USING TECHNOLOGY TO IMPROVE THE CURVE LEARNING OF BASIC NOTIONS IN ALGEBRA, CALCULUS AND GEOMETRY

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ABSTRACT

One of the main problems facing mathematics teachers in scientific and technical disciplines (Physics, Chemistry, Engineering, etc.) at universities or engineering schools when receiving first year students is the need of providing them with the capabilities required to understand advanced notions from the early beginning in order to be able of following the initial explanations of teachers talking about Physics, Mechanics, Chemistry: usually the first explanation starts by writing down a differential equation in the blackboard when students hardly understands correctly what a real number is !

The objective of this paper is to report how, firstly, a proper combination of technology (distance web learning through **WebCT** plus the Computer Algebra System **Maple**) and, secondly, a different way of presenting difficult notions concentrated more on the ideas than in the formalisms have been extremely useful in order to:

- Give the students the capability of understanding the initial explanations of teachers talking about physics, engineering, etc.;
- Reduce the gap between the mathematics explained at the secondary school and the mathematics expected to be known by a student when entering at the university (a critical problem in Spain from several years ago); and
- Provide to the students, in a very fast way, with a more solid set of math foundations to be used as an initial stratum.

This experience has been organized around a course of 60 hours (27 hours the first month, 21 the second one and 12 the last one) delivered at the very early beginning of the first year for Physics students at our university. It consists in ten modules of six hours each with three hours of explanations devoted to motivate and illustrate concepts and techniques plus three hours of practical problems with one of them including the using of Maple.

The tool used to control the individual progress of each student was **WebCT** through the realization of several questionnaires containing multiple-choice questions trying to identify initial misunderstandings or to detect unexpected difficulties.

KEYWORDS: Experimental Mathematics, Computer Algebra Systems, Basic Mathematical Training

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1. Introduction

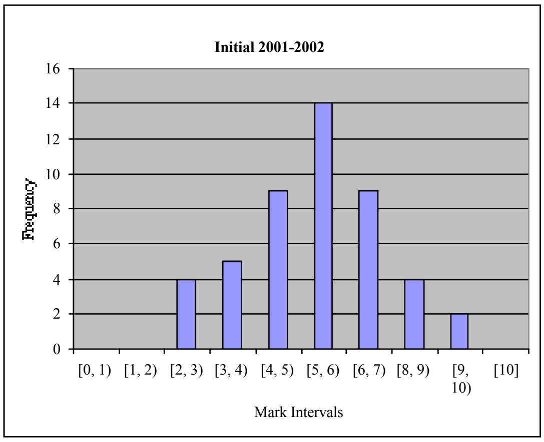
This paper is devoted to report an experience trying to solve (or alleviate) the problem of improving the mathematical stratum (in contents, in abilities and in comprehension) of first year university students in technical or scientific disciplines such as Physics, Chemistry or Engineering.

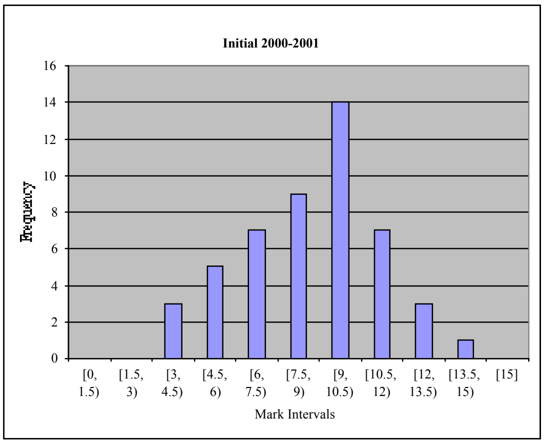
It is a common point, at least in Spain, agreed by almost all those professors teaching Mathematics to first year university students that they bring less mathematical notions (most of them assumed known by professors at university level), their ability to manipulate correctly (and, what is most important, in a coherent way) mathematical expressions is very poor and that the level of their understanding of basic notions is below the one required to be used without a previous remembering.

For the academic course 1999-2000, since the curricula of the Physics studies was updated and modified at that moment, it was decided to create a 60 hours course entitled *Laboratory of Mathematics* organized around ten modules of six hours each covering each module a concrete (and relevant) topic (see below for the concrete list of modules). Each six hours module has the following structure:

- The first three hours are devoted to present and motivate the relevant concepts mainly with examples and avoiding, if possible, complicated notations or mathematical language abuse such as $\forall\varepsilon>0\ \exists\delta>0$˜.
- In the next two hours two professors per group assist several groups of at most 30 students where they do, alone or in-group, a set of selected exercises (easy manipulation tasks) or problems (more complicated questions involving usually the joint use of several notions).
- Last hour, with the help of the Computer Algebra System **Maple**, is devoted to re-do some of the exercises or problems considered into the previous two hours or to illuminate and clarify through examples with a computational flavor some of the concepts regarded in the considered module.

This course has been already delivered twice (for 1999-2000 and 2000-2001) and it is compulsory for first year students of Physics studies. Upon arrival, and in order to adequate the course content to the new students, they answer a questionnaire in **WebCT** with between ten and fifteen multiple-choice (very elementary) questions aimed to detect unexpected misunderstandings or new non-known concepts. Next tables present the results obtained by showing a big proportion of students do not manage concepts such as line/point/plane relative position in 3D space or relative to the distribution of rational/real numbers in the real line.





Students are evaluated through the answering of two one-hour questionnaires of multiple-choice questions in **WebCT** plus the realization of a three hours written exam containing a set of selected problems involving each one the manipulation of several concepts and some capability of manual (and correct) manipulation of mathematical expressions.

According to the initially detected problems and to the requirements of other non-math professors involved into the first year of the Physics studies the ten modules were defined in the following terms:

1) Numbers and equations:
   - Representation and manipulation of numerical and algebraic entities: integer, rational, real and complex numbers; polynomials; algebraic fractions; equations and inequalities involving the absolute value.
2) Matrices and linear systems of equations:
   - Matrix and vector operations; rank; determinants; linear systems of equations (Cramer's rule, Rouche criteria, Gauss algorithm).
3) Sequences and limits:
   - Arithmetic and geometric progressions; convergence; limit calculus; series; sum ability (hypergeometric, arithmetic-geometric, the number e).
4) Functions and continuity:
   - Function characteristics (domain, graph, symmetries, periodicity, extremes, inverses, asymptotes; elementary functions (trigonometric, logarithms, exponentials, etc.); limits of functions; continuity.
5) Derivatives:
   - Geometrical and physical definition; derivatives calculus; max and min computation; Taylor series; computation of the graph of a function.
6) Integrals:
   - Geometrical and physical definition; primitive calculus; area, volume and length computations; numerical integration.
7) Differential equations:
   - Solution of a differential equation; exponential of a matrix; homogeneous ordinary differential equations with constant matrix.
8) Analytic geometry:
   - Points and vectors; coordinate frames; transformations; lines and planes; incidence and parallelism.
9) Euclidean geometry:
   - Scalar and vector product; distances and angles; polygons, solids, areas and volumes; conics and quadrics.
10) Data manipulation and visualization:
   - Interpolation; least squares; curves and surfaces (parametric, implicit, visualization).

Topics in blue represent those concepts completely new to the students.

All the generated material can be consulted by visiting the web page (in Spanish):

http://gesacapc22.gestion.unican.es:8000/public/lab201/index.html

where:

- the lecture notes together with the selected exercises,
- the Maple worksheets corresponding to the selected exercises, and
- several questionnaires

are available.

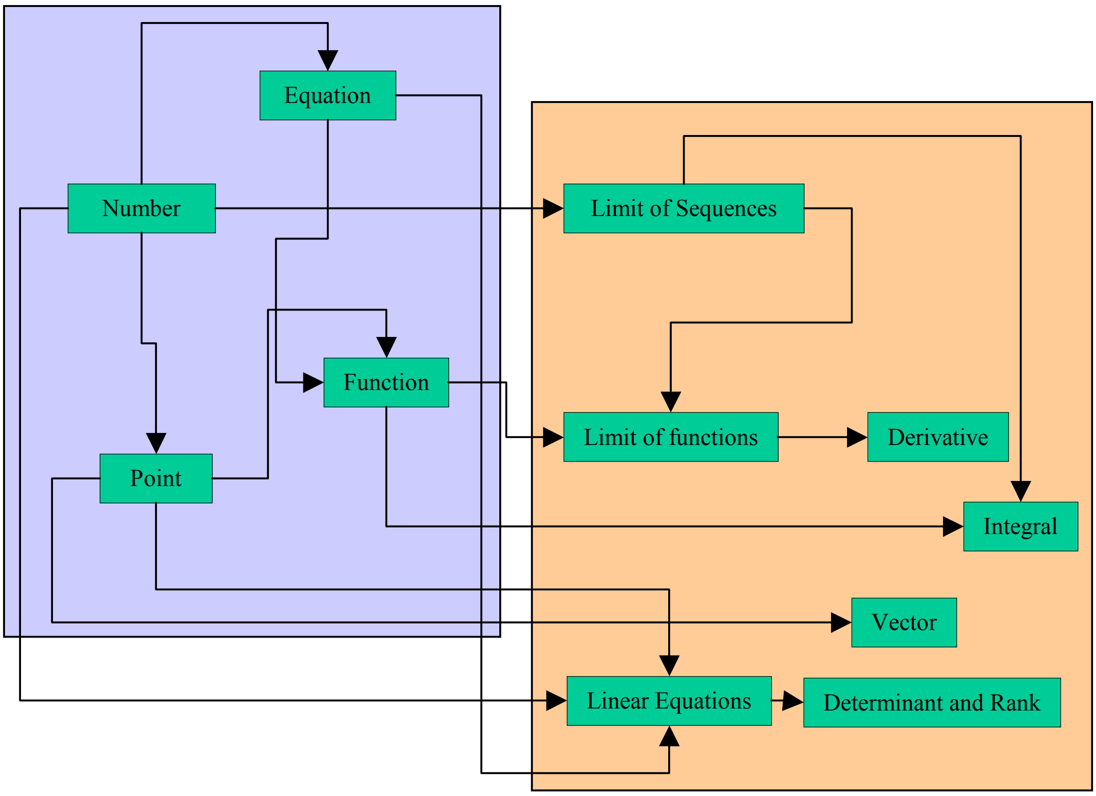
2. How to teach in an easier way difficult mathematical notions?

When arriving to the university students of scientific or technical disciplines have already heard about hard to understand mathematical concepts and, in fact, they have been evaluated in order to demonstrate their understanding and ability of manipulating such notions. Usually two different problems can be identified, with a non-very clear border, dealing with this question: either the concept is not correctly understood but its manipulation is rather acceptable or the concept is understood but not adequately manipulated. A third point to be addressed is the ability of using several concepts, initially not connected, to solve a concrete problem whose resolution requires the combined use of several techniques.

For example, it is very usual to find students with the ability of computing correctly derivatives or limits but without any clear idea about the meaning of what they are computing or without knowing why they are doing the computations in that way.

Our approach is concentrated around four building blocks: the notion of Number, the notion of Equation, the notion of Function and the notion of Point. If these concepts are not very well understood then the student will find big difficulties in order to follow not only other mathematical courses but also any other topics where Mathematics is the language and the tool (mechanics, dynamics, chemistry, etc.).

Around these four building blocks the different main concepts to be considered rotate as shown in the next diagram:



A first basic principle during all the course is to make explicit mention of when the basic building blocks are being used: for example when defining limit L of a sequence a_n the process where each a_n is closer to L than some $\varepsilon>0$ is presented as the concrete solving of a equation (inequality in this case) or the definition of determinant appears as an intelligent way of automatically solving a linear system of equations. In the same line students are shown that all the considered notions are strongly interconnected and thus, for example, the limit of a function f at a point α is introduced by considering the sequence $f(x_n)$ for any sequence x_n converging to α or the definition of definite integral appears as the limit of the sequences of areas approximating the desired to compute area below the graph of the considered function.

The second principle is devoted to provide motivations allowing the student to reproduce in many cases a formula or technique when it has been forgotten but it is needed: it is very easy to motivate how to compute the length of a curve by a very simple argument involving only the notion of integral as infinite sum plus Pythagoras Theorem.

Apart from the use of the building blocks as starting point to consolidate or introduce other notions, another fundamental objective of the course is to provide and improve the student's abilities concerning the formal and correct manipulation of mathematical expressions. For example, limit calculus is presented as a rewriting process where the initial sequence or function is presented in an equivalent form, where to read easily the value of the limit (in case it exists):

$$c_n = \sqrt{n^2 - n} - n$$

$$\Downarrow$$

$$c_n = \frac{(\sqrt{n^2 - n} - n)(\sqrt{n^2 - n} + n)}{\sqrt{n^2 - n} + n}$$

$$\Downarrow$$

$$c_n = -\frac{n}{\sqrt{n^2 - n} + n}$$

$$\Downarrow$$

$$c_n = -\frac{1}{\sqrt{1 - \frac{1}{n}} + 1}$$

This is done by the usual supervised mathematical training through exercises and problems plus the repetition of the latter with the help of the Computer Algebra System **Maple**. Next **Maple** session shows how a basic problem can be solved analytically, but visualizing at each stage what is going on, which is much more difficult to do (and time consuming) if a Computer Algebra System is not available.

Problem

Prove that if

$$f(x) = x^2 - 3x$$

then

$$\lim_{x \to 1} f(x) = -2$$

by computing for any $\varepsilon>0$ an interval around $x=1$, $(1-\delta,1+\delta)$, such that $f((1-\delta,1+\delta)-\{1\})$ is contained in the interval $(-2-\varepsilon,-2+\varepsilon)$.

```
> f:=x->x^2-3*x;
```

$$f := x \rightarrow x^2 - 3x$$

First the interval around 1 where the condition is verified is computed for ε=1/10.

> **solve({f(x)>-2-1/10,f(x)<-2+1/10},x);**

$$\{\mathrm{RootOf}(10\,\_Z^2 - 30\,\_Z + 19, .9083920217) < x,\ x < \mathrm{RootOf}(21 + 10\,\_Z^2 - 30\,\_Z, 1.112701665)\}, \{\mathrm{RootOf}(21 + 10\,\_Z^2 - 30\,\_Z, 1.887298335) < x,\ x < \mathrm{RootOf}(10\,\_Z^2 - 30\,\_Z + 19, 2.091607978)\}$$

Next the graph of f is displayed together with the lines $y = -2 - \varepsilon$, $y = -2 + \varepsilon$, $x = \alpha$ and $x = \beta$ with α and β the endpoints of the interval around 1 and verifying the required condition.

> **map(evalf,[solve(f(x)=-2-1/10,x)]);map(evalf,[solve(f(x)=-2+1/10,x)]);**

$$[1.887298335, 1.112701665] \quad [2.091607978, .9083920217]$$

After solving these two equations, which are α and β?

```
> display({plot(f(x),x=0.5..1.3,y=-2.2..-0.5,color=black,thickness=1),
        plot(-2-1/10,x=0.5..1.3,y=-2.2..-0.5,color=blue),
        plot(-2+1/10,x=0.5..1.3,y=-2.2..-0.5,color=blue),
        line([1.112701665,0],[1.112701665,-2.2],color=green,linestyle=1),
        line([.9083920217,0],[.9083920217,-2.2],color=green,linestyle=1),
        point([1,f(1)],color=magenta),
        line([1,-2.2],[1,f(1)],color=yellow,linestyle=3),
        line([0,f(1)],[1,f(1)],color=yellow,linestyle=3)},axes=BOXED);
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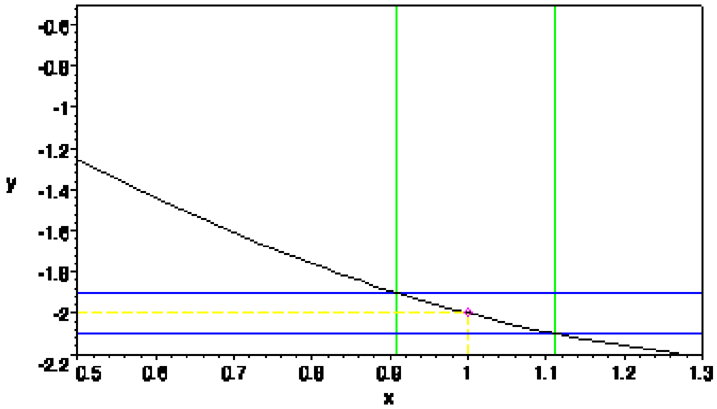
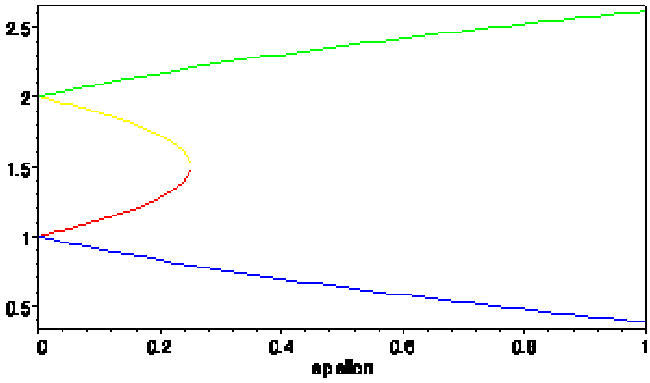


Finally the general case is solved. Study the roots in terms of ε giving the endpoints of the interval around 1.

> **sol1:=[solve(f(x)=-2-epsilon,x)]; sol2:=map(evalf,[solve(f(x)=-2+epsilon,x)]);**

$$sol1 := \left[\frac{3}{2} + \frac{1}{2}\sqrt{1 - 4\,\varepsilon}, \frac{3}{2} - \frac{1}{2}\sqrt{1 - 4\,\varepsilon}\right] \quad sol2 := \left[\frac{3}{2} + \frac{1}{2}\sqrt{1 + 4\,\varepsilon}, \frac{3}{2} - \frac{1}{2}\sqrt{1 + 4\,\varepsilon}\right]$$

```
>sol11:=plot(sol1[1],epsilon=0..1,color=yellow):
 sol12:=plot(sol1[2],epsilon=0..1,color=red):
 sol21:=plot(sol2[1],epsilon=0..1,color=green):
 sol22:=plot(sol2[2],epsilon=0..1,color=blue):
 display({sol11,sol12,sol21,sol22},axes=BOXED);
```



Solution: Use the previous computations to give a solution to the considered problem.

If ε is in the interval ($0, \frac{1}{4}$) then

$$f\left(\left(\frac{3}{2} - \frac{\sqrt{1+4\varepsilon}}{2} , \frac{3}{2} - \frac{\sqrt{1-4\varepsilon}}{2} \right) - \{1\} \right)$$

is in the interval (-2- ε ,-2+ ε).

If $\frac{1}{4} \leq \varepsilon$ then

$$f\left(\left(\frac{3}{2} - \frac{\sqrt{1+4\varepsilon}}{2} , \frac{3}{2} + \frac{\sqrt{1+4\varepsilon}}{2} \right) - \{1\} \right)$$

is in the interval (-2- ε ,-2+ ε).

3. About the mathematical impact of new technologies when used for teaching Mathematics

This section is devoted to show how the decision of using **Maple** (or any Computer Algebra System) to help the students in order

- to assist the understanding by providing an experimentation framework with visualization facilities; and
- to easily perform complicated computations

has several side effects that need to be taken into account as described later in this section.

The use of **WebCT** has also another implications derived from the use of internet for teaching but with a smaller impact concerning the mathematical contents of the course but remarking that the facilities provided by **WebCT** allows the teacher to easily control the individual progress of each student or to detect in advance unexpected misunderstandings.

For the material concerning the practical Maple sessions we consulted several texts available (see the references section to see a selection of the consulted textbooks) finding that

- Either there is no introduction to **Maple** (knowledge already assumed by the students) or, data structures & algorithms are freely used without providing the students with a minimal background to these Computer Science notions.
- When dealing with the computation of roots of polynomial equations (choosing the first significant example) Numerical Analysis enters immediately into the game; sometimes it enters before since LU or QR decompositions are explained for solving linear systems of equations. Of course our first year students do not know anything about floating-point numbers, errors (backward and forward), stability, etc. It is to be noted that those books using **Matlab** for Linear Algebra (for example Hill et al (1996),

Marcus (1993) and Smith (1997)) are more courses of Numerical Linear Algebra than even introductory Linear Algebra courses.

Note that no textbook was found fulfilling our requirements concerning, first, students entering into the first year at the university with a poor mathematical training and, second, without a previous knowledge of Computer Algebra.

Going from Mathematics+Technology to Mathematics

The decision of using a Computer Algebra System when teaching an introductory course of Mathematics to students of scientific or technical disciplines allows introducing the mathematical experimentation into the classroom. In many cases the using of **Maple** helps to the students to discover by themselves the definition of a mathematical concept: two canonical examples of this situation are the introduction of the derivative definition as a way of computing tangent lines to curves or the introduction of the integral definition as a generalization of the area concept to general curved domains.

Going from Mathematics+Technology to Numerical Analysis

Invoking the function *solve* in **Maple**, easily provides examples where no analytical solution can be computed: for example, to solve of a degree five polynomial equation does not have, in general, a closed form solution. **Maple** help automatically sends the user to invoke the function *fsolve* in order to get an approximation of a root for the considered equation in case this solution exists. Thus, in order to use properly Maple, students must have a (very basic knowledge) of what a floating-point number is and what it represents. Next step is the performing of elementary error analysis arising from, mainly, the solving of nonlinear equations in one unknown in order to check the goodness/badness solution provided by **Maple**.

Going from Mathematics+Technology to Computer Algebra

Due to the lack of previous training in Computer Algebra the first computer assisted sessions are devoted to learning the basics of **Maple**: numbers, polynomials, expressions, basic operations, vectors, matrices, etc. Very quickly, students started to use **Maple** as a powerful calculator able to solve problems otherwise impossible to solve by hand but also they are faced to what Computer Algebra is. Initially they separate two different kind of problems: those where the involved computations are purely symbolic (such as polynomial manipulations as the greatest common divisor, polynomial factorization or primitive determination) or numeric (such as root approximation or numerical integration). Special mention is made to the fact that both approaches must be used in a coordinated way since they are tools that Scientific Computing offers to the scientist or engineer to solve their problems.

Representation problems, which are a classical cornerstone in Computer Algebra, appear very often: if **Maple** is asked to compute the cubic root of –1 the answer always shocks the students:

```
> simplify((-1)^(1/3));
```

$$\frac{1}{2}+\frac{1}{2}I\sqrt{3}$$

If their first impression is to conclude that **Maple** has a bug (the statement that any software package has indeed bugs is clearly made explicit at the beginning of the course), this is not the case. This is the typical example where the reason why **Maple** returns this initially surprising result allows to introduce the discussion about how to define n-roots over the complex numbers or, without explicitly mentioning it, that many interesting complex valued functions are multi-valued. And that this is a very important problem in a discipline whose name is Computer Algebra.

Going from Mathematics+Technology to Data Structures, Algorithms & Programming

After the second or the third **Maple** session it is the right moment to explain several things that have been used implicitly: the notion of data structure (we have already used sequences, lists, sets, arrays, tables, strings, etc), the notion of algorithm (every worksheet is in fact the skeleton of one or several algorithms solving a particular problem) through the using of several **Maple** operators and functions and the different kind of tools **Maple** offers to the user in order to implement a procedure corresponding to a concrete algorithm (iteration, recursion, conditionals, etc).

It was initially planned that the computer assisted practical sessions must change their structure and no more prepared worksheets would be distributed: a concrete problem related with the current topics being discussed at that moment is distributed and the students may generate a worksheet containing the implementation of the algorithm solving the concrete problem proposed. But timing constraints have avoided up to this year to apply this initial plan.

4. Conclusions

The decision of using **Maple** into the practical sessions of the *Laboratory of Mathematics* course considered here has implied, first, a different way of presenting an introductory course of Mathematics for students of scientific or technical disciplines plus the inclusion and/or consideration from the early beginning of three new items into the curricula:

- An introduction to Computer Algebra through **Maple**.
- A short introduction to Numerical Analysis.
- An elementary introduction to Data Structures and Algorithms.

From the positive point of view it is important to remark that these three new items are inserted in a natural way since they are explicitly needed in order to make possible the using of the computer and **Maple** to solve some of the problems proposed and very closely related with the mathematical topics considered in the course. From the negative point of view it is clear that the time devoted to these three new topics is not used to deep inside some of the concepts of the course: it would be optimal if the students arrived in advance with the required knowledge of **Maple** and thus to avoid the spending of time in the first and third items. But this is difficult to achieve since this introductory course is taught at the very beginning of the first year of studies at the University.

From our experience, since it is very easy to motivate (and justify) the *soft* introduction of Computer Algebra, Numerical Analysis and Data Structures and Algorithms inside this introductory course, it seems to be a very convenient deal to include these topics as regular material but with a timing increase estimated in one more module. An extra advantage of this option would be the early introduction of these tools, which can be later used, into the teaching of other topics into the curriculum. It is worth to remark that it seems to be unavoidable the consideration of several topics (not usually classified as basic mathematical topics) from Numerical Analysis, Data Structures and Algorithms if a Computer Algebra System is to be used in the classroom.

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