CONNECTIONS WITHIN MATHEMATICS –
WHAT QUESTIONS SHOULD BE ASKED AND WHAT ANSWERS SHOULD BE GIVEN?

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ABSTRACT

Recognizing interconnections within mathematics is one of the main emphases of the NCTM's standards (2000): ”Thinking mathematically involves looking for connections, and making connections builds mathematical understanding.” From our experience we have come to realize that many teachers are not aware to the various interconnections exist within mathematics.

In this paper we describe a process in which the main purpose was to develop teachers' (pre-service and in-service) awareness to some interconnections and bring them to appreciate the importance of holding such view. The initial stages of the process were based on confronting the participants with questions regarding three concepts that have different representations in different contexts: 'a straight line', 'a parabola' and 'similarity'. The analysis of the data, obtained through a questionnaire and a discussion, showed that teachers tend to build in mind several isolated concept-images for a certain concept. Each concept image is formed in accordance with a specific context, and the dominant ones are those including algebraic properties of the related object. In advanced stages of the process we introduced answers to the questions in order to illuminate interconnectivity, and thus support the participants in creating a concept image that unites all the relevant aspects. Finally, the teachers were asked to reflect on the process they have gone through. The reflection showed that all the participants had acquired new mathematical and didactical ideas and that awareness to the importance of acquiring a connectionist view was formed.

KEYWORDS: teacher, connection, concept, geometry, algebra, definition, proof, straight line, parabola, similarity.

Introduction

Recognizing interconnections within mathematics is one of the main emphases of the NCTM's standards (2000): "Thinking mathematically involves looking for connections, and making connections builds mathematical understanding. Without connections students must learn and remember too many isolated concepts and skills. With connections, they build new understandings on previous knowledge".

We believe that in order to convey such a perspective it is essential that the teachers themselves possess a 'meta perception' of mathematics.

In this paper we describe a process in which the main purpose was to develop teachers' awareness to interconnections within mathematics and bring them to realize the importance of holding such view. The participants were two groups of mathematics teachers. One group consisted of seven in-service teachers (IST), each having at least five years of experience teaching high-school mathematics. The other one consisted of twenty-six pre-service teachers (PST) in their third year of academic studies towards the B.Sc degree in mathematics education. The process was based on confronting the participants with concepts that have different representations in different contexts. To our opinion, such concepts have the potential to emphasis interconnections within mathematics.

Theoretical background

In order to analyze the ways in which teachers comprehend various concepts we have found it constructive to gather two theoretical frameworks: the theory of concept image and concept definition and the theory of global and local coherence.

According to the theory of concept image and concept definition (Tall & Vinner, 1981), during the process of learning a certain concept, one builds a concept image and a concept definition in his mind. A concept image is the "*total cognitive structure that is associated with a concept*" and a concept definition is the "*form of words used to specify that concept*". One might hold a concept definition that does not correspond to its mathematical definition or is not linked to his or her concept image. Poor concept image means using a few prototypical examples of the concept while considering that concept (Hershkowitz, 1990). A somewhat richer concept image means basing judgment upon more prototypical examples and their mathematical properties. A full concept image includes a wide variety of examples associated with the concept and their properties.

Using the theory, we have found (Shriki & David, 2001) that teachers are able to demonstrate a full concept image while relating to a concept in a specific context (e.g. considering the parabola as a graph of a quadratic function). However, when they are asked to explain the connection between contexts, they are not always able to do so, and thus exhibit a poor concept image.

To explain this phenomenon we use the theory of global and local coherence. The theory relates to the way information is stored and retrieved from our memory. According to the theory, the tendency is to look for a lack of contradiction within a view (local coherence, LC) rather then for a lack of contradiction between possible views (global coherence, GC). One of the main questions derives from the theory concerns the factors that are the most influential in creating a certain view. Chi & Koeske (1983) found that the configuration and the structure of the associative network[1] determine whether or not one is able to utilize efficiently his or her knowledge. Shriki & Bar-On (1997) found that students' errors are not always a result of deficiencies in knowledge but can be sometimes attributed to a lack of GC view. The ability to create a GC view (and thus

[1]For more details see Anderson (1985)

producing correct answers) was influenced by the nature of the structures contained in the associative network: structures that united the required elements and the connections between them enabled generating answers from a GC perspective.

Unifying both theories, it can be argued that teachers might build in their mind distinct structures, each of them containing the required elements and connections for a specific context. They are capable of exhibiting a full concept image regarding each context in an isolated manner, but at the same time they exhibit a poor concept image regarding the concept as a whole. The reason for this apparent contradiction can be explained by the absence of a connection between those distinct structures.

Interconnections within Mathematics: The case of the straight line, the parabola and similarity

Various concepts and topics are taught in a cyclic manner in different contexts. Many teachers tend not to exhibit the connection between the contexts, and thus cause the creation of separate structures. In the following, we describe a four-phase process in which two groups of teachers (seven IST and twenty-six PST) were exposed to three examples of such concepts: the straight line, the parabola and similarity.

Phase I – The participants were asked to complete a questionnaire (Appendix A), and then to reflect on the task.

Phase II – Mathematical background was presented, and was followed by introducing questions. The questions were formulated in such a manner that would enable to initiate a conversation regarding the issue of interconnections.

Phase III – Answers were given and a discussion was held.

Phase IV – The participants were asked to reflect on the process.

SUMMARY OF THE ANSWERS RECEIVED FROM THE QUESTIONNAIRE

14 out of 26 PST suggested a ‘geometrical definition’ for the straight line. The most frequent suggestions were given: “The shortest path between two points” and “A collection of an infinite number of points”. All the IST responded correctly.

The answers obtained regarding the concept of parabola were quite similar to our previous findings (Shriki & David, 2001) [2].

The distribution of answers to the question that dealt with parabola’s similarity and the explanations that were given are summarized in Table 1.

[2] Twenty-one IST and thirty-three PST teachers participated in that study. They were asked to complete a questionnaire, which included questions 3-7 in appendix A. Analyzing the data gained from the description of the curves properties it was found that on average 38% of the PST and 48.82% of the IST teachers demonstrated a full concept image regarding each defined curve, in an isolated manner. When the teachers were asked to explain the logical connections between the definitions and to sketch a Venn-Diagram, only 1 PST and 2 IST were able to do it correctly, and thus the others expressed a non-connectionist view of that concept.

	Statement 3			Statement 2				Statement 1	Other
PST N=26	N=7 (26.9%)			N=11 (42.3%)				N=1 (3.8%)	N=7 (26.9%)
	N=3 They all have the same shape	N=3 They all have the same pattern $y=ax^2+bx+c$	N=1 It is possible to transform any parabola to another by means of translation, rotation, reflection and shrink/ stretch	N=4 Depends on the ratio between the coefficients of the algebraic pattern $y=ax^2+bx+c$	N=2 Depends on the points of intersection with the *x*-axis	N=2 Depends on the type of the extreme point	N=3 No explanation	No explanation	No reply
IST N=7	N=3 (42.9%)			N=2 (28.6%)					N=2 (28.6%)
	N=1 They all have the same pattern $y=ax^2+bx+c$	N=1 It is possible to transform any parabola to another by means of translation rotation, reflection and shrink/ stretch	N=1 No explanation	N=1 depends on the symmetry axis	N=1 No explanation				N=2 Depends on the definition of similar parabolas

Table 1

Summarizing the data we can argue that there is a strong tendency towards conceiving the three mentioned concepts in an **algebraic** manner, with almost no reference to the interconnections within each concept.

WHAT QUESTIONS SHOULD BE ASKED?

In order to emphasize the interconnections exist within each concept we confronted the teachers with questions we believe had the potential to bring them to rethink and rebuild their concepts images.

The concept of straight line

Mathematical background: Students first encounter the concept of a straight line as a fundamental concept in Euclidean geometry. Later on they learn to identify the graph of a linear function as a straight line when they learn basic concepts in analytic geometry. In other words, a fundamental concept in geometry appears as a ‘defined’ concept in algebra. As a result, we arouse the question of our ‘right’ to entitle this mathematical object by the same name (‘a straight line’) in both contexts.

What should be asked? Is it allowed to denominate a function like $y=3x+5$ by the name ‘a linear function’ and to entitle its graph ‘a straight line’? Why?

The concept of parabola

Mathematical background: In ninth grade, students in Israeli high school learn the concept of quadratic function, to sketch its graph and to find ‘special’ points. By that time, the word ‘parabola’ becomes a synonymous with ‘a graph of quadratic function’. As a consequence the statement: “the parabola is a graph of a quadratic function” develops into teachers’ and students’ dominant concept image regarding the parabola (Shriki & David, 2001). The definition of the parabola as a geometrical object[3] or as one of the conic sections is introduced only to eleventh grade students who learn in advanced classes of mathematics.

Based on the geometrical definition, teachers represent an algebraic pattern, but they restrict it only to the case in which the focus lies on the x-axis and the diretrix is parallel to the y-axis (the graph of an implicit function of the form $y^2=2px$, where $p\neq 0$; $p\in R$). The core of the instruction then becomes exclusively analytic, without any investigation of the geometrical characteristics of the parabola.

Translating the geometrical representation of the parabola into an algebraic representation, three distinct sets are constituted, in accordance with the 'direction' of the directrix (parallel to the x-axis, parallel to the y-axis, not parallel to the axis). The general equation is usually not presented in high schools[4].

What should be asked? Is the graph of quadratic function a parabola? Is the parabola a graph of quadratic function? Why? What subgroups of parabola are constituted by its algebraic definition? Is the graph of the function $y=x^n$, where n is an even number, n>2, a parabola? How can the parabola be built by using only geometrical means?

The concept of similarity

Mathematical background: In contrast to the two previous concepts, 'similarity' is not a mathematical object but a feature that connects between objects. Two objects can be either similar to each other or not. In Israeli schools the concept of similarity is introduced to students in ninth or tenth grade, but only in the context of triangles, thus teachers refer solely to the concept 'similar triangles'. As a consequence, there is no discussion regarding the various aspects of the concept 'similarity'.

What should be asked? How can we determine whether or not two polygons are similar? Is it possible to talk about similarity of other curves? Can two parabolas be similar? Two ellipses? Two circles? Two graphs of third degree functions?

WHAT ANSWERS SHOULD BE GIVEN?

This section includes the essence of answers we suggested to the questions above.

The concept of straight line

Why is the graph describing all the points that satisfy an equation of the form: $ax+by+c = 0$; $a^2 + b^2 \neq 0$ a straight line? Since 'straight line' is a fundamental concept in Euclidean geometry, answering the question one cannot use a definition. He or she should look for necessary and sufficient conditions for three points to lie on the same Euclidean line. Verification will focus on the case in which $a,b\neq 0$, since the other cases are much simpler. Figure 1 shows three points $A(x_a,y_a)$; $B(x_b,y_b)$; $C(x_c,y_c)$ which satisfy the equation $ax+by+c=0$. A line through point A, parallel to x-axis, and lines through points B and C, parallel to the y-axis, are drown. Points D and E designate their intersection.

We obtain:

$$\frac{BD}{AD}=\frac{y_b-y_a}{x_b-x_a}=\frac{\Delta Y_{AB}}{\Delta X_{AB}} \qquad ; \qquad \frac{CE}{AE}=\frac{y_c-y_a}{x_c-x_a}=\frac{\Delta Y_{AC}}{\Delta X_{AC}}$$

[3] See definition of curve Λ_1 in appendix A.

[4] An equation of the form $ax^2+2hx +by^2+2gx+2fy+c=0$ is called 'a second degree equation'. This equation describes a parabola iff $h^2-ab=0$.

The ratio $\frac{\Delta Y}{\Delta X}$ is constant (and equals $-\frac{b}{a}$) for every two points which satisfy the equation $ax+by+c=0$, and thus $\frac{BD}{AD}=\frac{CE}{AE}$. Using similarity of triangles or an inverse theorem of Thales, the three points A,B,C are on the same Euclidean line. The opposite direction is obvious, and is performed with the aid of an axis and by using Thales theorem.

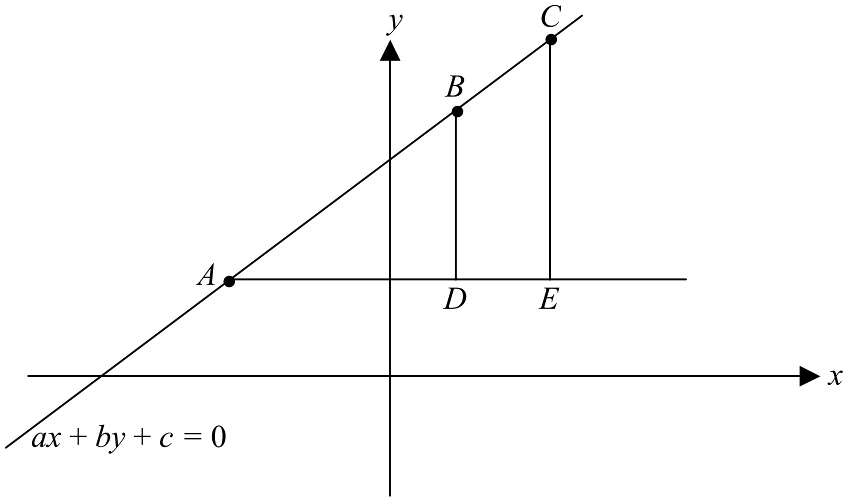

Figure 1

Discussing the need for justifying the 'right' to use the same name ('a straight line') for two concepts within two different contexts, and the proof itself, brought the participants to be aware of this obvious but somewhat neglected interconnections within mathematics.

The concept of parabola

In order to justify that the graph of a quadratic function is a parabola it is necessary to show that the graph of a quadratic function of the form $y=ax^2+bx+c$ ($a\neq 0$; $a,b,c\in R$) fulfills the requirements that are derived from the geometrical definition of the parabola.

The proof is exhibited in figure 2. Based on symmetry considerations of the graph of the function $y=ax^2$, suitable 'candidates' for the focus and the directrix are a point on the y-axis, $F(0,k)$, and a line parallel to the x-axis, l: $y=-k$, in accordance.

In order to find the value of k, we have to solve an equation that satisfies the geometrical condition

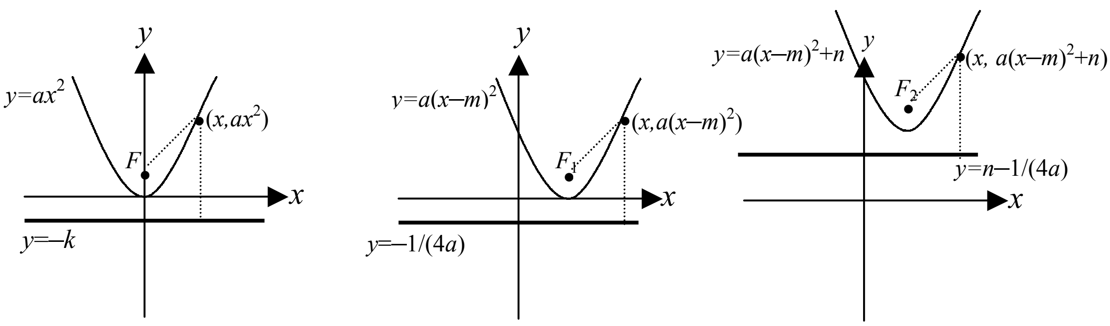

Figure A2 **Figure B2** **Figure C2**

Figure 2

(figure 2a): $(x-0)^2+(ax^2-k)^2=(ax^2+k)^2$. It follows that $k=1/(4a)$ and thus l: $y = -1/(4a)$; $F(0,1/(4a))$. The rest of the proof is left to the readers (figures 2b, 2c).

We have shown that the graph of any quadratic function is a parabola. It is important to note that the inverse is not true.

A similar proof is introduced in figure 3 regarding the graph of an implicit function of the form $y^2=2px$ ($p\neq 0$; $p\in R$).

$(x+k)^2=(x-k)^2+2px \Rightarrow$

$F(p/2,0)$; l: $x = -p/2$

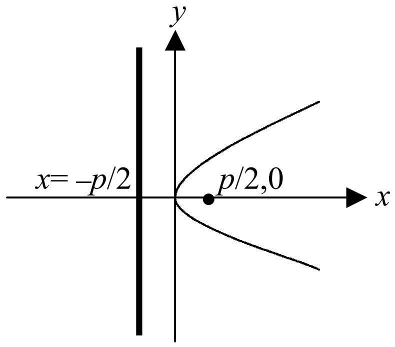

Figure 3

As we know, the graphs of functions of the form $y=x^n$, where n is an even number $n>2$, are all 'look like' parabola. Are they really all parabolas? Using the same process, we have to solve the equation: $(x-0)^2+(x^n-k)^2 = (x^n + k)^2$. We get: $x^2+x^{2n}-2x^nk+k^2=x^{2n}+2x^nk+k^2 \Rightarrow k=1/(4x^{n-2})$. Since the value of k depends on the selection of a point (x, x^n) on the graph of the function $y = x^n$, it can be concluded that those graphs are not parabola.

Following is a Venn-diagram, which describes the logical connection between the four sets of curves described in appendix A[5]:

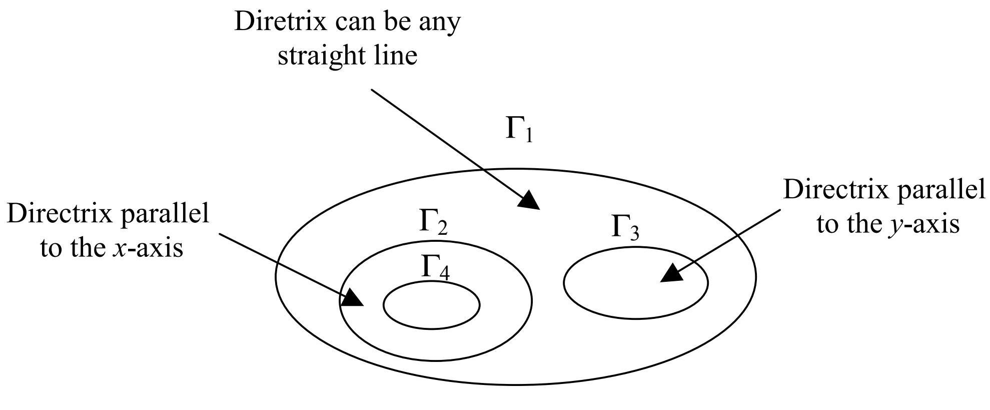

Figure 4

In order to emphasize the geometrical characteristic of the parabola we introduced two methods for receiving a parabola by using only 'geometrical tools' (see appendix B and C).

The concept of similarity

As it was mentioned, most teachers are not familiar with the definition of the concept 'similarity'. Furthermore, we have found out that they were not even aware to that fact, since they did not distinguish between 'triangles' similarity' and 'similarity'. The first step should be to

[5] A function of the form $f(x)=(ax+b)\cdot(cx+d)$; $a,c\neq 0$, $a,b,c,d\in R$, has at least one root, since its roots are determined by the roots of the two linear generator functions. Algebraically it means that if a quadratic function has no roots, it is not possible to write its pattern as a product of two linear patterns.

define ‘similarity’: Two figures are similar iff there is a transformation (translation, rotation, reflection and stretching/shrinking) or a composition of transformations that maps them into each other. If it is possible to map one figure into another by isometric transformations, those figures are congruent. According to that definition, it is obvious that all regular polygons with the same number of sides are similar, and that all the parabolas are similar. Algebraically, all congruent parabolas are obtained by manipulating isometric transformations on the graph of $y=x^2$, as shown in figure 2. For the graphs of the functions $y=ax^2$, $y=bx^2$, where $a\neq b$ a stretching/shrinking transformation should be manipulated (multiply the coefficient of the function $y=ax^2$ by b/a or the coefficient of the function $y=bx^2$ by a/b). The same can be done with any two parabolas.

Discussion

By the end of Phase I, the teachers’ reflection expressed confusion, embarrassment and even frustration for their inability to answer the questions correctly.

A further reflection was conducted at the fourth phase. It is interesting to note that the PST responses emphasized mathematical aspects, while those of the IST focused on didactical aspects. Most of the mathematical ideas were new to the PST, and they indicated that they had learned many new facts. They expressed their fear from teaching those subjects, and admitted to feeling ashamed for not knowing them. Many PST expressed their surprise of the fact that their teachers had never asked them those questions, and were never concerned about introducing the interconnections in mathematics.

The IST said they have never thought about the connection within each concept in the various contexts, and thus they have never felt the need to ‘justify’ using the same name of objects within different contexts. They stated that definitions and concepts should not be taken for granted, without rethinking and investigating them.

The analysis of the data we collected through all four phases of the process described above reveals the tendency to yield answers from a LC view. Most participants demonstrated isolated concept images, and could reason only in the framework of a single context. By the beginning of the process each participant held a certain concept image and concept definition regarding the mentioned concepts in the various contexts. We have found that the most common prototypical examples of those concepts carried algebraic characteristics.

Relating to this finding it is interesting to refer to the finding of a study, which dealt with reading comprehension of mathematical proofs by undergraduate students at the course of advanced calculus (David, 1996). It was found that students, while reading mathematical proofs, focused mostly on their algebraic parts. As a consequence they misinterpreted and misunderstood the mathematical text. We believe a further study is needed in order to explain this ‘algebraic tendency’ phenomenon.

The fact that the participants were not bothered by the absence of links between contexts for a specific concept can be well explained by the LC theory: In each context the appropriate part of the mental image of that concept is retrieved. It is quite obvious that the concept image of each distinct part of the concept has its ‘own independent life’, without any interconnections between parts, despite the fact that those different concept images share many components. The remarkable point is that those common components do not function as motivators for unifying all the distinct concept images of a specific concept under a new comprehensive image. Such an image could successfully confront the complementary as well as the contradictory aspects exist between each isolate image held for each context.

We are confident that all the teachers had learned new mathematical facts, and had begun to establish some interconnections between them. It is hard to tell whether old structures were broken and new ones were formed instead, but we believe the teachers will continue to inquire the concepts introduced and the connections between them, as well as develop a new look at other concepts, and thus create a GC view of mathematics.

REFERENCES

Anderson J. R. (1985): *Cognitive Psychology and its implications*, second edition, W. H. Freeman and company.

Chi M. & Keoske R. D. (1983): Network representation of child's dinosaur knowledge, *Developmental Psychology*, 19(1), 29-39.

David, H. (1996): *Reading comprehension of mathematical proofs (By first year students at the course of calculus),* Doctoral dissertation, Technion – Israel Institute of Technology.

Hershkowitz R. (1989): Visualization in geometry – two sides of the coin, *Focus on Learning Problems in Mathematics*, 11(1), 61-76.

Scher D. (1995): *Exploring conic sections with the geometer's sketchpad*, Key Curriculum Press

Shriki A. & Bar-On E. (1997): Theory of global and local coherence and applications to geometry, In E. Pehkonen (Ed.), *Proceedings of the 21th conference of the International Group for the Psychology of Mathematics Education*, University of Helsinki, Lahti, Finland, Vol. 4, 152-159.

Shriki A. & David H. (2001): How do mathematics teachers (inservice and preservice) perceive the concept of parabola?, In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th conference of the International Group for the Psychology of Mathematics Education*, Freudenthal Institute, Utrecht University, the Netherland, Vol. 4, 169-176.

Tall D. & Vinner. S (1981): Concept image and concept definition in mathematics with particular reference to limits and continuity, *Educational Studies in Mathematics*, 12, 151-169.

www.nctm.org (the official site of the NCTM).

Appendix A

1. How is a straight line defined in Euclidean geometry?
2. How is a straight line defined in algebra?

Following are four definitions of curves. Draw each curve and describe its properties[6]:

3. Given a line l and a point F not on the line. The curve Λ_1 is the locus of the points in the plane so that their distance from the point F equals their distance from the line l.
4. The curve Λ_2 is the graph of a function of the form: $y=ax^2+bx+c$, where $a\neq 0$; $a,b,c\in R$.
5. The curve Λ_3 is the graph of an implicit function of the form; $y^2=2px$, where $p\neq 0$; $p\in$ R.
6. The curve Λ_4 is the graph of a function which its pattern is a product of two non-constant linear patterns.

7. Sets Γ_i , i=1,2,3,4, contain all the curves of the form Λ_i, in accordance.
   Sketch a Venn-Diagram that describes the logical connections between the four sets of curves.

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8. Mark the statement you agree with, and explain your choice:
   There are no two parabolas that are similar to one another.
   There are some parabolas that are similar to one another, and there are some parabolas that are not.
   All the parabolas are similar to one another.

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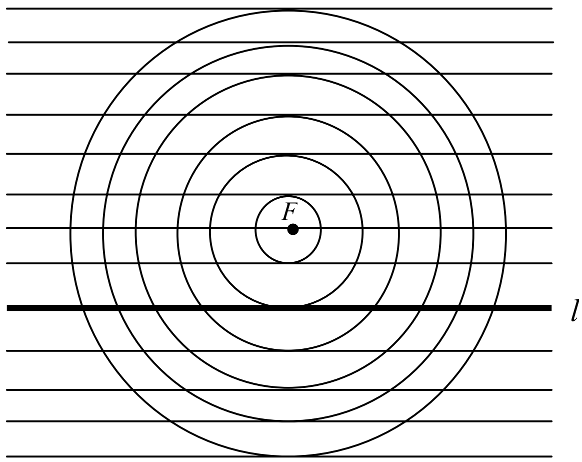
[6] This assignment was taken from Shriki & David, 2001.

Appendix B

Below is a sketch of concentric circles with a center at point *F*.
The distance between each two adjacent circumferences is one unit as well as the distance between two adjacent lines. One of the lines is designated as *l*.
Draw points that their distance from the point *F* equals their distance from the line *l*.



Assume you could repeat this process an infinite number of times, what would you get?

This activity demonstrates a simple process of constructing a parabola based on its geometrical definition.

Appendix C[7]

a. On a rectangular paper mark a point C near the bottom edge, and a point D on its edge (figure 1).
b. Fold the paper so that point D will unite with point C (figure 2).
c. Make a crease, open the paper, and mark the crease (figure 3).
d. Mark additional points on the bottom edge, and repeat the process (point C remains the same).

Questions:
- Assume you could repeat this process an infinite number of times, what curve do you think would bound the area where there are no creases? Explain your answer.
- What is the connection between that curve and the points C and D?

[7] This activity is based on Scher (1995).

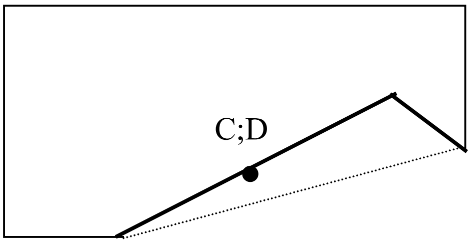


Figure 2 – folding the paper

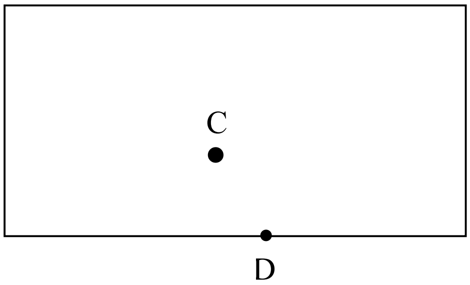


Figure 1 – marking points C and D

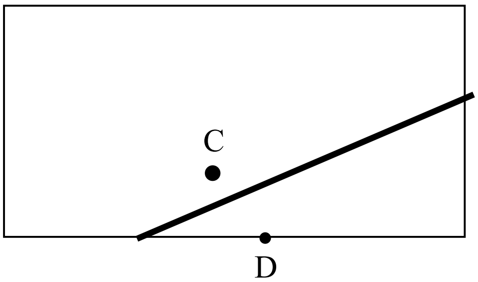


Figure 3 – marking the crease

What can you say about a line tangent to parabola?

The proof that the obtained curve is a parabola is beyond the scope of this paper.