

USING COMPUTER ALGEBRA TO ENCOURAGE A DEEP LEARNING APPROACH TO CALCULUS

Diana MACKIE
Napier University
School of Mathematics & Statistics
Sighthill Court
Edinburgh EH11 4BN
Scotland, UK
Email:d.mackie@napier.ac.uk

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ABSTRACT

The underlying concepts and proofs of introductory calculus involve difficult and abstract ideas that present a mountainous obstacle to many students. A tempting 'solution' for lecturers is to focus the teaching at this stage on techniques. This may have the advantage of ensuring acceptable pass rates but helps neither the students nor the teaching staff in the long term. Computer algebra systems offer both an opportunity and a challenge to present new approaches that assist students to develop better understanding of the basic concepts. They can be used to change the emphasis of learning and teaching of calculus away from techniques and routine symbolic manipulation towards higher-level cognitive skills that focus on concepts and problem solving.

Two of the key indicators of deep learning and conceptual understanding are the ability to transfer knowledge learned in one task to another task and the ability to move between different representations of mathematical objects. Computer algebra systems are multiple representation systems, that is, they have the ability to facilitate graphical, algebraic and numerical approaches to a topic.

The author will describe how carefully structured worksheets are used with Derive to ask questions then let the students provide the answers in such a way that they can construct their own knowledge. This allows learners to 'discover' rules, to make and test conjectures and to explore the relationship between different representations of functions and other mathematical objects using a blend of visual, symbolic and computational approaches. Students enjoy the power and versatility of computer algebra and are encouraged to become reflective, deep learners.

1. Deep and Surface Learning

Mathematics is used in real life to model systems and analyse data arising in science, engineering and the management sciences. In order to tackle real problems it is necessary to acquire skills of formulating a problem in mathematical terms, interpreting the solution and analysing its sensitivity, all of which require a good understanding of the underlying concepts of the topic. An exploratory attitude and expertise in investigative techniques are crucial to successful modelling or problem solving.

Conceptual understanding, flexible thinking and an exploratory approach are all indicators of deep learning. Deep learning is generally held to be at one extreme of the spectrum of approaches that students adopt towards learning. The other extreme is surface learning. In practice, most students fit somewhere in between.

Surface learning, as the name implies, involves simply 'scraping the surface' of the material being studied, without carrying out any meaningful processing of the content. Students who adopt such an approach are characterised by:

- concentrating solely on assessment requirements ;
- accepting information and ideas passively;
- memorising new ideas as a collection of rules without any attempt at integrating with the old ideas;
- failing to reflect on underlying purpose or strategy.

Students who adopt a deep approach, on the other hand, want to make sense of what they are doing and to build their own personalised knowledge structures. They tend to follow the general pattern of:

- endeavouring to understand material for themselves;
- interacting critically with content;
- relating ideas to previous knowledge and experience;
- examining the logic of arguments and relating evidence to conclusions.

Most mathematics educators would argue that they want their students to adopt a deep learning approach. They want students to develop mathematical insight and the ability to solve problems. Students need to be able to make sense of answers, to manipulate expressions mentally and to anticipate the likely outcome of a range of possible approaches. This kind of mental agility is an indication of deep understanding.

But experience shows otherwise, particularly when the topic being taught is calculus. The failure of large numbers of mathematics students to grasp the basic concepts of calculus is well documented. The underlying concepts and proofs of introductory calculus involve difficult abstract ideas, which perhaps explains why, traditionally, the focus at this stage is often on techniques. With sufficient practice, the majority of students can become competent users of the rules that enable them to differentiate or integrate a range of standard functions. They can also learn to apply these techniques to problems such as finding extreme values or the area under a curve. Far fewer students, however, can explain the underlying ideas of a limit, the differentiability of a function or integration as an infinite summation. Faced with the prospect of a large proportion of their students failing the final examination, many teachers abandon the attempt to develop conceptual understanding. The goal of the pragmatic lecturer is often for the student to develop skills in computational procedures, to apply the correct procedure in a given problem and to achieve good examination grades.

2. What's wrong with conventional teaching and learning methods?

Clearly the traditional style of teaching whereby calculus is presented as a logical exposition of definitions, proofs, techniques and applications isn't working as a means of encouraging deep learning. The reason it doesn't work is that presenting mathematics in a way that develops from formal ideas is not a sound approach, pedagogically. It does not attempt to build on the students' current knowledge structures.

Dubinsky (1991) asserts that, in his experience, students do **want** to understand concepts but if they don't understand what is being taught they lose interest and resort to surface learning techniques:

"As long as there is something for the student to think about, as long as he or she perceives that cognitive activity is leading to some sort of growth that could, eventually, lead to a solution of the problem, then there is little difficulty in maintaining the students' interest."

The experiential model of learning developed by Kolb (1984) stresses that learners must be actively involved in the learning process. He presents it as a four stage cycle - planning, doing, thinking and understanding. In mathematics learning this cycle can be interpreted as being involved in planning the learning outcomes, carrying out appropriate learning tasks and activities, and reflecting on what has happened leading eventually to relational understanding. The final stage involves linking actual learning experience with the underlying theories and thus integrating the new concepts into existing knowledge structures. It may, of course, take many loops of the cycle to reach the desired deep understanding of the abstract concepts.

Research has shown that appropriate sequences of learning and teaching designed to help the student actively construct concepts in this way can prove highly successful. Dubinsky (1991) describes the outcome of an introductory course in calculus in which students spent the first twelve weeks focussing on the underlying concepts before starting to practice techniques.

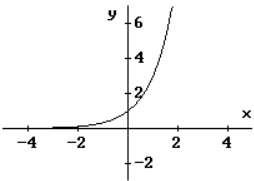
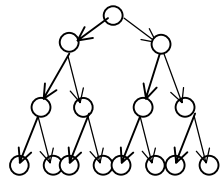
"By encouraging the students to think for themselves and to construct their own ways of handling concepts, it became apparent that they had integrated the ideas into their own knowledge structure."

Tall (1986a) argues that it is not necessary for the student to complete a large number of different examples of a new idea in order to develop relational understanding, as is popularly thought. It is the third and fourth stages of Kolb's learning cycle, reflection and understanding that lead to the conceptual understanding and abstraction. This can be achieved effectively by using a few carefully chosen examples to illustrate and explore important aspects of the concept and nurture the required reflective abstraction. Tall refers to "*The single, representative example which so often seems to be in the mind of the mathematician who understands a particular concept*" as the 'generic example'.

3. Multiple Representation

In the mind of a mathematician, the 'generic example' frequently exists in several different but linked representations. For example, an exponential function may be thought of in symbolic, graphical, geometric and numeric forms (figure 1) but, once the abstract concept has been grasped, the user can switch between the different representations with ease in order to retrieve the one most appropriate to the current problem.

Figure 1: Representations of an exponential function

Graphical	Geometric	Symbolic	Numeric												
		$f(x) = a^x$ or $\frac{dy}{dx} = kx$	<table border="1"> <thead> <tr> <th>x</th> <th>3^x</th> </tr> </thead> <tbody> <tr> <td>0</td> <td>1</td> </tr> <tr> <td>1</td> <td>3</td> </tr> <tr> <td>2</td> <td>9</td> </tr> <tr> <td>3</td> <td>27</td> </tr> <tr> <td>4</td> <td>81</td> </tr> </tbody> </table>	x	3 ^x	0	1	1	3	2	9	3	27	4	81
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Mathematicians use symbols both to **think** about mathematics and to **do** mathematics, to communicate ideas and to express results. Seeing a symbol or symbolic expression conjures up in the mind a mental image of what the symbol represents but students' and teachers' mental representation of a mathematical symbol can be very different. The teacher is able to select the mental image most appropriate to the particular task. A student might think of a function purely as an algebraic formula (process) whilst the teacher is thinking of a function as an object to be transformed by some operation such as differentiation or indefinite integration (Tall, 1991). This situation can lead to confusion and leave students unable to understand the teaching thus creating an obstacle to learning. Another confusing situation arises when a student tries to cope with several competing mental representations of the same concept.

Students who can only work in one representation often fail to solve a problem correctly. For example, even when they are required to sketch the graph of a function in one part of a question, they may ignore the 'evidence' of their own sketch in a subsequent part of the question in which they are using an algebraic representation of the function. Teaching and learning should aim to integrate competing representations into a single representation, sometimes called a 'multiple-linked representation' (Tall, 1991). This allows a person to use several different representations at the same time, switching from one to another when it is appropriate to do so.

According to Tall, there are four stages to the learning process:

- Using a single representation;
- Using more than one representation in parallel;
- Making links between parallel representations;
- Integrating representations and flexible switching between them.

It is the recognition of links between parallel representations and their common properties that leads to the formation of an abstract concept of the mathematical object or process. Once the abstract concept has been formed, its 'owner' can link back to any one of its concrete representations when required. This 'multiple-linked representation' state of thinking is an essential pre-requisite to deep understanding and underpins the flexibility required for problem solving.

4. How computers can help

With the traditional undergraduate curriculum, students do not often regard themselves as active participants in mathematical exploration. Rather they are passive recipients of a body of knowledge, comprising definitions, rules and algorithms. Computers offer a number of didactic advantages that can be exploited to promote a more active approach to learning. Students can become involved in the discovery and understanding process, no longer viewing mathematics as

simply receiving and remembering algorithms and formulae. (Shoaf-Grubbs, 1995). In particular, the computer provides opportunities for dynamic visualisation. Students can explore basic mathematical concepts from new geometrical and graphical perspectives. Several research studies have concluded that the visualisation features of computers can be used successfully to encourage multiple linked representations of mathematical concepts (for example Tall, 1986a, Schwarz, Dreyfus and Bruckheimer, 1990).

One software environment designed to promote the dynamic visualisation and exploration is 'Graphic Calculus' (Tall, 1986b) which uses visualisation to explore the concept of differentiability using the notion of 'local straightness'. If a function is differentiable at a point then, by magnifying the graph of the function on the computer screen, it eventually becomes like a straight line around the point. This can then be linked to numeric and symbolic approaches to give the notion of a derivative a computable meaning. Tall argues that the idea of local straightness is a more natural starting point for a student to understand differentiability than the concept of a limit. Using this approach, graphical methods 'lead' the analysis.

5. Computer Algebra

Visualisation with the aid of software such as Graphics Calculus or a graphics calculator can lead students to a greater understanding of concepts but this approach does not necessarily help them to cope with the corresponding symbolic representations. A computer algebra system (CAS) is a multiple-representation system; it has the ability to facilitate graphical, algebraic and computational approaches to a topic. It is therefore an ideal tool for directing learning towards multiple-linked representations of mathematical concepts. Through carefully designed activities students can investigate the links between different representations of objects, recognise their common properties and begin to construct their personal structures of mathematical knowledge. Student activities have to be designed with very detailed cognitive steps in mind. Appropriate teacher intervention will usually be required to ensure that the students follow through the required learning stages, in particular, the reflective thinking.

As an example, consider how students could use the computer algebra system, DERIVE to investigate the limit of $\frac{\sin x}{x}$, as x approaches 0, from various perspectives, which use different

representations of the limit. They can tabulate values of $\frac{\sin x}{x}$ for smaller and smaller values of x , plot a graph of the function over an interval around $x = 0$ and use the LIMIT command in DERIVE to evaluate $\lim_{x \rightarrow 0} \frac{\sin x}{x}$. Having demonstrated convincingly that the limit is 1, the

rigorous definition of the limit can be explored. The definition states that if $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$, then,

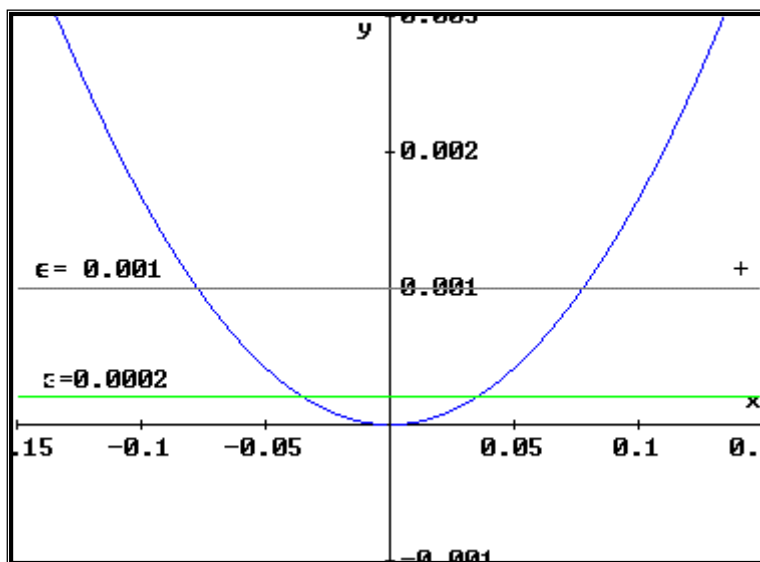
given any positive number ϵ , there is a corresponding number δ such that $\left| \frac{\sin x}{x} - 1 \right| < \epsilon$ whenever

$0 < |x| < \delta$. In order to investigate this graphically, plot $\left| \frac{\sin x}{x} - 1 \right|$ for $-1 < x < 1$. Then, by

superimposing the graph of $y = 0.001$, say, the points where the two graphs intersect show the value of δ when $\epsilon = 0.001$, as shown in figure 2. The meaning of 'given any positive number ϵ , there is a corresponding number δ ' can be clearly demonstrated by experimenting with

different values of ϵ . Each stage in the investigation should be followed by opportunities for recording results, reflective discussion and, if necessary, further exploration.

Figure 2: Exploring the Definition of a Limit



CAS activities in which students are asked to construct examples that satisfy certain constraints can also be used to encourage exploration of concepts, to focus on the links between different representations and to develop reflective thinking. The following simple example helps students to make connections between the derivative of a function and the slope and shape of its graph:

Define 4 non-linear functions, two that have a positive first derivative and two that have a negative first derivative in the range -10 to 10 . Plot all 4 functions on a graph and describe the shape of each curve. How can you tell from the graph where the derived function is positive?

Research has shown that the ability to transfer what has been learnt from doing one mathematical task to doing a similar one is more likely when the learner has been helped to discover the rule for doing the first task. (Sotto, 1995) A computer algebra system is an ideal tool for allowing the learner to ‘discover a rule’ or to make a conjecture and then prove or disprove it. The tedious repetitive manipulation is automated and the learner can concentrate on the results. One obstacle to learning can be the teacher who is determined to ‘teach’ or at least to tell the student everything so that there is nothing left for them to construct for themselves. Support materials should concentrate on providing activities and asking questions then letting the student provide the answers through reflection. In the following exercise, DERIVE is used to help the learner discover the rule for differentiating functions such as $\sin(kx)$ and $\cos(kx)$ as an introduction to the Chain Rule for differentiation.¹

*Use DERIVE to obtain the derivatives of $\sin(x)$, $\sin(3x)$, $\sin(6x)$, $\sin(-2x)$ and $\sin(-2.5x)$. Deduce the general formula for the derivative of $\sin(kx)$, where k is a constant. Use the **Limit** command in DERIVE to investigate $\lim_{h \rightarrow 0} \frac{\sin(k(x+h)) - \sin(kx)}{h}$. Does the answer agree with your own rule? (Similarly for $\cos(kx)$).*

¹ Many similar examples can be found in Learning Mathematics through DERIVE, (Berry, Graham and Watkins, 1996)

Write down the derivatives of the following functions and check them with DERIVE: $\sin(2x)$, $\sin(-4x)$, $\cos(3x)$, $\cos(0.5x)$

6. Development of problem solving skills

A typical student approach to problem solving is to find a suitable worked example to mimic then carry out the computation. Clearly this strategy is limited by the extent of the students' memory bank of similar problems and inhibits flexible thinking. A better approach is to consider alternatives, experiment, conjecture and test, then analyse the results. A computer algebra system can be a major factor in developing an exploratory approach to learning mathematics and, in particular, investigating problems from multiple representational perspectives. Using the computer to produce graphs, carry out calculus operations or perform repetitive calculations, students can be encouraged to make and test conjectures, to consider alternative solutions and to tackle open-ended problems. Removing the burden of manipulation and computation allows students to spend time on these other activities. This approach can make the study of mathematics more enjoyable, more relevant and more rewarding (Mackie, 1992).

Most elementary courses concentrate on closed form solutions; approximation methods are only mentioned briefly. This is due to algebraic limitations and the tedious and extensive calculations required to obtain or analyse approximate numerical solutions. As a result, modelling is usually restricted to artificially constructed examples and leads students to question the applicability of what they are doing. The use of computer algebra removes these restrictions and allows students to solve real problems using a combination of numerical and algebraic techniques.

7. Disadvantages

The ideas expressed so far in this paper present a very positive role for CAS in mathematics education. Are there any disadvantages? Norcliffe (1996) warns that most revolutions are double-edged and that the computer revolution brings many challenges, concerns and dangers to mathematics. One of Norcliffe's concerns is that students' algebraic manipulation skills will deteriorate if they are allowed to rely on computer algebra but that these skills are an essential foundation for mathematics. Many other educators would agree that there are fundamental mathematical skills that are essential even when technological tools such as computer algebra systems are available. There is, however, no widespread agreement on what exactly these core skills are. It seems clear that students must still spend time developing manipulation skills. The potential of computer algebra lies in its ability to improve conceptual understanding and problem solving.

In a report of a project involving the use of DERIVE with A-level students to investigate the link between infinite summation and anti-differentiation, Terence Etchells (1993) describes how a lesson can go wrong if the students have an insufficient understanding of the underlying mathematical concepts:

“A very didactic problem with CASs is that they can produce meaningless expressions. Students are punching keys and performing operations on expressions that have no meaning; they are producing mathematical nonsense.”

A similar example from the author's own experience occurred with a class of students using DERIVE to investigate the effect of the constant k in the exponential function e^{kx} . Two students

did not use the symbol e for exponential e (despite a clear reminder in the worksheet) and thus could not display the graph. At least these students realised that something was wrong. A more 'serious' mistake was the failure by one student to use brackets correctly and she was therefore investigating the functions: e^x , $e^2 \cdot x$ and $e^{0.5} \cdot x$ without realising her mistake.

Teacher support and appropriate intervention is crucial to correct mistakes of this nature and enable students to make the important links. Judging the right amount of help at the right time is a skill acquired through practical experience. Students must be allowed sufficient time to learn the language and features of DERIVE before using it to enhance their learning.

8. Conclusion

"It would be a mistake to incorporate CAS into our courses primarily as exercise solvers, whilst continuing our present orientation towards carrying out algorithmic computations. Our goals, expectations, assignments and classroom instruction need to change in order to maximise the opportunities offered by modern technology." (Small & Hosack, 1976)

Although written some time ago in the early days of computer algebra in education, the view of these authors is still valid. Many of the changes, which they predicted, have not yet happened but are still necessary. Students measure the importance of an activity by the amount of time devoted to it. At present most of their time is spent practising routine skills. Perhaps it is not surprising that students view mathematics as a collection of formulae (to be memorised) and "to do maths" is to compute. If more routine computation is done on a computer more time is available for concentrating on concepts, motivation, applications and investigations. Computation will be seen as a means rather than as an end in itself.

The power of computer algebra goes beyond routine computation. It has the potential to facilitate an active approach to learning, allowing students to become involved in discovery and constructing their own knowledge, thus developing conceptual understanding and a deeper approach to learning.

A computer algebra system is a tool not a self-contained learning package or encyclopaedia of mathematical knowledge. It is the way in which it is presented to and used by students that determines its ability to influence learning. Much emphasis these days is placed on student-centred learning and less on the teaching but teaching and learning are equally important. It is necessary to first understand the learning process and then design teaching and learning activities to achieve these. Only then will students become deep learners.

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