EXAMS AND COMPUTER ALGEBRA SYSTEMS

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KEYWORDS : computer algebra systems, Maple, Mathematica, problem solving, exams

ABSTRACT

In the French system of CPGE (undergraduate level), CAS (Computer Algebra System) is used as a mathematical aid. The ability of students to use CAS as a tool in a real mathematical activity is relatively easy to test if you are tutoring them in the context of their research projects. It is not the same in exams. On these occasions, the examiner chooses the question and the examinee has just an hour to tackle it.
In our presentation, using examples from French exams in Maple, we will endeavour to show the various pitfalls to avoid and how an examiner can become able to assess the ability of students to use CAS as a mathematical tool.

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In order to show how we are tackling the problem of using Computer Algebra Systems (CAS) at exams, we have to describe the system where we are teaching (CPGE), what are the goals of using CAS in it and what kind of exams we are talking about.

THE CPGE

The French “Grandes Ecoles” system was the result of the French Revolution (1789-1799). Before that time, the University was mainly concerned with Theology. For some reasons that are not relevant here, it appears that the only way to change this predominance was to create a new kind of university. These were called “Grandes Ecoles”. As they were cut from the clerical tradition of the official university, they became more interested in real life. Two centuries later, they remain closer to industry than the university. The main usual criticism of the latter is its remoteness from the real world. The CPGE are the undergraduate level of the “Grandes Ecoles”. Nowadays, this represents 40 thousand students and 1200 professors of mathematics (of a population of 60 millions) working on the same national curriculum. Since the exams are national, it is difficult to change that aspect. Nevertheless, despite this centralised side, they are dispersed in large and small “lycées” throughout country (“lycées” also deal with the secondary education of pupils aged 15 to 18). Their size depends on the population of the city where they are, the larger ones being in the largest cities. Some of them can have just about 40 students, while others such as Janson de Sailly (Paris) have one thousand or more. After two or three years in CPGE, the students usually pass their exams (roughly 85%). Then they can choose to enter one or other “Grande Ecole” depending on their exam results, those with the highest ones can choose whatever they want (usually the “Ecole Polytechnique” or the “Ecole Normale Supérieure”), the other ones with the lowest score results taking whatever places are left available. Since the Revolution, the practical success of our system has turned it into a rather selective one. In fact, most of the best science students avoid the university to try to enter one of the “Grandes Ecoles” so they begin their studies in the CPGE even if they finish elsewhere.

COMPUTER ALGEBRA SYSTEMS AND THE CPGE

In CPGE, the official idea is that CAS should be used as a problem-solving tool. This means that the students have to deal with abstract ideas and CAS will perform the calculations after that. It is quite easy to check the ability of students to use CAS in this way during the year. You have just to give them usual problems (at home in one week) and exercises (at the blackboard in one hour), wait and see what they have done. The final interview is sufficient to understand how and why they use (or do not use) CAS.

The two CAS in use in CPGE are Maple and Mathematica but we mainly use Maple V (Release 5). Programming is taught but without recursion (see [1] and [2] for an example of this teaching). To be more precise, we use a kind of micro-Maple limited to a list of operators and functions (see table 1) even if the students have the right to use every function they know. So, in this paper, we will focus on it. Nevertheless, our remarks remain relevant for other versions of this software as well as for Mathematica and others.

COMPUTER ALGEBRA SYSTEMS AT EXAMS

At exams, the written part takes place at the same time for thousands of students nationally. So, on the one hand, it is difficult to provide them access to computers, on the other hand, we cannot allow

them to use their own ones (whatever their size are) because it may be a source of fraud. Thus, we have decided to test their abilities to use CAS at the oral part of the exams.

So, during this part, a computer with Maple and Mathematica is available in the examination room. The examiner provides technical assistance. For example, he can answer to questions as: “what is the instruction for computing an integral?” or “how can I re-initialise Maple?" He is not supposed to judge the examinee through the questions he asks. The fact that a computer is in the room does not mean students have to use it. Using CAS or not using it is their choice. So, the first difficulty is to recognise that CAS may be helpful in a particular question. Anyway, the students will be judged on their ability to do mathematics, not on their knowledge of software. Thus, we have to ask them to tackle usual exercises where CAS can be used but is neither necessary nor sufficient.

We have chosen to discuss our criteria on examples (see [3] for others) because it is difficult to consider this question in the abstract. Our philosophy concerning the use of CAS at exams is to use classical exercises but to keep only those where CAS may help without doing everything. Nevertheless, without applying it on examples, this simple idea does not take its full sense.

COMPUTER ALGEBRA SYSTEMS AS VIDEO GAMES

In a lot of cases where real programming is not needed, students can produce results without understanding a thing on the matter. For example in the two following exercises, a basic technical expertise is enough to find the right answer:

Exercise 1: Compute $\lim_{x\to 0}\dfrac{2\tan x-\tan 2x}{x(1-\cos 3x)}$.

Exercise 2: Compute $\int_0^{+\infty}\dfrac{\mathrm{d}x}{1+x^4}$.

Testing student's ability to find the right results ($-\dfrac{4}{9}$ and $\dfrac{\pi\sqrt{2}}{4}$) in such cases is not far from testing their ability to play a video game. In fact, the only thing to know is how to encode the mathematical expressions written above in the Maple language. So, you cannot tell much about the student's mathematical knowledge through their way of doing such exercises.

COMPUTER ALGEBRA SYSTEMS AS PROGRAMMING LANGUAGES

A solution to avoid the use of CAS as a kind of video game is to ask students to program them on mathematical examples. So, the following exercises concern programming. They have been chosen to show the boundary between fair and unfair exercises of this kind at the mathematical part of exams.

Exercise 3: Two distinct natural numbers are called amicable if the sum of the proper divisors of one number equals the other. Write an algorithm finding all couple of amicable numbers smaller than 1500.

Exercise 4: Write a function *f* returning the sum of the cubes of the digits of an integer *n* in decimal expansion. Find the *n* such that $f(n) = n$.

Exercise 5: Write a function returning the index of the maximum value of a sequence of real numbers $(u_1, u_2, \ldots, u_n)$.

Exercise 6: Find a method to compute x^{10} with just four multiplications. How many multiplications are necessary to compute x^{55}? Generalise and write the corresponding function.

Exercise 3 is very simple if you know that there is a function computing the sum of the proper divisors of a natural number in the number theory package of Maple (it is called sigma). It is rather difficult if you are limited to micro-Maple. As we cannot forbid the use of the functions available in full Maple, such an exercise must be avoided. A consequence is that, for every exercise, we have to check if it cannot be solved by the use of just one magical function out of our micro-Maple (result: 220, 284 and 1184, 1210).

A good solution of the first part of exercise 4 involves recursion (using the mathematical property: $f(0) = 0$ and if $n = 10q + r$ then $f(n) = f(q) + r^3$). To solve the second part, you have to realise that a solution has at most four digits and then to try all of them which is easy (result: 0, 1, 153, 370, 371 and 407). Thus, this exercise is not a bad one to test mathematical ability at undergraduate level but it involves a real knowledge of programming. So, as it is not the main goal of our teaching, this kind of exercises must be avoided or kept for a second question to apparently good students in this domain, just to check how far they can go.

Exercise 5 and a lot of the same kind (as sorting and searching for example) must be avoided as they are just programming exercises. They must be reserved for the computer science exam. Exercise 6 is at the frontier of this kind. May be it is better to avoid it too.

Anyway, in this paragraph, we see that to avoid the testing of the ability to play video games, we test the students on their knowledge of computer science. The goal is missed. If this kind of exercises can be used for those who are obviously good, they must be kept to them.

COMPUTER ALGEBRA SYSTEMS AS TRAPS

Another way of testing whether students really know what they are doing when using CAS may be to give them examples where CAS results are wrong. Such cases are not difficult to find, especially in computing integrals:

Exercise 7: Compute $\int_0^{+\infty} \frac{\mathrm{d}\,x}{x^2 - a^2}$ if it exists.

Exercise 8: Compute $\int_{-1}^{1} \frac{\sin a \;\; \mathrm{d}\,x}{1 - 2x\cos a + x^2}$.

Exercise 9: Compute $\int_{-\infty}^{+\infty} \frac{e^{ix}}{1+ix}\mathrm{d}\,x$.

In exercise 7, Maple gives a clearly impossible answer ($\frac{i\pi}{2a}$). It is really easy to tell that this result is wrong because it is not real. So, the reaction of a student on this kind of output may be interesting. In fact, a lot of them have a bad tendency to believe everything they see on a computer screen.

In exercise 8, Maple gives the correct but amazing result:

$$\frac{\sin a\left(-\arctan\frac{-1+\cos a}{\sqrt{1-\cos^2 a}}+\arctan\frac{1+\cos a}{\sqrt{1-\cos^2 a}}\right)}{\sqrt{1-\cos^2 a}}$$

To simplify it properly in order to find the right answer ($\frac{\pi}{2}$ if sin $a > 0$, $-\frac{\pi}{2}$ if sin $a < 0$ and 0 if sin $a = 0$), you must have some knowledge on the inverse tangent function.

In exercise 9, Maple gives 0 as the answer (and Mathematica $\frac{\pi}{e}$). Both are wrong but there is no obvious reason for students to doubt any of these results (the right one is $2\frac{\pi}{e}$).

It is easy to imagine a number of traps of this kind. If there are really strong reasons to doubt the result of CAS, they can provide an opportunity to test the understanding of students but in other cases, they must be forbidden. Thus, at exams, we prefer to avoid such exercises even if some of them are very interesting. An implication of that is that the examiners have to really use CAS to solve their exercises before the examination just to check they are not mining the road to the solution without realising that.

COMPUTER ALGEBRA SYSTEMS HELPING INTUITION

The best way of finding good exercises is to take a look at our use of CAS in real mathematical life. One of its uses is to help our intuition. Here are some examples of exercises where they can be used in this way.

Exercise 10: Find the complex numbers z such that $\frac{z^2}{2z+3i}$ is imaginary. Represent them in the complex plane.

Exercise 11: Let Γ be the curve given in parametric form by: $M(t)\begin{cases} x = 3t^2 \\ y = 2t^3 \end{cases}$. Find the locus C of the point from where the tangents to Γ meet at right angle. What kind of curve is C? Determine the position of C relatively to Γ.

Exercise 12: Let $A = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix}$. Compute A^n.

Exercise 13: Let $f_n(x) = \frac{\sin nx}{n\sqrt{x}}$, $x \in (0, +\infty)$. Does the sequence (f_n) converge uniformly?

Exercise 14: Consider the sequence (u_n) defined by: $u_0 = a$ and $u_{n+1} = \frac{3}{2u_n^2 + 1}$ for all $n \geq 0$, $a \in \square$. What can you say about $\lim_{n \to +\infty} u_n$?

If students understand the exercise 10 properly, they can plot the solution directly (see figure 1). Of course, more can be said on the curve on figure 1. The examiner will judge students on their ability to recognise and to prove that the curve is composed of a straight line and a circle.

In exercise 11, CAS is useless to find a parametric representation of C ($\begin{cases} x = t^2 - 1 + \dfrac{1}{t^2} \\ y = \dfrac{1}{t} - t \end{cases}$) but very useful to plot it (see figure 2). With a minimum knowledge, students can suspect that this curve is a parabola. In the same way, they can see that the two curves are tangent. Then they have to prove these two properties.

In exercise 12, through some computations done with the help of CAS, it is quite easy to see that $A^n = \begin{pmatrix} a_n+1 & a_n & a_n & a_n & a_n \\ a_n & a_n+1 & a_n & a_n & a_n \\ a_n & a_n & a_n+1 & a_n & a_n \\ a_n & a_n & a_n & a_n+1 & a_n \\ a_n & a_n & a_n & a_n & a_n+1 \end{pmatrix}$ for some a_n. Then, through multiplying this matrix by A, CAS help to find the law $a_{n+1} = 6\,a_n + 1$, the result follow ($\dfrac{6^n - 1}{5}$) but it needs a minimum of mathematical knowledge.

Without visual aids, most of the students do not see that the sequence of exercise 13 converges uniformly. With CAS, they generally see that but it is more difficult to prove it. For that purpose, CAS is useless. This exercise does not miss the goal but is rather difficult and must be left for further testing of an apparently good student.

In exercise 14, a lot of students see incorrectly that $\lim_{n\to+\infty} u_n = 1$. The reason lies in their interpretation of the drawing of the graph of the function f defined by $f(x) = \dfrac{3}{2x^2+1}$ (see figure 3). Starting from any number a, they felt that the sequence approached 1 in absolute value and oscillated from one side of 0 to the other. At this step, it is not too bad but what can be very upsetting is when they are able to prove it! With a better understanding, they went to examine the graph of f o f (see figure 4). Generally, not only they see the correct result (two limit points $\dfrac{3 \pm \sqrt{7}}{2}$ if $a \neq \pm 1$, the limit 1 occurs only if $a = \pm 1$) but they are able to isolate the properties to be proved in order to prove it (CAS can be helpful for this purpose). In this example, we note that "seeing" requires a lot of knowledge in mathematics. As the previous exercise, this one must be left for a further testing.

In conclusion, all these exercises (10 to 14) are rather good to test mathematics ability even if they are not all of the same level. In all of them, CAS is used to "see" the correct result and to see requires a lot of knowledge in mathematics.

COMPUTER ALGEBRA SYSTEMS TAKING CHARGE OF CALCULATIONS

In a lot of cases, CAS can be used to take charge of calculations. In this kind of use, it is important to choose exercises where the students have either to analyse the results or to find the right calculation to do.

Exercise 15: Compare $2+2\sqrt{2}$ and $\sqrt{5+2\sqrt{6}}+\sqrt{9-2\sqrt{6}-4\sqrt{5-2\sqrt{6}}}$.

Exercise 16: Find the zeros of the polynomial $P = x^4 - 2x^3 + x^2 - 2x + 1$. Represent P as a product of irreducible polynomials over the real domain.

Exercise 17: Let $M=\begin{pmatrix} a & 0 & 0 & 0 & b \\ 0 & a & 0 & b & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & b & 0 & a & 0 \\ b & 0 & 0 & 0 & a \end{pmatrix}$ where a and b are complex numbers. On which condition is M diagonalisable?

Exercise 18: Let a, b and c be three real numbers and $M=\begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$. Find the matrices M such that $M^2 = I$.

Exercise 19: Solve the following differential equation: $xy'+(1-x)y=\frac{xe^x}{x^2+1}$. Plot some integral curves. Is there any continuous solution on $\mathbb{R}$?

Exercise 20: Let C is the curve given by $M(t)=\begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix}$. Show that the osculator planes to the curve at three different points and their plane have a common point.

In exercise 15, an approximation of the difference gives a probable answer (the numbers are equal). To prove it with CAS, you have to use a special function not in our micro-Maple (radnormal) thus this kind of exercise is a trap because the students have no simple reason to realise that $\sqrt{5\pm2\sqrt{6}}=\sqrt{3}\pm\sqrt{2}$ even if this equality is really easy to prove. This kind of example shows that we have really to check if an exercise can be done with the use of our micro-Maple only. As a matter of fact, knowing the property, it is very easy to believe that CAS will be a good help but it is not always the case.

In exercise 16, CAS find the zeros of P ($\frac{1\pm\sqrt{2}}{2}\mp\frac{1}{2}\sqrt{-1\pm2\sqrt{2}}$) easily (with allvalues, solve is not enough). The only mathematical problem is to assemble them in polynomials over the real field (result $x^2+\left(\sqrt{2}-1\right)x+1,\ x-\frac{1+\sqrt{2}}{2}\pm\frac{\sqrt{2\sqrt{2}-1}}{2}$).

In exercise 17, students can think that CAS gives the answer directly but, in fact, they have to check that the eigenvectors given are linearly independent. Computing the determinant of the proposed vectors, we find that it is not the case if $a + b = 2$.

In exercise 18, CAS gives M^2 and help to transform the matrix equation in a system of three equations ($\begin{cases} a^2+2bc=1 \\ b^2+2ca=0 \\ c^2+2ab=0 \end{cases}$) which can be solved with CAS (result: $\pm I$, $\pm\frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$).

In exercise 19, it is easy to plot a lot of integral curves (see figure 5) but the continuous solution is not easy to spot among them without a theoretical study. For that, the students have to write the general solution ($y(x) = \dfrac{e^x\left(\frac{1}{2}\ln(x^2+1)+A\right)}{x}$) and to realise that the only point of discontinuity is 0 and to compute its limit at 0 (see figure 6).

In exercise 20, with CAS, it is possible to write the equation of the osculator plane at $M(t)$ and of the plane passing through points with parameters a, b and c ($f(t) = 3t^2 - 3ty + z - t^3$, $g(x,y,z) = (ab + bc + ca)\,x - (a+b+c)\,y + z - abc$). Then, they can solve the system of equations $\begin{cases} f(a)=0 \\ f(b)=0 \\ f(c)=0 \end{cases}$ and substitute the solution into g.

In conclusion, most of these exercises (16 to 20) are rather good for testing the ability of doing mathematics with CAS. Exercise 15 is not a good choice because it is not of the same level whether you know a magical function or not. This kind of cases must be avoided, as they are unfair

CONCLUSION

Through these examples and counter examples, we can see that the exams in CPGE test the official way of using CAS taught in CPGE (see [4]). The exact opposite of this way of thinking can be found in [5]. Our main problem is that we cannot use it in the written part for the time being. CAS should be available all the time to allow students to become completely accustomed to using CAS but there are two problems firstly the high price of small portable computers and secondly the possibility of fraud. Thus the testing is done only at the oral part of the exams.

Our general idea to choose exercises is to use classical ones but to keep only those where our micro-Maple (see table 1) may help without doing everything. A good exercise must not give a decisive advantage to those who have a knowledge of functions out of our micro-Maple or of recursion because it will be unfair to the others. Thus, exercises involving too much programming must be reserved as further tests for those who are obviously good. For the same kind of reasons, traps as cases where Maple gives a wrong answer must be avoided or reserved for further testing. An implication of that is that the examiners have to really use CAS to solve their exercises before the examination. To finish with the question, we will stress two points. Firstly, experiments where the students have to see something are rather good to test mathematical ability because "seeing" requires a lot of mathematical knowledge. Secondly, exercises involving parameters are often good because CAS do not discuss particular cases. So, in this kind of exercises, the students have to understand and to interpret the results of CAS.

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TABLE AND FIGURES

table 1

^	$	allvalues	display	for	local	Pi	series	while
*	&*	assume	do	I	map	plot	simplify	with
–	( )	coeffs	dsolve	If	map2	plots	sin	
+	–>	conjugate	eigenvals	Im	matrix	pointplot	solve	
=	{ }	convert	eigenvects	infinity	max	polynom	sqrt	
<	[ ]	cos	eval	int	multiply	print	subs	
>	< >	D	evalc	inverse	nops	proc	sum	
,	<=	denom	evalf	iquo	numer	product	taylor	
;	>=	det	evalm	irem	numeric	Re	trace	
"	' '	diag	exp	limit	op	restart	transpose	
/	:=	diff	expand	linalg	Order	RETURN	unapply	
:	Abs	Digits	factor	linsolve	parfrac	seq	vector	

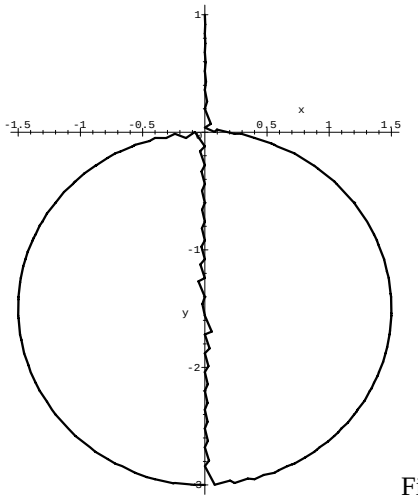

Figure 1

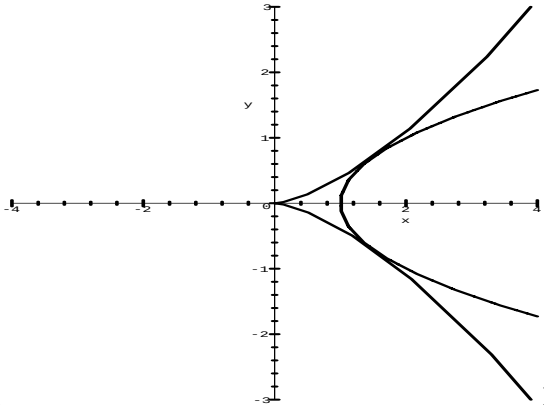

Figure 2

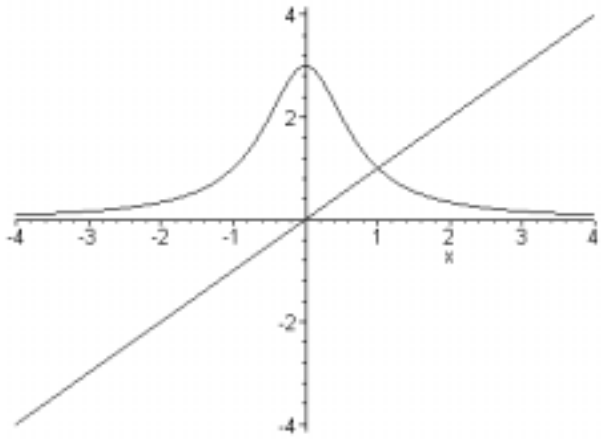

Figure 3

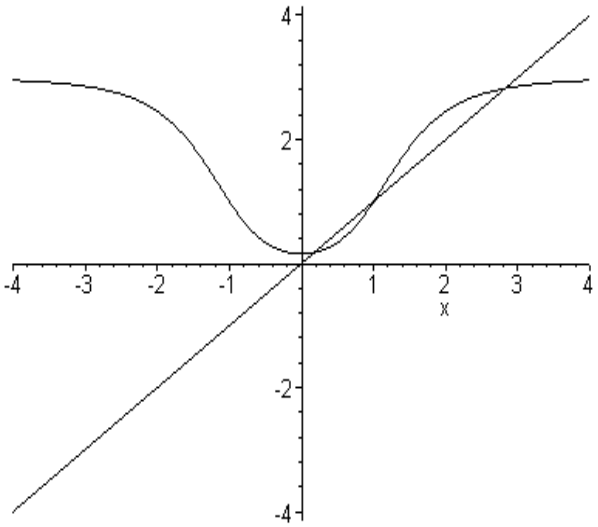

Figure 4

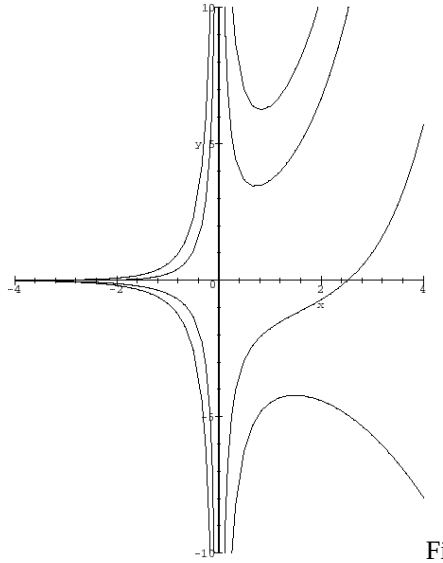

Figure 5

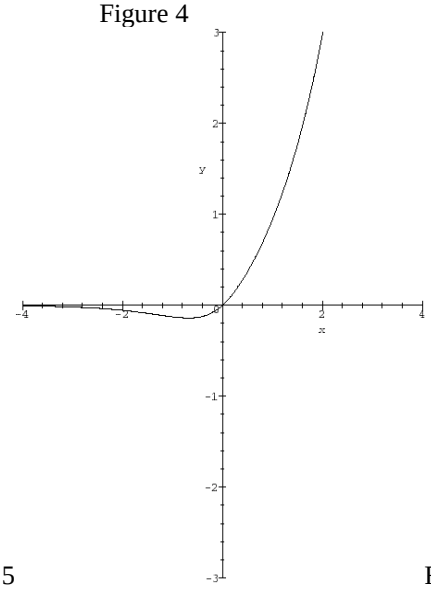

Figure 6