PARADOX AND PROOF

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ABSTRACT

The definite integral is a major topic in Calculus with many student difficulties. In [6], we have traced the development of understanding as it progresses and found a curious phenomenon. When unable to proceed along a particular schema, students introduce a heuristic that helps them bridge the gaps in their understanding. In this particular situation such a gap is filled a change of a unit from that of a rectangle to that of a line segment of 0 width, the indivisible. They follow with the computation involving the sum of heights of infinitely many line segments to obtain the area under a curve - the definite integral. We suggest approaches to channel their thinking - a guided heuristic that confronts students with concrete physical scenarios where similar manner of reasoning leads to a contradiction. Using Zeno's paradoxes of the race between Achilles and the tortoise, we begin the process of introducing students to a directed heuristic. We follow with the mathematical context, using a construction by John Wallis [14], to provide a mathematical framework within which the intuition of indivisibles can connect with the notion of the area of two dimensional regions.

1. Introduction

The literature abounds with student difficulties on the Calculus topics of limits and definite integrals [4, 6, 7, 8, 11, 13]. Difficulties with limits in fact manifest themselves in topics such as the definite integral, the derivative of a function, etc. We will focus our attention in this paper on the students' difficulties with respect to one key concept, viz., that of estimating the area under an irregular curve by using the method of Riemann sums. The reason for the focus on the definite integral is an interesting phenomenon observed by us in [6], which is briefly described as follows.

In addition to the standard idea held by some students of approximating the area by sums of rectangles inscribed under a given curve, other students also saw the area as a sum of line-segments from the abscissa to the curve. More precisely, instead of seeing the area as a limit of the process of taking partial Riemann sums, they see it as a sum of the ordinates of the function in question. A historical study of the concept of the area under a curve reveals that the image held by the students corresponds in its general outline to the viewpoint presented by Archimedes in *The Method*, Cavalieri [3] in the *Geometria Indivisibilibus* and John Wallis in *Arithmetica Infinitorium* [14]. It is fascinating that this intuition of area has persisted on its own, without formal instruction, till the present day.

The method used by students' is their "heuristic" just as it was Archimedes' heuristic. A *heuristic* in our context is an approach used by students that makes sense to them. This approach need not necessarily be the way taught in class, nor need be mathematically precise but from the point of view of the student it is a way to see the solution to the problem under consideration. The dictionary meaning of a heuristic is "relating to exploratory problem-solving techniques that utilize self-educating techniques (as the evaluation of feedback) to improve performance".

Heuristics are commonly used in computer science, sometimes as the only solution to an existing problem. A heuristic solution is used in practical problems when no other known solution exists, or when a complete or more exhaustive search is too expensive. In the embracing of mathematics education reform in the past ten years however, heuristics have played a very important role. Most of the Computer Algebra Systems used in almost all "reform" mathematics classrooms are an outgrowth of the heuristic principle. Heuristics have been used by mathematicians since or before the time of Archimedes. It was Archimedes' heuristic proof that led him to the more rigorous mathematical proof of finding the area under a parabola [1].

Now, in the situation under consideration namely that of finding the area of a region by means of the method of Riemann sums students betray an interesting confusion while explaining the construction (excepts below), the confusion between two different types of "units" to measure that area, the rectangle and a line segment. A similar kind of confusion, is responsible according to Grunbaum [15], for creating the Achilles and Tortoise paradox of Zeno.

In general, it is difficult for students to notice the fact that they have indeed confused two entirely different quantities, when they are actually carrying out the process, or when they reanalyze their work. However, if the situation is posed to them in terms of a physical scenario such as a race between Achilles and a tortoise, clarifying the confusion becomes much easier.

2. Statement of the problem.

Consider the student excerpts below.

(In the excerpts below, I indicates an interviewer, and C, D are students)

Excerpt 1

[1] I: How do you go from a Riemann sum to make it equal to the 2/3 we got here?

[2]C: Make these rectangles infinitely small, smaller, smaller and smaller, I mean almost until **[3]** they're a line, they're a unit and then you are just adding up these units and the smaller—this **[4]** empty area is the more exact estimation until you get to a point where there is no empty space **[5]** to be accounted for that gives an exact number.

Excerpt 2

[6] I: How would you get the closest . . .

[7] D: Um...

[8] I: . . . possible area?

[9] D: Closest possible area would be by taking the length of a line segment from the x axis to the

[10] function itself. And that would give you an infinitely many ...and many areas to add up. And

[11] that's what the definite integral gives you. It just allows you, you know, to be able to work **[12]** with basically a rectangle with no width, just height.

In the first excerpt, the student starts from the interval on which the function is defined and progressively reduces the width of the subintervals, and suddenly makes a jump to an entirely different unit, the line segment without width, or the ordinate of the function. That corresponds to the conceptual jump between two different intuitions, from the continuous one to the discrete one (lines 2, 3). This student conducts the process of infinite subdivision. The means to carry out this infinite subdivision starts out by using the rectangles as "units", however, a conceptual jump occurs, after which the units used are the line segments or ordinates of the function. The aim, of course, is to reduce the error between the estimated area and the actual area. The conceptual jump is not apparent to the student at the time it is carried out or later. In the world of paradoxes, an analogous situation is the race between Achilles and the tortoise.

In the second excerpt above, the student, also in an attempt to reduce the error between the actual area and the estimation, starts out by explaining that the error would be reduced by taking the infinite sum of the heights of line segments to get a two-dimensional area. Here, the summation of units of one kind, namely the line segments of 0 width are assumed to generate area of a two dimensional region.

We build our instruction based on the heuristic created by students to guide them toward a solution that is in agreement with their heuristic but is at the same time mathematically rigorous.

Notice the problems of the student C. He departs from the Riemann construction, most probably because the student's grasp of the limit concept and of its role in the construction is weak. That leads him to abandon the summation of two dimensional units of rectangles to get two dimensional area and to use an inappropriate unit - the line segment or the indivisible. Next he sums these one dimensional units to get a two dimensional area. Therefore, on one hand, we have to discourage the jump between the different intuitions and we use the analysis of Achilles and

Tortoise for that purpose (Section 3). On the other hand, we have to provide the student with the mathematically correct path to express the process of infinite subdivision [16]. Finally, we can also provide the student with the mathematically correct path between the summation of the lines and the area under the curve using Wallis-Cavalieri construction [Section 4].

3. Two Paradoxes

Let's look at the steps taken by students. Students' construction could be stated in the following steps:

1. The region whose area is to be determined is irregular.
2. Fit regular shapes (rectangles) in the region.
3. Regular shapes result in over-fitting or under-fitting, and involve an error.
4. Reduce width of regular shapes.
5. Error is reduced but still exists.
6. To eliminate error, consider line segments to be the regular shape.
7. Add heights of regular shapes.
8. Sum of the heights is the area of the irregular region.

Students proceed to determine the area systematically until they arrive at the end of step 5. At this point the concept that they should use is the limit of the numerical sequence corresponding to their visual image (steps 3 and 4). However, instead of applying the limit, they resort to using line segments. It is step 6 above that results in a paradoxical situation, in which the spatial units of rectangles are replaced by the discrete units of line segments of 0 width - the indivisibles. The danger in inappropriate coordination of the continuous and discrete elements is well illustrated by

the Paradox of Achilles and the Tortoise, which states: Suppose Achilles runs ten times as fast as the tortoise and gives him a hundred yards start. In order to win the race Achilles must first make up for his initial handicap by running a hundred yards; but when he has done this and has reached the point where the tortoise started, the animal has had time to advance ten yards. While Achilles runs these ten yards, the tortoise gets one yard ahead; when Achilles has run this yard, the tortoise is a tenth of a yard ahead; and so on, without end. Achilles never catches the tortoise, because the tortoise always holds a lead, however small. But – as we know full well – Achilles will soon pass the tortoise. We agree here with Grunbaum who [15] locates the source of the paradox in the imposition of the discrete structure of locations of Tortoise onto the continuous structure of the motion of Achilles. For if we restate the problem in a familiar language of "word problems", saying, find the distance x from starting point and the time when Achilles passes the Tortoise, there is no paradox. Instead we solve the familiar equation $\frac{100+x}{V_A} = \frac{x}{V_T}$, where V_A is the speed of Achilles and V_T is the speed of the Tortoise. The only discrete element here, the moment when Achilles is passing the Tortoise, is not imposed from outside but is an intrinsic element of the structure of the problem. Thus, the importance of the proper conceptual framework to avoid the paradoxical situation becomes, we hope, apparent to the student.

4. Wallis - Cavalieri construction

Pedagogically, the discussion of the paradoxical elements inherent in students' reasoning has as its goal to warn them of possible conceptual and epistemological difficulties along the taken path. An alternate route is to provide (prior to treatment of the definite integral), students' familiarity with the concept of the limit of a sequence and by coordinating that concept clearly with the geometric construction of Riemann sums [16]. Here, on the other hand, we provide a mathematically correct framework in which a bridge can be established between the intuition of lines and areas of two dimensional regions.

The Wallis-Cavalieri technique [17] is a development based on the work of John Wallis, of a method to calculate the area bounded by $f(x)=x^2$ on [0,1] [14]:

We construct a sequence of the Wallis-Cavalieri ratios	which are decomposed by the Wallis method into the estimated limit and an additional term	
$W_1 = \dfrac{0^2+1^2}{1^2+1^2} =$	$\dfrac{1}{2}$	$= \dfrac{1}{3} + \dfrac{1}{6} = \dfrac{1}{3} + \dfrac{1}{6\times 1}$
$W_2 = \dfrac{0^2+1^2+2^2}{2^2+2^2+2^2} =$	$\dfrac{5}{12}$	$= \dfrac{1}{3} + \dfrac{1}{12} = \dfrac{1}{3} + \dfrac{1}{6\times 2}$
………………………..	…….	……………………..
$W_n = \dfrac{0^2+1^2+2^2+\ldots+n^2}{n^2+n^2+\ldots+n^2}$	$\dfrac{2n+1}{6n}$	$= \dfrac{1}{3} + \dfrac{1}{6\times n}$

Therefore $\lim W_n = \dfrac{1}{3}$ *as n* $\rightarrow \infty$ and is the area of the required region.

The ratios $W_n = \dfrac{0^2+1^2+2^2+\ldots+n^2}{n^2+n^2+\ldots+n^2}$ can be seen as resulting from the expressions $\dfrac{\sum_{i=0}^{i=n} (\frac{i}{n})^2}{n+1}$ where the numerator represents the sum of squares of the ordinates of the function at n+1 points, while the denominator represents the sum of n+1 corresponding coordinates of the function g(x) = f(1) = 1 (recall that $f(x) = x^2$). Thus visually, the formula represents the ratio of the sum of n+1 lines under the curve to the sum of the corresponding lines in a circumscribing unit square.

In a slightly more general case when the domain of the function is [0, b] we will have the expressions $W_n = \frac{\sum_{i=0}^{i=n} (\frac{bi}{n})^2}{f(b)(n+1)}$. A corresponding Riemann sum on that subdivision is $R_n = \frac{b}{n}\sum_{i=1}^{i=n} (\frac{bi}{n})^2$, and the relationship between the two is $W_n = \frac{n}{b\times f(b)(n+1)} R_n$.

Therefore, as n→ ∞, lim W_n =

$$=\lim \frac{n}{b\times f(b)(n+1)} R_n = \frac{1}{b\times f(b)} \lim \frac{n}{n+1} \lim R_n = \frac{area\, under\, the\, curve}{b\times f(b)}.$$

Since $b\times f(b)$ has the interpretation of the area of the circumscribing rectangle, indeed we have here the ratio of areas of two continuous regions. Thus, in some metaphorical way, the intuition which sees the area as the sum of lines acquires a credibility of its own in a mathematically correct framework.

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