THE IMPROPER INTEGRAL. AN EXPLORATORY STUDY WITH FIRST-YEAR UNIVERSITY STUDENTS[1]

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ABSTRACT

In this paper we analyse the answers of a group of first-year university Mathematics students to a questionnaire, with the aim of determining the difficulties they have when carrying out non-routine tasks related to improper integrals.

Among our research questions, we distinguish the followings: *How do students react when they have to face up to tasks of a non-algorithmic type, questions of reasoning and non-routine questions in the topic area we are involved in? In which system of representation do they feel more comfortable? Are they conscious of the paradoxical results they can achieve? Are they able to articulate different systems of representation in questions related to improper integrals? Do they establish any relationship between the new knowledge with the previous one, particularly the one related to definite integrals, series and sequences?*

The questionnaire consisted of nine questions including not only calculus tasks and determining the convergence of given improper integrals, but also intuitive questions and some paradoxical results too (for example, a figure with an infinite longitude which closes the same area as the unit circumference, or an infinite figure with a finite volume). We particularly asked the students to interpret most of the results they had obtained.

Answers given by the students to each of the questions were categorized, which allowed us to reach some partial conclusions to our research. The answers obtained also allowed us to decide on the selection criteria for choosing students to be interviewed.

From analyses carried out, we can conclude that there are students who have difficulties in articulating the different systems of representation, and have problems in connecting and relating this knowledge as a generalization of previous concepts, such as definite integrals, series and sequences.

KEYWORDS: Improper integral, registers of representation, articulation, transference.

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1. Introduction

The concept of improper integral is first came across by Mathematics Degree students during the second half of their first-year studies while covering the subject *Mathematical Analysis II*. From then on, our students continue to come across the concept, basically, for example, when calculating integral transforms and Fourier series.

The main aim of our research is to design a teaching sequence for Improper Integrals using a *Computer Algebra System* (CAS); we therefore considered it necessary to carry out an exploratory study to identify obstacles and difficulties faced by First-Year Mathematics students when learning the contents related to improper integration. We are also interested in detecting certain errors and difficulties, which arise when making conversions between the algebraic and graphic registers or when handling elements within a single register (Duval, 1993).

With these ends in mind, we drew up a questionnarie in which we included a group of non-routine questions in order to then discover the level of students' understanding, where understanding is taken from the point of view of Duval's theory of systems of semiotic representation (Duval, 1993). The research questions posed refer mainly to the handling of algebraic and graphic representation. They are as follows:

- How do students react when faced with non-algorithmic type questions, which entails reasoning and non-routine questions about improper integrals?
- Which system of representation do they feel more comfortable in?
- Do students make any geometric interpretation of the results they obtain?
- Can they articulate different systems of representation in questions related to improper integrals?
- Do the students relate this new knowledge to their previous knowledge, especially regarding definite integrals? And do they relate it to their knowledge of series and sequences?

2. Theoretical Framework

As stated, we made use of the theoretical framework designed by Duval (1993) to evaluate the levels of students' knowledge when both algebraic (formal) and graphic systems of representation come into play. With the help of this theoretical framework, and once the answers to the questionnaire were analysed, we were able to design a competence model that would allow us to classify the levels of understanding when only these two systems of representation come into play. We then selected six students to be interviewed.

The core of our theoretical framework (and the part used in designing the questionnaire) is based on Duval's ideas of construction of knowledge (1993): we consider it necessary to distinguish between a mathematical object and its representation to achieve Mathematical understanding. And in order to attain this aim, different semiotic representations of a mathematical object need to be used. Duval defines these representations as follows:

> Semiotic representations are productions made up of the use of signs that belong to one system of representation, which has its own constraints of meaning and function.
>
> A geometrical figure, a text in natural language, an algebraic formula, a graph are all semiotic representations that belong to different semiotic systems.

Duval goes deeper into this idea until finally defining a semiotic register of representation:

> A semiotic system can be a register or representation if it allows three cognitive activities related to semiosis:

1) Formation of an identifiable representation as a representation of a given register.

2) Treatment of a representation, which is the transformation of the representation within the same register where it was formed. Treatment is a transformation that is internal to a register.

3) Conversion of a representation, which is the transformation of the representation into another representation in another register where the whole or part of the meaning of the initial representation is preserved. Conversion is a transformation external to the original register.

Duval notes that, as each representation is partial with respect to what it represents, interaction between different representations should be considered absolutely necessary to form the concept.

However, several authors, such as Hitt (2000), feel that not only are the transformation tasks within a register of representation important, but also equally important is the opposition between examples and counter-examples.

As we are also interested in the connections students make to their previous knowledge, the concept of transference also plays an important role (see Hitt, 2000; 2002).

3. Methodology

Subjects

The questionnaire was completed by a group of thirty-one students- thirteen male and eighteen female – at the end of the second half of the 2000-01 course, and was undertaken during a class session. Participating students were taking all or some of the First-Year subjects offered in the Mathematics Degree Course, especially the subject *Mathematical Analysis II*. Few of the students were found to be studying First-Year Mathematics for the first time.

Students had one hour to answer the questions set.

The questionnaire

The questionnaire was made up of nine questions, including not only calculus tasks (Items 3, 4 and 5) and improper integrals convergence (Items 4 and 7), but also intuitive questions (Items 2, 8 and 9) and some paradoxical results (Items 3, 4 and 6). Especially, students were asked to interpret most of the results they achieved (Items 3, 4, 6 and 7). The texts of the questions are given in the Appendix.

Analysis of the answers to the questonnaire

We can now look at and analyse the results obtained:

***Item 1*:**

As in our study we also include the transference that might be undertaken from previous knowledge to new knowledge (improper integrals), we thought it fundamental to analyse what conception students had of definite integrals.

Students' answers were categorised into several groups of answers. Some of these groups are included in others. The most striking results for the answers to this item are:

- Twenty-nine students (93.54%) mention that this can be used to calculate areas, but only four students (12.90% of the total and 13.79% of the set) mention the sign of the function.
- A mere four students (12.90%) explicitly mention that the interval [*a, b*] is finite.
- Five students (16.12%) speak of previous calculation of the indefinite integral in order to speak of the definite, and a further five (not the same) mention Barrow's Rule.
- Seven students (22.58%) appear to conceive of definite integrals as an operation.

- Five students (16.12%) refer to partitions and three (of these) to Riemann Sums.
- Four students (12.90%) refer to some condition of continuity for *f(x)* and only five students (16.12%) note that this is bounded.

We can see that a large percentage of students consider the integral to be an area, but it seems few take into account the circumstances when it is definitely an area and when not, thus confirming results from other research (e.g. Hitt, 1998).

***Item 2*:**

A graph of a function tending to infinite at one of the ends of integration is given. However, its integral is finite.

Our aim is to check whether the presence of a graph (graphic register) will confuse students as to their knowledge of criteria and theorems and make them think that the area is infinite.

Based on analysis of their answers we can note:

- Only eleven students (35.48%) state that nothing can be said a priori, recognising that it might be finite or infinite.
- Eleven students (35.48%) say that this integral represents the area beneath the curve.
- Five students (16.12%) say that it will be positive, and one of them adds that the area is always positive.
- Four students (12.90%) feel more inclined to think that it might be infinite, although they recognise a priori that this cannot be known.
- It is significant that thirteen students (41.93%) do not say anything about the value of the integral; they either describe the function or the integral, but do not say anything else.
- Two students (6.45%) state that they would have to solve it to know the result.
- Three students (9.67%) say that it cannot be solved because it fails to fulfil certain conditions.
- Only one student (3.22%) leaves the question unanswered.

We had believed that the brief text would produce concrete answers, but it seems that this brevity has generated a wide variety of answers. This should be taken into account if this question were to be used in another experiment.

A relatively low number of students (eleven) answered by affirming that nothing could be known a priori, while an equally low number of thirteen students (fourteen, including the student who failed to answer at all – to which should be added the students who affirm that it cannot be solved, among others) wrote nothing at all about the value of the integral.

Furthermore, a large number of students attempted to carry the text over into the algebraic register: six students classify it as an improper integral, thirteen describe the behaviour of the function of the integrand, two attempt to separate the integral into subintervals in order to calculate, while two attempt to solve it.

***Item 3*:**

This question was set to find out whether simple integral calculation leads students to forget the aim of the calculus. Furthermore, we wanted to see if they correctly use calculus techniques for improper integrals[2]. Our objective was simply to create a cognitive conflict by making them face an infinite surface with the same area as the unit circumference. The answers attained were as follows:

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[2] Therefore, it was essential for us to choose a case which we considered simple.

- Only fourteen students (45.16%) correctly solved the integral. Two of those students who managed to solve it (14.28% of the group) did not use symmetry.
- Only eight students (25.80%) clearly expressed that the area was π[3]. And only one student (3.22%) related the area beneath the graph and the circumference.
- Seventeen students (54.83%) failed to write a single explanation about what they did and interpreted nothing. Two students (6.45%) did write something, but did not answer the questions.
- Ten students (32.25%) did not finish the calculus and, in total, twelve students (38.70%) did not manage to calculate the area[4].
- Three students (9.67%) seemed unable to integrate the function.
- And another three students (9.67%) integrated at intervals such as [0, 2] or [0, 4], perhaps misled by the drawing.
- Another three students (9.67%) wrote down the equation for the circumference or drew it, while one student considered its area as an integral.
- Four students (12.90%) did not answer the question at all.

***Item 4*:**

Given the form of the second integral, we believed students would easily work out that this was the volume that produced the surface determined by the first integral and that this volume would be infinite, as a consequence of the result obtained in the first integral.

As they needed to calculate two integrals, the types of answers were highly varied. The most striking results are as follows:

- Twenty-two students (70.96%) calculated the first integral right. However, fewer students (fourteen – 45.16%) calculated the second one right.
- A mere ten students (32.25%) express that one integral is the area of a function and the other the volume, and only three students (9.67%) clearly state that the area could be infinite and the volume finite.
- Five students (16.12%) were unable to solve the second integral, or attempted to solve it using incorrect methods.
- The two students (6.45%) who separated the integrals did so wrongly.
- Eleven students (35.48%) did not interpret anything; we include here those students who did not solve the integrals correctly.
- Two students (6.45%) concluded that there was no relation between both integrals. One of them had realised that one was the area and the other the volume of the same function.
- Only two students (6.45%) left this question unanswered.

It seems that even "simple" calculation of integrals causes the students problems. They also appear reticent when asked for an interpretation of the calculation carried out. This is mirrored in the fact that of the fourteen students who calculate both integrals, only ten express that one represents an area and the other a volume; and only three of these attempt to state clearly that it is a matter of a figure with an infinite area but which produces a solid of finite volume.

[3] In other words, they explicitly interpreted that the integral calculated was the area beneath the graph.

[4] We refer to students who in fact tackled the question. The students who left it unanswered are not included here. In total, sixteen students failed to attain the value π.

***Item 5*:**

In this case the original text of the question[5] was altered, as we were also interested in finding out how many students would think it was right.

This question can be worked on in two registers: in the algebraic, the function of the integrand can be said to have singularities at the interval of integration, so that the solution mechanism is faulty; and in the graphic register, it can be reasoned that the function is strictly positive at the interval of integration, so that the integral cannot be negative.

Our analysis of the answers is as follows:

- Twenty-two students (70.96%) clearly state that the function is problematic in the origin. But not all students realise that, if this is the case, the integral cannot be tackled as they did so. In fact, five of these (22.72% of the twenty-two students) believe that Barrow's rule has been well applied.
- Only sixteen students (51.61%) say that the integral should be split in two to calculate it correctly.
- Only twelve students (38.70%) are thought to have explained clearly how to solve the integral correctly.
- Three students (9.67%) correctly went about the solution of the integral, concluding that the area is infinite.
- A mere four students (12.90%) took note of the symmetry of the function. Of these, only two correctly solved the integral.
- Six students (19.35%) drew the function.
- Six students (19.35%) assigned absolute values to the result so that it was positive, as the integral "*is an area*".
- Three students (9.67%) believe that the trouble lies in the fact that it was necessary to add the integration constant.
- The answers of the eight students (25.80%) who stated that there was no error were considered contradictory, as they later looked for some mistake. In total, nine contradictory answers (29.03%) were found.
- Only one student (3.22%) left the question unanswered.

***Item 6*:**

Analysis of the answers gives the following results:

- Only ten students (32.25%) correctly went about the question and calculated the right result for the integral. Of these, only seven (70% of the group) interpreted the result (correctly or incorrectly).
- Five students (16.12%) failed to give any type of interpretation or explanation (three of them – 60% of the group – solved the integral correctly).
- Before calculating the integral, only one student (3.22%) stated that the figure does not have to enclose an infinite volume, although it increases when the figure is prolonged. And another student (3.22%) concluded that the volume would go on increasing, but did not specify whether it would reach a boundary or not.
- Six students (19.35%) concluded that the volume of the figure would be the same if we went on prolonging it.
- One student (3.22%) said that the figure did not enclose any volume because it did not cut the axis.

[5] Eisenberg & Dreyfus (1991)

- Fourteen students (45.16%) left the question unanswered. Perhaps it would be more reliable and clearer if the percentages given for this item were expressed as a function of the number of students who actually answered the question.

***Item 7*:**

In this question a relationship between series and integrals is clearly shown; once more, students need to combine the graphic and algebraic registers to bring out all the richness of this relationship. Also, while in Item 4 both integrals are interpreted in space, in this case interpretation is made two-dimensionally.

The question was asked in order to check whether students would be able to transfer their knowledge of series to this new situation, and if they can do so, how do they do it.

The types of answers obtained are as follows:

- Only one student (3.22%) correctly interpreted each of the integrals, using a graph for the first and a similar graph for the second. This student was later interviewed.
- Three students (9.67%) calculated both integrals and concluded that the first diverged while the second did not.
- Another three students (9.67%) calculated the integrals but added no comments. In total, six students (19.35%) calculated them.
- Four students (12.90%) appear to have confused the behaviour of the function with that of the integral. For example, Student 8 calculated both integrals and wrote: "*As we can see when solving the integrals, the function* $f(x)=\frac{1}{x}$ *tends to* ∞ *at the interval* $[1,\infty)$ *and the function* $\frac{1}{x^2}=f(x)$ *tends to* 1 *at the interval* $[1,\infty)$".
- Twelve students (38.70%) left the question unanswered. Perhaps the percentages given for this item should only be given taking into accout the actual number of students who in fact worked on the question.

We can see how most students prefer to work in the algebraic register. When they are asked for graphic production, or asked to use a specific graph, few do so. Also, many students do not have a clear idea about the relationship expressed in this item between series and integrals.

***Item 8*:**

This question was asked using only the graphic register. We intended to check whether students were able to enrich their knowwledge in this register by using what they already knew in the algebraic register (after solving the previous items).

Some of the results obtained were:

- Sixteen students (51.61%) say that it is false and put forward a counter-example or reasoned it out. One of the students (6.25% of the group) reasoned wrongly and one student says it is false, but writes nothing else. This student calculated both integrals in Item 4, but found no relationship between them.
- Nine students (29.03%) put as a counter-example Item 4; two students (6.45%) Item 7, and two students (6.45%) put forward a mistaken counter-example, the case of the function in Item 6.
- The nine students (29.03%) who said that text is true had not solved Item 4. There were also three students (9.67%) who calculated the integrals in Item 4, without any interpretation, but could not solve this item.

- Four of the students who said that it was true (19.90% of the total and 44.44% of the group) attributed the properties of the area to the volume; two (22.22% of the group) did not make any reasoning, and three (33.33% of the group) reasoned it out through integral properties. Furthermore, two of them (22.22% of the group) put forward an example to prove the property.
- Five students (16.12%) left the question unanswered.

It appears logical for students, in order to conceive of a figure with a finite volume, to think of this as closed and bounded. This obstacle seems to be strongly related to the lack of articulation between registers and will be the focus of further research work.

***Item 9*:**

This is the only question in which the numerical register was explicitly used. Once more, we attempted to check whether students would use what they knew in the algebraic register, as the area had been calculated under practically the same function in Item 2. We therefore used another scale for the graph so that the similarity was not so obvious.

Many of the answers were "brief", including little justification, and this might be because they had already calculated the area or because the numerical values given in the table might have been familiar.

Some data we can take from their answers are:

- Only two students (6.45%) clearly state that it is the same as Item 3.
- Only three students (9.67%) say that the total area is π. Curiously, only one of these students had solved Item 3, and one student carried out the calculation right, but had not solved Item 3.
- Thirteen students (41.93%) used a dynamic pattern and said that the area "tends", "nears" or "draws close to" π.
- Five students (16.12%) left the question unanswered; three of them (60%) had solved Item 3.

4. Implications for Interviews

Although from the very start, conducting interviews had been seen as useful in order to carry out this study, this idea became a necessity once the students completed the written tests and the answers obtained analysed. We are aware of the interest and richness brought about by a qualitative, rather than merely quantitative, analysis of this type of study.

Of the nine questions that made up the questionnaire, five were chosen for the interview, and a new one was added by means of which we intended to produce more transferences between knowledge of sequences and that of improper integrals, using the graphic register. Students were selected on the basis of the overall results to the questionnaire and/or on the basis of some answer, which we considered significant. A total of six students were interviewed, being almost a fifth (19.35%) of the original total.

In further studies, we will comment in detail on the results of the interviews and our quantitative anlysis will be complemented by a qualitative one.

5. Conclusions

After reading the answers to the thirty-one questionnaires, we can plainly see that students prefer to work in the algebraic register and that the use of non-routine questions, as well as those questions where they are asked to reason and justify their answers, disorientates them.

The main difficulties we detected were due to a lack of meaning or knowledge regarding previous concepts (such as convergence, sequences and definite integrals).

As noted (Item 1) we can see that, even as an area, students have no clear sense of the integral concept. Indeed, they appear to conceive of the definite integral ALWAYS as an area, so that they interpret this as the sum of the integral of the function in the parts where it is positive plus the absolute value of the integral of the function in the parts where it is negative.

Very few students have a clear idea of, or can explain, the conditions of having a finite interval and a function bounded in it. It seems they have learnt that the definite integral "is an area", although this is not made clear in their definitions. Also, many of the students make a merely algorithmic use of this concept.

Intuitive type questions lead to unclear answers (Item 2). Also, students attempt to focus on them formally (also Item 7). They also fail to interpret the results, although expressly asked to do so (Items 3, 4).

In general, we can see that students are not accustomed to combine several registers in order to interpret results (Item 4), or fail to use the graphic register when asked to do so (Item 7). Other difficulties which arise when solving the questions derive from a lack of coordination between both registers.

To answer our initial questions, we can state that students are not accustomed to non-routine type questions and that this type of question can disconcert them. Also, in spite of a lack of formal tools, evident among many students, they prefer to work in the algebraic register (or are limited to this), in spite of their sometimes being asked to use or produce a graph. Therefore, it appears that, generally speaking, they are unable to articulate information between these two registers. The tendency to restrict themselves to the algebraic register impedes graphic interpretation of many results.

Finally, with regard to transferences of previous knowledge, we can see that in general, no such transference takes place (for example, in Item 7, in the transferences between the series object and the improper integral object). Moreover, we have discovered that in many cases this previous knowledge is not altogether complete, thus limiting transference.

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APPENDIX

Item 1:

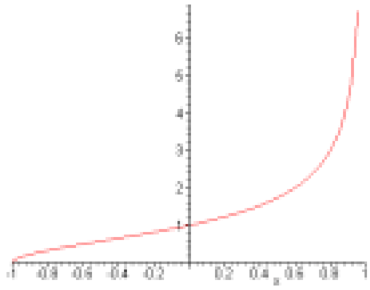
How would you explain to a classmate the meaning of $\int_a^b f(x).dx$?

Item 2:

The following graph represents the function

$y = \sqrt{\dfrac{1+x}{1-x}}$.

What can you say a priori about the value of $\int_0^1 \sqrt{\dfrac{1+x}{1-x}}dx$?

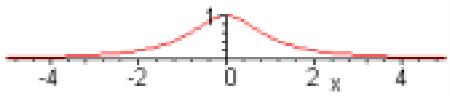


Item 3:

We know the circumference with radius 1 encloses an area of π squared units.

The graph we show you next is the graph of the function $y = \dfrac{1}{1+x^2}$. Check that the area this curve encloses with the OX axis is π too.

Interpret the result.



Item 4:

Calculate the value of the following integrals: $\int_2^{\infty} \dfrac{1}{x-1}dx \qquad \pi.\int_2^{\infty} \dfrac{1}{(x-1)^2}dx$

Interpret geometrically the results obtained. Is there any relation between both integrals?

Item 5:

Can you find any mistake in the following reasoning?

$$\int_{-1}^{1} \frac{dx}{x^2} = \int_{-1}^{1} x^{-2}dx = \left.\frac{x^{-1}}{-1}\right|_{-1}^{1} = \left.\frac{-1}{x}\right|_{-1}^{1} = \frac{-1}{1} - \frac{-1}{-1} = -2$$

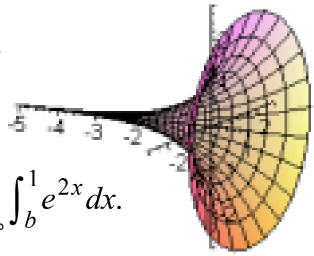
Item 6:

The following figure shows the result of rotating the curve $y = e^x$ around the OX axis.

If we continued prolonging the picture towards the **left** (towards - ∞), which volume do you think it would enclose?

To check your intuition, we will calculate it. Each of the circular sections has a radius of e^x. Therefore, the area of each section is

$$A(x) = \pi.(\text{radio})^2 = \pi.\ (e^x)^2 = \pi.\ e^{2x}.$$



If we sum all the areas we will obtain:

$$V(x) = \int_{-\infty}^{1} A(x)dx = \int_{-\infty}^{1} \pi.e^{2x}dx = \pi.\lim_{b\to -\infty}\int_{b}^{1} e^{2x}dx.$$

What happens? Interpret the result.

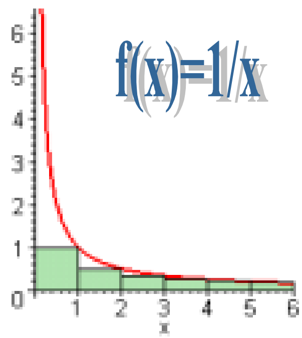
Item 7:

We know that $\sum_{n=1}^{\infty}\frac{1}{n} = \infty$ and $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$.

In view of this results, what can you say about the value of

$\int_1^{\infty}\frac{1}{x}dx$ and $\int_1^{\infty}\frac{1}{x^2}dx$?

Use the graph provided.



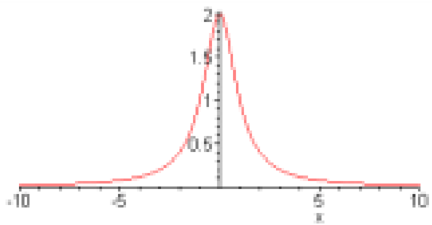
Item 8:

Is the following reasoning true or false? Why?

" If a region has an infinite area, then the solid formed when rotating that region around one of the axis has an infinite volume".

Item 9:

The following graph represents the function $y = \frac{2}{1+x^2}$.



We provide the table with the value of the integrals

$\int_0^n f(x)dx$ for different values of n. What do you think it will happen if we continue increasing n?

N	100	200	300	400	500	600
$\int_0^n f(x)dx$	3.121593320	3.131592736	3.134926012	3.136592664	3.137592658	3.138259324

700	800	900	1000
3.138735512	3.139092654	3.139370432	3.139592654