the orthogonal and the Lorentz group of transformations respectively. This is in fact the idea behind the pre-relativistic derivations of the LT by Larmor and Lorentz (papers reprinted in Schaffner 1972, part II, sections 9 and 11 and in Sommerfeld 1952, paper II; see also Whittaker 1951, pp.31-33). This point can be used in connection with (1) above, in the sense that they are dual to each other.

(4) Conventionally, parallelism and the concept of a connection on a differentiable manifold are introduced in a rather abstract and unnatural way. Weyl's geometrical interpretation of the basic physical principle of GR, namely the *Equivalence Principle* (all bodies in free fall in an infinitesimal region of spacetime have the same acceleration), leads to a natural definition of parallelism that is equivalent to its modern abstract definition (see (e)

above and Siu et al. 2000, §8.4.8). The proof can be structured as a sequence of exercises in tensor algebra and differential geometry.

(5) On the other hand, Weyl's analysis, mentioned in (e), led him, to consider that spacetime has a conformal rather than a metric structure, to identify the conformal factor with the electromagnetic potential (a physically wrong but mathematically fruitful idea!-see just below) and to introduce the concept of gauge transformation. It was the first, very early attempt to develop what much later came to be known as a gauge field theory, especially in connection with gauge invariance of electrodynamics. The similarities stressed by Weyl between the geometrical concepts of GR and the dynamical concepts of electromagnetism were elaborated later and led to important developments in differential geometry and its relation to Physics, namely, the theory of connections and of fibre bundles and the formulation of gauge field theory (Pais 1982 pp.339-340, Cao 1997 §9.1). Although this is a rather advanced subject, Weyl's procedure can provide a natural introduction to concepts and methods, which are equally used by mathematicians and physicists and which play an equally important role in pure Mathematics and in Theoretical Physics.

3.2 Differential Equations, (Functional) Analysis and Quantum Mechanics

It is well known that since Newton's time, differential equations have always been one of the main links between Mathematics and Physics, leading to important developments both in analysis and in the concise and fruitful formulation of physical theories. It is perhaps less known that many important concepts of functional analysis originated in the study of quantum theory (QT) and conversely, that it was only through its concepts and methods that a deep understanding of atomic phenomena became possible.

Below, I outline only a few, but fundamental points of this really complex, continuous and deep interrelation that has been so fruitful both mathematically and physically.

Some crucial historical elements[5]:

(a) Already in the 18th century it was realized that there is a close formal similarity between Fermat's *principle of least time* in geometrical optics, and Maupertuis' *principle of least action* in classical mechanics. In the 1830's, on the basis of this similarity, Hamilton formulated the two disciplines in a unified way and developed a general method for solving 1st order partial differential equations (PDE) that became central in the formulation and solution of mechanical problems as well (Hamilton-Jacobi method).

(b) About 90 years later, Hamilton's ideas stimulated de Broglie to take the aforementioned formal similarity as an indication of a deeper relation between mechanical and optical phenomena and to predict the wave nature of atomic particles. Schrödinger, in turn, further elaborated this idea, and arrived in 1926 to the formulation of wave mechanics.

(c) In the 1920's, atomic physics was a complicated mixture of classical mechanics and electrodynamics with additional semi-empirical rules and heuristic arguments. People were trying hard to develop models of atomic phenomena and to understand their mathematical structure. Heisenberg in 1925 developed a kind of algebraic manipulation of atomic quantities, in analogy with Fourier series operations, the novel idea being that in this manipulations, the Fourier frequencies and coefficients were *doubly* indexed as a consequence of the so-called *Ritz combination principle* in atomic spectroscopy. It was immediately realized by Born that Heisenberg's calculus was just the algebra of (infinite, in

[5] References to the original literature and to secondary sources can be found in Tzanakis, 1998, 2000, 2001.

general) matrices. This led to matrix mechanics, the first formulation of modern quantum mechanics (QM).

(d) After Schrödinger's formulation of wave mechanics in 1926, physicists were puzzled by the existence of two conceptually and mathematically very different theories of atomic phenomena (matrix mechanics and wave mechanics), which nevertheless gave identical, empirically correct results. Schrödinger himself and von Neumann tackled the problem. Both showed that the two theories were mathematically equivalent. Von Neumann's approach was more rigorous and led him to introduce for the first time the concept of an abstract (separable) Hilbert space, to show that all such spaces are isomorphic and to resolve the puzzle by making clear that the two physical theories were based on two different, but isomorphic such spaces (l^2 and $L^2(\mathbf{R})$ respectively).

Points (a)-(d) can be integrated into teaching in a number of ways, depending on the course given, its emphasis, the time available etc. I will give some possibilities below, in which the emphasis is on Mathematics, rather than Physics:

(1) The least action principle and the principle of least time, constitute natural examples of variational principles, leading to mathematically interesting and physically relevant equations, the Hamilton-Jacobi equation, which is central in classical mechanics, and the so-called eikonal equation of optics. On the other hand, they are generic examples, in the sense that it is possible through them to establish a general result in the theory of PDE's: the solution of a 1^{st} order PDE is *equivalent* to the solution of a system of first order ODE's, the so-called Hamilton's system of the associated canonical equations (Courant & Hilbert 1962 section II.9, Gel'fand & Fomin 1963 section 23). As it is well known, this result is of central importance both in the theory of differential equations and in mechanics. In fact, one can proceed very close to Hamilton's and Jacobi's approaches to illuminate the subject from two different, but important view points (Dugas 1988, part IV, ch.VI, Klein 1928/1979, pp.182-196).

(2) Schrödinger's elaboration of Hamilton's mathematically unified treatment of classical mechanics and geometrical optics mentioned above, was based on *arguing by analogy*: If classical mechanics is mathematically similar to geometrical optics, and since geometrical

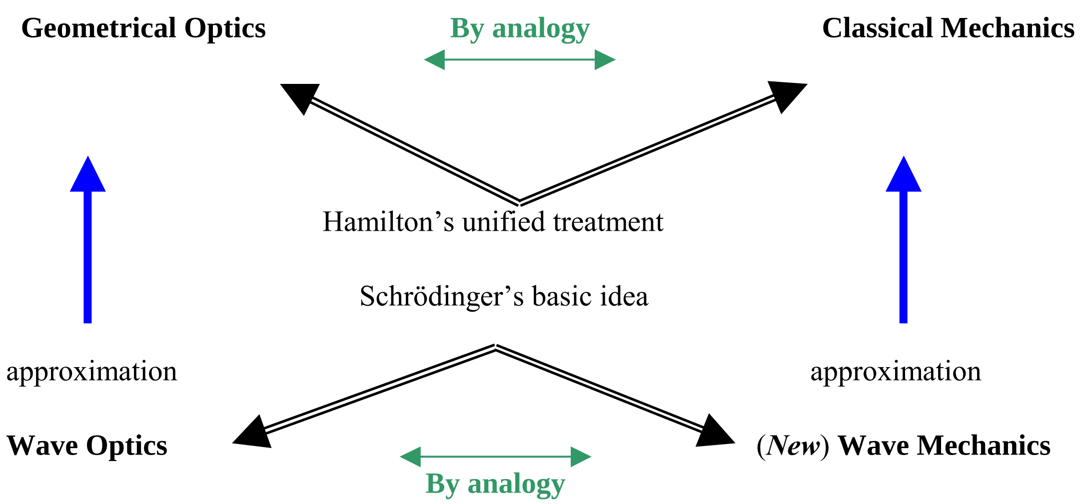

Figure 1: A schematic representation of Schrödinger's reasoning by analogy

optics is an approximation to wave optics, perhaps classical mechanics as well is an approximation to a wave mechanics, which is similar to wave optics in the same sense that classical mechanics is similar to geometrical optics. In this way, Schrödinger's equation results as the mechanical equivalent of the wave equation, schematically given in figure 1.

This derivation can be presented in relation with (1) above (for a detailed treatment, see Tzanakis 1998). This is a characteristic example that makes clear the important role of *analogy as a mode of reasoning* of great heuristic value (for a detailed discussion both in Mathematics and in Physics, see Tzanakis & Kourkoulos 2000, §2). Another such example is provided by Heisenberg's approach described in (c) above and schematically represented in figure 2 (see also Heisenberg's own account in Heisenberg 1949/1930, appendix, which can be used for didactical purposes in a slightly restructured form, as well as, his original paper reprinted in van der Waerden 1967, paper 12).

Classical quantities q, p

Fourier representation
Single index frequencies
$\nu_k = k\omega$

$\nu_k + \nu_l = \nu_{k+l}$

Quantum quantities q, p
Looking for a
New representation because of
Double index frequencies
ν_{nm}
obeying ***Ritz principle***
$\nu_{nl} + \nu_{lm} = \nu_{nm}$

By analogy

$q(t) = \Sigma\, q_k \exp(\mathbf{i}\, \nu_k t)$ ⟷ $q(t) \sim (q_{nm} \exp(\mathbf{i}\, \nu_{nm} t)$

Operations

By analogy

$p+q \rightarrow (p_k+q_k) \exp(\mathbf{i}\, \nu_k t)$ ⟷ $p+q \rightarrow (p_{nm}+q_{nm}) \exp(\mathbf{i}\, \nu_{nm} t)$

$pq \rightarrow (\Sigma_l\, p_l\, q_{k-l}) \exp(\mathbf{i}\, \nu_k t)$ ⟷ $pq \rightarrow (\Sigma_l\, p_{nl}\, q_{lm}) \exp(\mathbf{i}\, \nu_{nm} t)$

By analogy
+
Ritz Principle

Hence, $pq \neq qp$ which leads to
Heisenberg's uncertainty relations!

Figure 2: A schematic representation of Heisenberg's reasoning by analogy

Although analogy seems to play a central role as a discovery pattern both in Mathematics and in Physics, no enough attention is usually given to it in teaching. These examples are important in this respect as well.

(3) Many of the basic concepts of functional analysis can be motivated in a natural way (i..e. avoiding logical gaps), in the context of questions and problems of atomic Physics. The concept of an abstract Hilbert space mentioned in (d) above is a characteristic one. Schrödinger's formal proof of the equivalence of matrix and wave mechanics may serve as a

general motivation to look for a more rigorous proof. This in turn leads to appreciate the significance of the existence of a (complete) orthonormal basis and the various equivalent conditions (e.g. Parseval's relation). Von Neumann's approach is very clear and can be followed closely (von Neumann 1932). Other examples can also be given, like the concepts of a hermitian and of aunitary operator and their generalization, a normal operator, the concept of spectrum and important theorems associated with these concepts that followed as a result of von Neumann's work on the foundations of QM (for more details, see Tzanakis 2000 §3.4).

3.3. Statistical Mechanics, Dynamical Systems and Ergodic Theory

A somewhat more advanced example, which shows the deep and fruitful influence that Physics can exert on the development of new mathematical concepts, methods and theories is provided by the historical development of statistical mechanics and ergodic theory. Only some aspects of this subject are briefly discussed below.

Some crucial historical elements:

(a) Implicit to the work of Boltzmann (in 1871, 1884, 1887) and Maxwell (in 1878), is what later became known as the *ergodic hypothesis*, a desired basic property of the systems with a large number of degrees of freedom studied in statistical mechanics: The phase space trajectory of a mechanical system passes through *every* point of its energy surface.[6] If this conjecture were true, the phase space average of any quantity would coincide with its time average along the trajectory of the system. This was an utterly important conclusion in the foundations of statistical mechanics. It was gradually realized that this hypothesis leads to contradictions on the basis of important mathematical theorems according to which space-filling curves cannot be smooth as required in (statistical) mechanics (Sagan 1994). In an attempt to overcome this obstacle, the Ehrenfests in 1912 distinguished the ergodic hypothesis from the *quasi ergodic hypothesis*, according to which the phase space trajectory of the system is a *dense subset* of the energy surface, hoping that the latter could offer a better foundation of statistical mechanics (Ehrenfest & Ehrenfest 1912/1990 §10 and note 98).

(b) The formulation of the quasi ergodic hypothesis and the significance in the context of statistical mechanics of the coincidence of phase space and time averages (what in fact later was taken as the definition of an ergodic system), were the basic motivations for the important investigations of Birkhoff, von Neumann and Hopf that led to the proof of the first ergodic theorems. It was a crucial starting point for the development of what later became known as ergodic theory (Sklar 1993, §5.II.1, Farquhar 1964 ch.3).

(c) The stability of the solar system was an old problem investigated by many great mathematicians since the 18th century. It was a main motivation for the study of periodic motions in N-body systems and more generally in dynamical systems and it strongly influenced the development of the qualitative theory of differential equations, especially through Poincaré's work on celestial mechanics and Birkhoff's investigations on general dynamical systems that paved the way to the development of the modern theory of dynamical systems (Poincaré 1957, Birkhoff 1927; for a short review, see Moser 1973, §§ I.1, I.2).

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[6] It is not clear to what extent Boltzmann and Maxwell thought of this hypothesis as a fundamental element of statistical mechanics (Boltzmann 1954, pp.11-12 and footnote on p.297). Apparently, it was the Erhenfests' review of 1912 that stressed the importance of the ergodic and quasi-ergodic hypotheses and the difficulties inherent to them (Ehrenfest & Ehrenfest 1912/1990).

(d) The existence of (quasi) periodic motions of dynamical systems was an important problem systematically investigated by Poincaré in connection with the stability of the solar system and more generally with the N-body problem. Kolmogorov made significant progress in 1954, by proving the existence of such motions under quite general conditions and contrary to what one would expect if dynamical systems were ergodic (see (b) above). Kolmogorov's ideas were elaborated by others, especially Siegel, Arnold and Moser[7] and led to a revitalization of classical mechanics in the last 40 years, by fruitfully combining concepts and methods of such diverse fields like measure theory, differential equations, topology and differential geometry. Conversely, new, essentially physical, concepts, like ergodicity, mixing property and entropy of a dynamical system etc, were introduced that further enhanced the development of ergodic theory and dynamical system theory, into an interdisciplinary domain that touches upon many diverse areas of pure and applied mathematics and theoretical physics; e.g. probability and measure theory, differential topology, number theory, statistical mechanics, fractal geometry etc.

Ergodic theory and the theory of dynamical systems are certainly advanced topics and at most an introduction to their basics can be incorporated in an undergraduate curriculum. Even the definition of its most basic concepts, like an abstract dynamical system or ergodicity, are rather technical and require some knowledge of various areas of Mathematics (e.g. measure theory, differential geometry, topology etc). My main point is that even these basics cannot be grasped properly if their introduction is *decontextualized* as it is usually done. On the contrary, their introduction in the proper context in which they have naturally appeared historically, namely, in connection with specific, difficult and physically important problems, can greatly enhance their understanding.

(1) Most of the basic concepts of ergodic theory have a deep physical meaning and were introduced in an effort to understand specific physical problems. Thus, ergodicity of a dynamical system (in the sense of the coincidence of phase space and time averages), its entropy, the mixing property etc can be motivated by the ergodic problem in statistical mechanics (see (a) and (b) above) and Boltzmann's probabilistic definition of (macroscopic) entropy of a physical system in terms of microscopic quantities.

(2) The importance of the ergodic hypothesis in statistical mechanics constitutes a natural (but of course, not the only) framework for discussing the interesting and independent subject of space filling curves (e.g. Peano's curve) and the associated deep problems of the definition of the concept of dimension, especially in connection with the fascinating concept of a fractal and its relation to ergodic characteristics of specific dynamical systems (see e.g. Falconer 1990, ch.13).

(3) The significance of the stability of the solar system is self-evident, even in a general cultural context. A historical introduction to this subject, in which the nature of problems and achievements are explained without proof, is helpful (cf. (c) above). If students appreciate the difficulty of these problems *and* their physical importance, they can also appreciate better why one has to work out and understand in detail the dynamical behaviour of many, somewhat artificial, low dimension systems.

(4) For a long time classical mechanics was considered as a dead research domain. It served only as a subject for introducing basic mathematical methods of Physics, or as a first

[7] Kolmogorov 1954/1978, Arnold & Avez 1967, Siegel & Moser 1971, Moser 1973 are the standard references containing the basic theoretical concepts, mathematical tools and results. See Sklar 1993, §5.II.3.

step of a theory that one had to overcome in order to understand phenomena beyond the everyday world (in the atomic or astronomical scale). This picture has changed in the last 40 years, at least as far as research is concerned (cf. (d) above). This came as a result of two different, but dual in character and interconnected lines of research: (i) On the one hand, there was a struggle for understanding the ergodic properties of specific physical systems. Often, this was done in the hope to show that in some sense ergodic systems are the majority of physically relevant multidimensional systems, hence that dynamical motions are mainly non-periodic for systems with many degrees of freedom. (ii) On the other hand, there has always been a continuous interest in and research on the stability of motion of specific dynamical systems and the determination of (quasi) periodic trajectories. For a long time it was believed that this could be true for systems with a few degrees of freedom. We now know that none of the beliefs underlying research along (i), or (ii) is strictly true (there are low dimension ergodic systems and "a lot of" periodic motions in systems with many degrees of freedom). This is a fascinating development in Mathematics, some elements of which can be given to our students, illustrating (i) and (ii) above by means of elementary examples taken from such diverse fields like classical mechanics, riemannian geometry, or number theory.

4. Final Remarks

All three examples support points (a), (b) in §1.2.(2), although lack of space does not allow a detailed epistemological analysis in support of these points. At the same time, the basic historical facts presented for each example, constitute a natural framework for introducing new mathematical concepts and methods and by linking them to mathematically relevant and physically important questions and problems, which may serve as a meaningful motivation for students. They can be adapted so that they become didactically appropriate and they can form the basis for the development of teaching sequences in which many technical details are incorporated in the form of exercises, problems and small research projects. Details can be found in the given references.

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