# SEMANTIC UNDER-LOADING: THE LESSON OF LOGS

**Michael BRENNAN**
Waterford Institute of Technology
Waterford
Ireland
mbrennan@wit.ie

.

## ABSTRACT

The semantics of some of the most fundamental elements of arithmetic and algebra, rather than either modes of learning or teaching, or the conceptual complexity of the elements themselves, may be barriers to understanding. In the treatment of logarithms there is an over-supply and under-use of terms that describe the same concept. *Power, index and exponent* supposedly are synonyms and are invoked when *logarithms* are defined. There are supposedly distinct theories, "The Rules of Exponents" and "The Laws of Logs".

Questions on logarithms in algebra are unpopular, although understanding the nature of logarithms does not seem to be a prerequisite for applying logarithms to numerical problems. Nonetheless the mystification that arises from poorly specified and "under-loaded" symbols in the theoretical treatment of logarithms must result in a disheartening loss of understanding.

Students' difficulties with the semantics of elementary mathematics need to be acknowledged where possible and remedies sought, at undergraduate level if necessary. To this end a case is made for abandoning the term *logarithm*, despite its longevity, and for rationalising the terminology in the area of powers and exponents.

**Keywords:** mathematical education, logarithm, under-loading

# 1. Introduction

"You cannot teach logarithms to illiterates"
- *"Stand and Deliver", Warner Brothers, 1988*

In June 2002 logarithm tables will have been used for the last time in a public examination in the Republic of Ireland when grade 10 students take the Junior Certificate Mathematics Examintion. This will have brought to an end more than a century of association between log tables and school arithmetic.

Thirty years ago *every* student who took the Republic's Leaving Certificate Examination, (the grade 12 school-graduation exam at "Honours" and "Pass" levels), was examined not only in the use of log tables but in his/her knowledge of the properties of logs. The following is part of a specimen question from 1970:

"$\log_{10} 2 = x$  $\log_{10} 3 = y$.
Express in terms of $x$ and $y$, (i) $\log_{10} 6$, (ii) $\log_{10} 24$ (iii) $\log_2 3$ (iv) $\log_{10} \sqrt{8}$ "

In the years since then the number of students staying on to take the Leaving Certificate exam has grown from 30% to 90% of the population cohort, and their range of ability has widened. This has necessitated the introduction of syllabi at three ability-levels, (Higher, taken today by 17%, Ordinary by 73% and Foundation by 10% of the cohort) and only at the highest level of these do questions such as the one quoted above appear. Significantly, not only were the grade 8 class of September 2000 (the grade 10 class of 2002) the first to be allowed to use calculators, but the "theory" of logs was at the same time removed from all three Junior Certificate syllabi.

Questions on logs are unpopular. The Irish Chief Examiner's Reports for the Junior Certificate exam of 1996 and 1998, and the Leaving Certificate exam of 2000 show that logarithms are the least or second least popular question on the exam papers and that this is an annual trend.

Some teachers find logs difficult. A team of experienced U.S. textbook authors, Hornsby and Lial, recently conceded in background material to their College Algebra book:

"Without a doubt, the concept of the logarithm is one of the most difficult for algebra students to grasp. The authors of your text admit that even they did not fully understand the concept until taking follow-up courses and teaching logarithms in their classes! So if you find this topic difficult, don't feel as if you're alone".

This paper examines possible reasons for the unpopularity and difficulty of logarithms.

# 2. The Tradition

It is helpful to look at how logs are introduced in textbooks.

There are two distinct modes for doing so, the algebraic and the analytical. Typically in the latter, the *base a exponential function* $y = a^x$ is firstly defined, and then $y = \log_a x$ ($a>0$, $a \neq 1$) is defined to be the inverse of $y = a^x$ (Finney *et al.*, 2001).

This paper concentrates on the alternative, algebraic, treatment because of its potential for being more intuitive than the analytical, and because it is more likely to have been the mode of introduction to the topic experienced by students before they enter third-level education.

In the tradition of Irish and British school-mathematics, Hall and Knight's *Elementary Algebra for Schools*, first published in 1885, is a seminal book. The eighth edition published in 1907 was, along with its companion text *A School Arithmetic* by Hall and Stevens, reprinted more than 30 times, and remained with schools all the way to the New Maths of the late 1960s.

Before going further note that to denote, for example, the "2" in the symbol $x^2$, the term "index" is more common in Irish and British schools than the internationally-used "exponent".

The Eighth Edition of Hall and Knight defines *log* as follows:

"The **logarithm** of any number to a given base is the index of the power to which the base must be raised in order to equal the given number. Thus if $a^x = N$, $x$ is called the logarithm of $N$ to the base $a$".

Hall & Stevens (1908) had a similar definition and followed it with**:** "Since every logarithm is an *index*, it follows that the rules which govern the use of logarithms are deducible from the laws of indices".

But within a few years we had:

"The logarithm of a number to a given base is the power to which the base must be raised to equal the number" (Jones, 1913)

Note the omission of "index".

From the New Maths era onwards came:

"Logarithms are another form of indices…The logarithm of a number to a base is the power to which the base must be raised to give the number" (Holland and Madden, 1976)

"The logarithm of a positive number N to the base *a* is defined as the power of *a* which is equal to N" (Bunday and Mulholland, 1983).

"The *logarithm* function is the inverse of the index function " (Solomon, 1997)

"**Logarithm** is another word for **power**, **exponent** or **index"** (Sherran and Crawshaw, 1998)

"Logarithms are mirror images of exponentials…the logarithms are the exponents" (Strang, 1992)

" $\log_b x$ is defined to be that exponent to which b must be raised to produce $x$" (Anton, 1999).

It need hardly be said that the intended meanings of the above quotations are all the same. The language however is inconsistent. A logarithm is variously defined as an index, exponent, or power. While the first two of these are synonymous, they are not synonymous with the third.

Also, while a logarithm is indeed an exponent the adjectives "logarithmic" and "exponential" are given opposing (inverse) meanings.

## 3. Power

The greatest confusion surrounds the word "power". *Power* and the phraseology associated with it are so embedded in the language of algebra since the sixteenth century that they are difficult to deconstruct.

8 is a power. It is, among other things, the 3rd power of 2.

The use of the ordinal, 3rd, appears safe until we start "raising" things. When we read $8 = 2^3$ as "8 is 2 raised to the 3rd power", "*raised to the*" points to the superscript 3, as if *it* were the power as well as being the "exponent".

Repeated use of the cardinal, "8 is 2 to the power (of) 3" or "raised to the power 3" leads to the situation, common at present, in which *power* is used as another term for the exponent, and the original meaning of power as a product of copies of the base is ignored. Yet this meaning is recalled in usages such as "$x + x^2 + \dots$ higher powers of x".

Similarly, while $\log_2 M = x$ is correctly called the "log form" of the relationship between M and x, its inverse, $M = 2^x$, is often referred to as the "index or exponential form" (which should mean log form!) - when its correct name is "power form".

Because of the identification, in textbooks and teaching, of the concept of an exponent/index with the term "power" it is not hard to deduce from terminology in current use that

logarithm = exponent = **power** = antilogarithm

and that

index form = inverse of log form = inverse of index form

The fact that students do not protest at this lack of consistency and clarity in the language is not proof that they are comfortable with it. The evidence points to the contrary. It is tempting, if flippant, to say that it is possible to teach logarithms *only* to illiterates. A likely result of the inconsistency is that the students lose confidence in their ability to understand the logarithm concept, and settle for engaging with logs at the procedural level only.

# 4. Teaching Logarithms

In an effort to counter-act the inconsistency of the language a group of 40 first-year college computing students were given a basic course in powers and indices (exponents), with substantial drilling in the "rules of indices", $a^m a^n = a^{m+n}$ and so on.

The students were then introduced to the terms used to describe the *power equation* $8 = 2^3$, comprising the power (8, or $2^3$), the base (2) and index (3). "Logarithm" ("log" for short) was given as another, synonymous name for the index when the base was > 0 and not 1.

Next a natural-language description of $8 = 2^3$ was progressively transformed to mathematical "shorthand":

**The index of 8, when 8 is written as a power of the base 2, is 3.**

**The index of 8 when 8 is written to the base 2 = 3.**

**The log of 8 when 8 is written to the base 2 = 3.**

**log of 8 to the base 2 = 3.**

**$\log_2 8 = 3$** ("log form" or "index form")

Drill was then given in transforming between *power form* and *index form* equations. Subsequent work emphasised the identification of log with index by reinterpreting a subset of the rules of indices as rules of logs and giving drill-exercises in these.

Practice was also given in the application of logs to solving equations containing an unknown exponent, in particular to finding *n* in the compound growth/decay formula $A = P(1 + {}^{r}/_{100})^n$.

Six weeks later as part of a wider test the students were asked the following:

Q1. What is a log?

Q2. What are logs useful for?

The answers to Question 1, summarised in Table 1 show that 15 of the students returned at least the answer "an index" or "index or power", one student (who had started the Higher Leaving Certificate course but switched to the Ordinary Level) gave a more complete definition ("a log is the inverse of a power"), and four others defined a log only as a "power" but may have meant "index".

In Table 2 the answers to Question 2 are tabulated against the answers to Question 1, and show that in their knowledge of a *usage* for logs those who could not say what a log is, performed no differently from those who could.

Of the four students who defined a log as a power, two could describe a use for logs and two could not. When these students are shared evenly between the two groups who got Question 1 right, the outcome is Table 3, which merely confirms the evidence of Table 2.

Four of the students in the class had passed Higher Leaving Certificate Mathematics before entry to the computing course. None of these four was able to describe what a log is, suggesting both a lack of understanding of the concept by them at school, and an indifference to learning a new treatment of the topic at college.

Of course being able to answer that “a log is an exponent/index” is only a beginning, but it will now be argued that it is a most important realisation.

# 5. Can logs become popular?

The experience of the author from teaching at both second-level and third-level is that while logs will probably never move to the top of the popularity list in examination questions, they can certainly move off the bottom.

The language problem must first be solved. Napier's term *logarithmus* (logos+arithmos, a "reckoning number"), in the form *logarithm* or *log* does not offer an intuitive notion of the role that a log performs, in the way that, say, *index* (which 'points' to how often a base is multiplied by itself), *base,* and to an extent *power* do for their roles. A suggestion is made later for abandoning the term "logarithm" altogether. Until that happens it needs to be introduced carefully. The key to this lies in admitting to our students that "logarithm" is an superfluous word, hallowed by tradition, for a concept with which they are familiar - the 3 in $x^3$. They already know this object as the "exponent" or "index". The freedom to interchange the words "logarithm" and "index (exponent)" is to be their lifeline.

Students should be shown the semantic transformation from $8 = 2^3$, with which they are familiar, to $\log_2 8 = 3$. Next should come drill in changing between $A = x^n$ and $\log_x A = n$. The "laws of logs", such as logAB = logA + logB, can be presented as mere rewordings of the rules of exponents.

Students should come to feel that in using a redundant word like "log" they are only humouring their teacher/lecturer. They will not be intimidated by drill-exercises such as "Expand $\log_a \frac{P^3Q^4}{R}$ as a sum or difference of logs" or its opposite "Write $2\log_4 D + 3\log_4 F - {}^1/_2\log_4 G$ as a single log". At all times they know that there is an unbroken thread from the land of *logs* to the familiar ground of *indices/exponents*. It is a "thread" of two strands: the translation at any stage of *log* to *index,* and the skill to switch fluently between a *log form* equation and its *power form.* By this means students can be led into a "labyrinth" of questions such as the one quoted from 1970, and equations requiring logs for their solution. Hopefully they will reach a stage where the thread is no more than an underlying confidence, a feeling that <u>if required</u> they could get back safely to indices/exponents! And this is enough. It should by then be as easy to go forward as to go back, but at least they can advance without the insecurity that ill-defined terminology brings and without the suspicion that their leader, the teacher, isn't sure of the ground.

# 6. Changing the Terminology (i): *Index* for *log*

In an ideal world none of this remedial work would be necessary. Mathematics would not have redundant terminology, technical terms whose work could be done by terms already in the field.

The job of *logarithm* could be done by the under-worked (hence "under-loaded") term *index/exponent.*

The logic of this is that the word *logarithm* should be dropped from Mathematics. But how easy is it to replace?

Despite its restricted use internationally *index* would be a better candidate to take the place of *log*, than would *exponent*.

$81 = 3^4$ could be semantically transformed in the style shown earlier to $index_3\, 81 = 4$, or for short, $ind_3\, 81 = 4$, or $Ind_3\, 81 = 4$. The form $index_a$ would require that the base $a$ be > 0 and not 1, and a reason would be given for this (This is not so strange, the fraction $^a/_b$ of primary school becomes restricted by b $^{1}$ 0 in Junior school).

When the base is 10, i.e. *d*ecimal, the form *Ind* 10000 = 4 could be used.

When the base is the *n*atural number e, the form *In* 5 = 1.609… could be used.

In reverse, $Ind_3\, 81 = 4$, the *index* form, transforms to $81 = 3^4$, the *power* form.

Rules of Indices having been justified, $x^n.x^m = x^{n+m}$ would imply that the index of a product equals the sum of the indices of the factors, or

Ind AB = Ind A + Ind B

(Strang (1992) uses a direct approach in this sense to justify the properties of logs).

Of course if *logarithm* were dropped from algebra there would be consequences for the language of functions:

In an ideal world one might attempt a new terminolgy:

$x^4$ is a *variable-base power* function. $4^x$ is a *fixed-base power* function, as is $e^x$.

Let $e^x$ be distinguished by the label *ebasepower* function, or *ebp* for short, (both labels have no more syllables, or letters, than *exponential*):

$$ebp(x) = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$$

"Exponential decay" would become "fixedbasepower(ful) decay", and "growing exponentially" would be "growing fixedbasepower(ful)ly".

*fixedbasepower* functions and *indicial* functions would be inverses, replacing *exponential* and *logarithic* functions.

# 7. Changing the Terminology (ii): *exponent* for *log*

The difficulty in choosing *exponent* to replace *logarithm* would lie in the fact that variations of *exponent* are already in use and have well-entrenched meanings.

*exp(x)* is the so-called exponential function $e^x$: it has the property that

$$exp(A + B) = expA.expB.$$

If *exponent,* or *exp* for short were to replace *log,* the first law of logs would become

$$exp(AB) = expA + expB,$$

which would be hard for teachers to adjust to.

Furthermore, while the exponential function $e^x$ might become the *ebasepower function* described above, the current *logarithmic function* would become the *exponential function*, which would cause headaches for existing mathematicians and the literature.

# 8. Acknowledging the Difficulties

But something needs to be done if logs are to become more digestible than they have been for the past 100 years. With the abandonment of logs in arithmetic there is time and space to improve their accessibility in algebra. The notation in current use gives rise to the mystification of logs, and their popularity is as low as ever.

As educators we should not stand idly by. Either we purge the notation of its redundancy at an early stage or we engage in remedial work at third level. Part of the remedial work must be to acknowledge the difficulties of the material in the manner of Hornsby and Lial above. Such openness in mathematical education can both have a reassuring effect on the student ("I'm not so stupid after all") and act as a challenge to him/her to rise to the task required.

In the case of logs, an acknowledgement of (and an apology for) a non-intuitive and careless terminology should be part of every introduction to the topic.

# 9. Conclusion

The theory of logarithms as currently presented in algebra creates difficulties for many students, although the application of logarithms to numerical problems is more easily pursued. If the mathematical community were prepared to make a beginning on untangling the language surrounding logs it would be interesting to assess the effect of this on the understanding and application of logarithms at undergraduate level. While a start could be made by reducing to one the number of names for an *exponent* and using this name (with the usual restriction) to replace *logarithm*, rationalisation of terms on the scale suggested above may be too radical to hope for.

Yet not to attempt *some* change is to be complacent about a terminology that is non-intuitive, archaic and inconsistent, and to accept with resignation that logarithms are for comprehension by the more able students but not by the majority.

**REFERENCES**


-Anton, H., 1999, *Calculus – A New Horizon*, New York: Wiley.

-Bunday, B.D., Mulholland, H., 1983, *Pure Mathematics for Advanced Level, 2nd Ed*, London: Butterworths.

-Chief Examiner in Mathematics, 1996,1998 *Junior Certificate Examination*, Dublin: Department of Education and Science.

- Chief Examiner in Mathematics, 2000, *Leaving Certificate Examination*, Dublin: Department of Education and Science.

-Department of Education and Science, 1969-70, *Leaving Certificate Examination Mathematics Ordinary Level, Specimen Paper II- Set B*, Dublin.

-Finney, R.L., Weir, M.D., Giordano, F.R., 2001,*Thomas' Calculus, 10th Ed,* Boston: Addison Wesley, Longman..

-Jones, H.S., 1913, *A Modern Arithmetic*, London: Macmillan.

-Hall, H.S., Knight, S.R., 1952, *Elementary Algebra for Schools*, London: Macmillan.

-Hall, H.S., Stevens, F.H., 1959, *A School Arithmetic*, London: Macmillan.

-Holland, F.J., Madden, A.D., 1976, *HM Mathematics 4*, Dublin: The Educational Company of Ireland.

-Hornsby, E.J., Lial, M.L., 1999, *A Graphical Approach to College Algebra, 2nd Ed.*, New York: Addison Wesley Longman. *Chapter Resources: Author Tips*, http://www.awlonline.com/precalculus/hornsbylial/algebra/tips.htm#5.

-Sherran. P., Crawshaw, J., 1998, *A Level Questions and Answers - Pure Maths*, London: Letts Educational.

-Strang, G., 1992, *Calculus*, Wellesley MA: Wellesley-Cambridge.

| Student answer: | “Index”, “Index or power”,… | “Power” | “Inverse of a power” | Other (term, base, number, etc) | No answer |
|---|---|---|---|---|---|
| Number of students | 15 | 4 | 1 | 10 | 10 |

**Table 1: Replies to Question 1: What is a log?**

| | Q1. “Index” or “Inverse of a power” | Q1. Other |
|---|---|---|
| Q2: Gave full or partial example | 12 | 18 |
| Q2. No example | 4 | 6 |

**Table 2: Comparison of answers to Question 1, and Question 2 (“Give an example of what logs are useful for”).**

| | Q1. “Index”, “Power” or “Inverse of a power” | Q1. Other |
|---|---|---|
| Q2: Gave full or partial example | 14 | 14 |
| Q2. No example | 6 | 6 |

**Table 3: Adjusted comparison of answers to Question 1 and Question 2.**