WHAT IS MODERN IN "MODERN MATHEMATICS"? HOW SHOULD MODERN TEACHING REFLECT THIS?

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ABSTRACT

It is commonly held that what distinguishes modern mathematics is the availability of high-speed electronic computers and pocket calculators with graphical capabilities. Consequently, mathematics is usually taught in schools and undergraduate courses as if Euler and Gauss were our contemporaries, with electronic gadgets replacing tables, slide-rules and sketching.

We discuss cultural changes in mathematics over the past two hundred years overlooked by such an approach, specifically the rise of rigour and algebra. These have altered the face of mathematics, providing a deeper understanding of many important results, by providing a unified, coherent setting.

At the same time, these developments have increased the power and scope of mathematics, by enabling it to deal with non-quantitative problems, by making many computations accessible to computers and by making it more applicable to other disciplines.

We use, in particular, the Fundamental Theorem of Calculus as an illustration and offer a programme for teaching calculus in a manner which accommodates these developments and eases the student's path to further studies.

1 Background

The purpose of this paper is to provide a sketch of cultural changes in mathematics over the past two hundred years, using in particular the Fundamental Theorem of Calculus as illustration. Brevity dictates that details be missing.

Such discussion as this paper intends to foster is urgently needed, for at the dawn of the 21st century undergraduate and high school mathematics are generally still taught as if Gauss and Euler were contemporaries, rather than historical figures whose bicentenaries have been all but forgotten.

Two main strands are discernible in mathematics, often perceived to be in conflict, namely *solving problems* and *constructing theories*. The history of mathematics clearly shows they are symbiotic. For trying to solve previously intractable problems has frequently led to advances in available theory or the development of new theory. Newton's Differential Calculus is a prime example. Equally, purely theoretical advances have frequently — and unanticipatedly — led to solutions of problems in areas to which no significant connection had been suspected. The application of the theory of fibre bundles to arbitrage, of cohomology to number theory and of Hopf algebras to theoretical physics (in the guise of quantum groups) spring immediately to mind. A propos the last example, in [2], Dieudonné was still able to write of the connections with the natural sciences of category theory (p.246) and homological algebra (p.180) "None at present". How dramatically and quickly that changed!

Moreover, we have witnessed the same theory has arise independently and contemporaneously from both practical needs and "purely speculative" considerations. For example, Russell and Whitehead's philosophical programme in *Principia Mathematica* and Dehn's interest in knots both led to the word problem in group theory.

However, this symbiosis does not mean that there is no clearly discernible trend in the history, development and culture of mathematics.

The overriding trend in mathematics over the last two centuries has been increasing rigour and increasing "*algebraisation*": There has been a systematic formalisation and axiomatisation of calculations and arguments. This is not due to a theological addiction to formalism, but the inevitable consequence of several developments: Disturbing paradoxes were found to be immanent in mathematics as practised, and several cherished expectations were dashed by the cold light of irrefutable reason.

The discovery of non-Euclidean geometry by Bolyai, Lobachevski and Riemann, the proof by Galois of the unsolvability of polynomial equations of degree higher than four and the discovery of the paradoxes of set theory shook mathematics to its very foundation, destroying the cherished certitude of established beliefs and confident expectations.

New phenomena have been observed which were unthinkable observed without the guidance of mathematically formulated theory. The discrepancy between the prediction from our theoretical model and the actual recorded observations of our planetary system led to the postulation of previously undiscovered planets, and the subsequent search, guided by the theoretical predictions, resulted in the discovery which confirmed the theory. Who would have thought of trying to measure the bending of light rays around Mercury had Einsteinian theory not predicted it?

Mathematics had provided coherent explanation of events, processes and observations in terms of elegant and insightful theory.

Instead of abandoning mathematics, a rigorous, axiomatic formulation was sought, devoid of the pitfalls while preserving the powerful theories and theorems. By and large, the scientific and philosophical community was convinced that the major results were correct, even though some of the reasoning used to justify them was flawed.

So, after centuries of primary concern with problem-solving, mathematics was forced to concern itself increasingly with constructing theories, not just in an ad hoc manner to justify particular techniques to solve particular problems, but in a considerably more serious and catholic manner.

An enduring legacy of this development is that by becoming more fundamental, mathematics today is more abstract and rigorous, thereby becoming more applicable and more broadly applied. Electronics, meteorology, modern "financial management", not to speak of computing and computers, quantum theory and relativity theory are all inconceivable in their current forms without modern mathematics.

Paul Dirac's observation in 1931 ([3] p. 368) has lost nothing of its aptness.

> *"The steady progress of physics requires for its theoretical formulation a mathematics that gets continually more advanced. This is only natural and to be expected. What however was not expected by the scientific workers of the last century was the particular form that the line of advancement of the mathematics would take, namely, it was expected that the mathematics would get more and more complicated, but would rest on a permanent basis of axioms and definitions, while actually the modern physical developments have required a mathematics that continually shifts its foundations and gets more abstract. Non-euclidean geometry and non-commutative algebra, which were at one time considered to be purely fictions of the minds and pastimes of logical thinkers, have now been found to be very necessary for the description of general facts of the physical world. It seems likely that this process of increasing abstraction will continue in the future and that advances in physics is to be associated with a continual modification of the axioms at the base of mathematics rather than with a logical development of any one mathematical scheme on a fixed foundation."*

2 The Fundamental Theorem of Calculus Revisited

The Fundamental Theorem of Calculus serves well to illustrate Dirac's point.

If $f : [a, b] \longrightarrow \mathbb{R}$ is continuous, then the function

$$F : [a, b] \longrightarrow \mathbb{R}, \quad x \longmapsto \int_a^x f(t)\, dt \tag{1}$$

is continuous on $[a, b]$ and differentiable on $]\,a, b\,[$ with

$$F'(x) \;=\; f(x) \tag{2}$$

for $a < x < b$.

This was originally an astounding theorem, for it demonstrated that two apparently unrelated problems — finding a function whose derivative is a given function and finding the average value of a given function — have a common solution. It also provided a link between the then new differential calculus, and integration, which had been known since antiquity in the guise of the "method of exhaustion".

One immediate consequence is the use of results from differential calculus to provide strategies and techniques for integral calculus, as we discuss below.

One direction in which calculus subsequently developed is multivariate calculus, from which we quote some results — Stokes' and Gauss' theorems — in a convenient form.

Let $\boldsymbol{F} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be differentiable. Let S be a suitable oriented surface in $\mathbb{R}^3$ with boundary ∂S, which we take with the induced orientation. Let V be a suitable oriented solid region of $\mathbb{R}^3$ with boundary ∂V, which we take with the induced orientation. Then

$$\int_{\partial S} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_S \nabla \times \boldsymbol{F} \cdot d\mathbf{s} \tag{3}$$

$$\int_{\partial V} \boldsymbol{F} \cdot d\mathbf{s} = \int_V \nabla \cdot \boldsymbol{F} dv \tag{4}$$

where $\mathbf{r}, \boldsymbol{s}$ and v denote line, surface and volume elements.

Thinking of F' as the gradient of F, and agreeing that integrating a function on a finite set consists of summing its values on that set, we obtain a reformulation of the Fundamental Theorem of Calculus.

Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be differentiable. Let I be an oriented interval whose boundary ∂I is taken with the induced orientation. Then

$$\int_{\partial I} F = \int_I \nabla F \cdot d\mathbf{r}. \tag{5}$$

Thus, Green's, Stokes' and Gauss' theorems are merely "higher dimensional" — more "complicated", as Dirac would say — versions of the Fundamental Theorem of Calculus. Alternatively we may view the Fundamental Theorem of Calculus as a special case or application of the others.

Subsequent developments took a different direction, with far more profound consequences. They consisted of the investigation, primarily due to Poincaré, of the word "suitable" in the above theorems. Poincaré studied *simplexes* and their *boundaries*. A k-simplex in $\mathbb{R}^n$ is the convex hull of $k+1$ points. However the boundary of a simplex is not a simplex, but rather a *chain* of simplexes.

Writing σ for a simplex, $\partial\sigma$ for its boundary, ω for a *differential form* and $d\omega$ for its *differential*, the above theorems all assume the form

$$\int_{\partial\sigma} \omega = \int_\sigma d\omega. \tag{6}$$

[Recall that a differential 0-form is just a smooth real valued function $f(x_1, \ldots, x_n)$ and its differential is the 1-form

$$df := \sum_j \frac{\partial f}{\partial x_j} dx_j \tag{7}$$

If $\omega = f dx_{j_1} \ldots dx_{j_k}$ is a differential k-form, then its differential is the $(k+1)$-form

$$d\omega := \sum_i \frac{\partial f}{\partial x_i} dx_i dx_{j_1} \ldots dx_{j_k} \tag{8}$$

with the convention that $dxdx = 0$.]

The work initiated by Poincaré involved the notions of *homology, homotopy* and *Betti numbers* to describe regions of $\mathbb{R}^n$ and he introduced a group, the *fundamental group*, to characterise when the integrals of a function along two paths between the same two points necessarily coincide.

As a result of the insights and influence of Emmy Noether, these notions were recognised to be best formulated in terms of homology, cohomology and homotopy *groups*.

Cartan was able to provide a sound algebraic foundation for what physicists and engineers had been practising with differential forms, often to the consternation of mathematicians.

Finally, as a consequence of the work of de Rham, we would now say that the differential forms form a *chain complex*, that is, a sequence of abelian groups $\{\Omega^k \mid k \in \mathbb{N}\}$ and homomorphisms $d^k : \Omega^k \to \Omega^{k+1}$ $(k \in \mathbb{N})$ with $d^{k+1} \circ d^k = 0$. In the language of differential forms used by physicists and engineers: "Every exact form is closed". [Recall that the differential k-form ω is called closed if $d^k\omega = 0$ and it is called exact if it can be written as $d^{k-1}\sigma$ for some $(k-1)$-form σ.]

In the case of $\mathbb{R}^3$, $d^1 \circ d^0 = 0$ is the familiar statement from vector calculus "curl-grad $= 0$" or $\nabla \times \nabla f = 0$, and $d^2 \circ d^1 = 0$ is the statement "div-curl $= 0$" or $\nabla \cdot \nabla \times \mathbf{v} = 0$.

Heuristically, the Fundamental Theorem of Calculus and its generalisations in Equation 6 state that the de Rahm complex is dual to the simplicial complex of suitable spaces.

The k-th homology group of a chain complex is the kernel of the k-th homomorphism factored by the image of the preceding one, so that

$$H^k(\mathbb{R}^n) := \ker d^k / \mathrm{im} d^{k-1}. \tag{9}$$

It provides a measure of the extent to which closed k-forms are exact. The Fundamental Theorem of Calculus yields the statement

$$H^1(\mathbb{R}) = 0. \tag{10}$$

It and the more general versions together yield

$$H^k(\mathbb{R}^n) = 0 \tag{11}$$

for all $n, k \in \mathbb{N}$ with $k > 0$.

This purely algebraic statement lands us where mathematics has progressed since Euler. The increased abstraction has necessitated and been made possible by algebraisation and the development of new, abstract and often non-quantitative theories such as the modern theories of abstract algebra, functional analysis, integration theory, harmonic analysis, axiomatic set theory, algebraic topology, differential geometry, algebraic geometry and category theory.

Rather than making modern mathematics more remote from applications and applicability, this abstraction has had the opposite effect: Mathematics is now applied to more disciplines than ever before, with previously intractable problems now solved. Fermat's Last Theorem furnishes the most recent example of a problem elementary enough for a layman to understand, whose solution has required the application of some of the most abstract mathematical theories, which, on the face of it, have no bearing on the problem and in ways which few had envisaged.

Moreover, because of this algebraisation has meant many calculations which formerly required skill and/or ingenuity have been reduced to routine manipulations and algorithmic procedures, suited to solution by computers.

3 How Does This Affect the Way We Should Teach Mathematics?

It would be patently absurd to appeal to de Rham cohomology in a first course on calculus! But it is equally absurd to present such a course as if de Rham cohomology belonged to the realm of science fiction.

The principal difficulty with many current "elementary" and "introductory" courses is that by hiding — or, at best, just ignoring — rather than revealing and emphasising the unity and coherence of modern mathematics, they impede both understanding and subsequent advancement.

Mathematics courses, especially "low level" ones, are all too frequently taught as collections of computational tricks and arcane formulæ to be remembered just long enough to pass the next examination.

Students often complain of needing to forget pictures and "intuitions" acquired in first calculus courses when learning multivariate calculus and having to start all over again.

Students typically perceive calculus and algebra as mutually exclusive, as must be expected from the way they are usually taught.

I believe that calculus is best taught algebraically, given the historical development outlined very briefly above. That history actually illustrates two different ways in which the algebraisation of mathematics in general, but especially calculus, has occurred.

Firstly, it has become clear that calculus is the study of $\mathcal{F}(X)$, the real algebra of real valued functions on the subset X of $\mathbb{R}^n$ and of certain of its sub-algebras.

Secondly — and this is the more recent development — we now assign to each of the subsets X of $\mathbb{R}^n$ a collection of algebraic invariants which reflect many of the significant properties of X.

Clearly this second aspect can hardly be introduced into a first course on calculus, although it might be alluded to in informal discussion of what a student would meet eventually, when pursuing mathematics far enough.

But I have tried basing the calculus courses on the first aspect, viewing calculus as the study of the real algebra, $\mathcal{F}(X)$, of real valued functions on the subset X of $\mathbb{R}^n$ and of certain of its sub-algebras. A first course in calculus typically considers only the case $n = 1$ with $X \subseteq \mathbb{R}$. It is the central notion of a limit which distinguishes calculus from "pure" algebra and calculus may be regarded fruitfully as the study of how limits interact with the algebraic structure of $\mathcal{F}(X)$.

This has the immediate pedagogical advantage of providing context for the central theorems, thereby dispelling much of the mysteriousness and lack of motivation students so often criticise. Moreover it leads naturally to other topics. Differential geometry, for example, may be fruitfully regarded as the study of the sub-algebra, $\mathcal{C}^\infty(X)$, of $\mathcal{F}(X)$ consisting of all smooth real-valued functions defined on the manifold X. While this seems to lack geometry, it is an amazing fact that under mild conditions, the manifold can be recovered from the algebraic structure of $\mathcal{C}^\infty(X)$! Of course computation in differential geometry relies on the algebra.

It also demonstrates that much of the work can be reduced to "mere" symbolic computation and done by machine, the crucial step being the determination of how limits behave and how they interact with the algebraic operations on $\mathcal{F}(X)$.

Of course, I do **not** commence teaching calculus by announcing to students that we shall be spending the course studying the real algebra $\mathcal{F}(X)$, just as I would definitely **not** say to pupils at the outset of learning elementary arithmetic at primary school that they will be learning about the ring of integers!

Rather, I introduce the structure piece by piece, establishing the relevant properties. The course is guided by the algebraic structure, so that **after** the course, a student meeting the notion of an algebra and homomorphism of algebras can readily recognise that the calculus course provided examples of algebras and homomorphisms between them. After all, upon being told what a ring is, any pupil who has a good mastery of elementary arithmetic should appreciate that the integers form a ring.

Specifically, we define a sum and product on the elements of $\mathcal{F}(X)$ "point-wise", that is to say, given functions $f, g : X \longrightarrow \mathbb{R}$, we define

$$f + g : X \longrightarrow \mathbb{R}, \qquad x \longmapsto f(x) + g(x) \tag{12}$$

$$f.g : X \longrightarrow \mathbb{R}, \qquad x \longmapsto f(x).g(x) \tag{13}$$

At this stage it can be fruitfully pointed out that in fact this also includes multiplying a function by a real constant, for we may identify each real number c with the constant function $X \longrightarrow \mathbb{R}$ which assigns c to each and every element of X. In this way we may regard $\mathcal{F}(X)$ as an extension of the set of real numbers, very much the way that the integers form an extension of the counting numbers, the rational numbers of the integers and the real numbers of the rational ones.

If the range of g is a subset of X, we can also define the *composition* of g and f,

$$f \circ g : X \longrightarrow \mathbb{R}, \quad x \longmapsto f(g(x)). \tag{14}$$

These operations provide the algebraic structure on $\mathcal{F}(X)$. Applying them to the functions

(i) $e_c : X \longrightarrow \mathbb{R}, \quad x \longmapsto c$

(ii) $p_1 : X \longrightarrow \mathbb{R}, \quad x \longmapsto x$

is sufficient to define all polynomial functions on X.

If we add

(iii) $s : X \longrightarrow \mathbb{R}, \quad x \longmapsto \sin x$

(iv) $p_{-1} : X \longrightarrow \mathbb{R}, \quad x \longmapsto \frac{1}{x}$ as long as $0 \notin X$,

we have all the rational functions and trigonometric functions.

Finally if we include

(v) $exp : X \longrightarrow \mathbb{R}, \quad x \longmapsto e^x$, together with the inverse functions to all of the above whenever these are defined, then each elementary function — and therefore all those studied in calculus courses — arise by means of recursion. In other words, these few "basic" functions, together with the three algebraic operations *generate* all the functions met in a first calculus course. These functions are frequently referred to as *elementary functions* and we write $\mathcal{E}(X)$ for the set of all elementary real valued functions defined on X. Clearly, $\mathcal{E}(X)$ is a sub-algebra of $\mathcal{F}(X)$.

It is therefore enough to study the behaviour of these five functions as long as we know how the algebraic operations behave with respect to the other operations studied in calculus.

The behaviour of our basic functions with respect to taking limits is easily determined directly "from first principles" in the case of the first three and with appeal to the properties of the functions in the case of the last two.

As to the relationship between limits and our algebraic operations, it is easy to show that

$$\lim_{x\to a}(f+g)(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) \tag{15}$$

$$\lim_{x\to a}(f.g)(x) = [\lim_{x\to a} f(x)].[\lim_{x\to a} g(x)], \tag{16}$$

as long as both $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ exist.

Moreover, if $\lim\limits_{x\to a} g(x) = k$ and $\lim\limits_{y\to k} f(y) = \ell$, then

$$\lim_{x\to a}(f\circ g)(x) = \ell. \tag{17}$$

Of course we must specialise to a sub-algebra of $\mathcal{F}(X)$ to ensure that the limits exist and this again illustrates a common procedure in mathematics: When something does not work in complete generality, restrict attention to the cases where it does and investigate the minimal restrictions required as well as the reason(s) for the failure in complete generality.

Furthermore, this leads to a very natural way of distinguishing continuous functions, for these are precisely the functions for which we can evaluate $\lim\limits_{x\to a} f(x)$ by "plugging in", that is, by evaluating f at a. Such functions form a sub-algebra, $\mathcal{C}^0(X)$, of $\mathcal{F}(X)$.

The corresponding results in the case of differentiation are no more difficult in the case of our basic functions, and the rules for the behaviour of differentiation with respect to the algebraic operations are precisely the linearity, Leibniz rule and chain rule every student meets. Once again we must restrict the set of admissible functions to which we can apply this operation.

We have the following table for this more restricted set of functions.

Limits	**Differentiation**
linearity	linearity
rule for products	Leibniz rule
rule for composites	chain rule

We can therefore evaluate limits, differentiate explicitly any function which is generated by our five basic functions using the three algebraic operations. Thus we have, in effect, an algorithm for evaluating the limits of, or equally for differentiating, the class of functions generated by our basic functions and the algebraic operations:

1. Express the function in terms of our basic functions using only the three algebraic operations.

2. Write down the limit/derivative of each of the basic functions appearing.

3. Execute the algebraic operation on the appropriate limit/derivative corresponding to each algebraic operation on the functions.

It "only" remains to translate this into your favourite programming language.

The case of integration is more interesting. Our basic functions can be integrated directly with ease and, using the Fundamental Theorem of Calculus, we derive the linearity of the integral, integration by parts and integration by substitution corresponding to our three algebraic operations.

We can tabulate the relationships as follows.

Limits	**Differentiation**	**Integration**
linearity	linearity	linearity
rule for products	Leibniz rule	integration by parts
rule for composites	chain rule	integration by substitution

But there are differences as well between differentiation and integration.

Whereas we specialised as we went from just functions to continuous functions and again when we went from continuous functions to differentiable ones, we do not continue this line of specialisation by passing to integrable functions. On the contrary, while every continuous function (and *a fortiori* every differentiable one) is integrable, not every integrable function is continuous.

Moreover difficulties arise if we insist on explicitly expressing integrals purely in terms of our basic functions and the three algebraic operations. For while differentiation maps our special subset, $\mathcal{E}(X)$, of $\mathcal{F}(X)$ to itself, integration does not. The above algorithm cannot be used to calculate integrals instead of taking limits or calculating derivatives.

This inconvenience has several didactic advantages.

a. It provides a natural example of a problem which cannot be solved algorithmically.

b. It naturally raises questions leading to further and deeper study of mathematics and opens the way to more applications. It provides, for example, motivation for Taylor series and Fourier series as techniques for evaluating otherwise inaccessible integrals by using readily computable ones and limits, demonstrating once more the power of the central notion of calculus.

c. It illustrates another situation common in mathematics: We push our available techniques as far as we can and then seek to find other means when we are confronted with situations beyond the scope of our current techniques and methods. Moreover, the obstacles and difficulties we meet often prescribe specifications for the new techniques and methods we need to develop.

What more can we wish for as teachers of mathematics?

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