ALGEBRAIC CALCULATOR TECHNOLOGY IN FIRST YEAR ENGINEERING MATHEMATICS

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ABSTRACT:

Algebraic calculators have made minimal inroads to most Engineering mathematics courses in Australia. Indeed, many still forbid normal graphics calculators in assessment despite their wide usage in the school systems, which feed the undergraduate courses. This is curious as even the algebraic calculator technology is no longer very new and reminds us of the resistance to change in undergraduate mathematics teaching.

Currently we are developing an engineering course in product design, which combines traditional course objectives with handheld CAS. For several years now, our Engineering students have used Mathematica from second year of course (although not in tests) and normal graphics calculators are used in all work in first year. The emphasis on facts and skills in the extant course means that over 60% of examination questions previously given in the first year course could be solved much more simply using an algebraic calculator.

The transition period requires that the traditional course be essentially maintained, partly to ensure student mobility between engineering courses, but some topics are modified for the new course and assessment is independent. Current engineering textbooks usually restrict themselves to traditional algebraic and calculus approaches, although graphics calculators are now more commonly used. Indeed many of these textbooks explicitly state opposition to the extension of CAS within the framework of the traditional course. This forces the provision of resources in-house to service a CAS approach to engineering algebra and calculus.

In this paper we discuss the introductory course and its implementation problems, illustrating how algebraic calculators can solve basic questions in a normal course, and how the calculators may be used in the future.

Keywords: Technology, CAS, Engineering

Introduction

Algebraic calculators have become quite common over the past decade, yet they have had minimal impact on either senior secondary or entry level university courses in Australia. This is despite increased use overseas – especially within Europe. This may seem surprising in view of the lead-time of taking up technology, which occurred in education with scientific calculators, graphics calculators and personal computing. Some of the controversy associated with the widespread adoption of scientific calculators and, more recently, conventional graphics calculators is outlined in Tobin, (1998a). In particular the ability to extend these conventional graphics calculators to algebraic work is noted. One consequence of this is students already can have very differential access to algebraic assistance with their present calculators, an existing equity consideration.

However there are some significant issues raised by this new algebraic technology which either did not arise before or which differ in degree.

Three particular issues stand out. Algebra is a pet love of the mathematics community, especially the teaching component, and linked inextricably with the issue of understanding in mathematics. To ease a student's path in algebra is seen by some as the ultimate 'cop-out'. The actual manipulations afforded by algebra appear to many as the essence of mathematics and skill in these manipulations marks true comprehension in the subject. In fact it is algebra which provides the mysterious and powerful language of mathematics and ensures its rituals are the preserve of the cognoscenti. Mathematical thinking and algebraic manipulation are too often seen as equivalent. This perspective ignores the very different levels of ability in algebra which are called for in everyday use of mathematics. There is always a need for some experts but functional ability in others is often sufficient. In any case solution of equations in algebra often requires good understanding with or without CAS as noted in Ball, (2001).

A second critical issue is that algebraic learning and use has grown over time to suit a curriculum which was very different. Thomas (2001) makes a distinction between two important aspects of algebra – its role in process of solution and its role in conceptual understanding. The CAS can assist very directly in the former, yet much of current algebra teaching is directed to this. The challenge is to exploit CAS in expanding student skills in understanding the basic concepts. This is a nontrivial issue. Despite widespread hopes that the new technology options would enhance understanding by what Kaput (1992) called multiple, dynamically linked representations of a concept, no hard evidence exists yet that this has been achieved (However see Boers and Jones, 1994). One reason may be that the tools are just grafted onto an extant course, which is invariably necessary in political terms, or that the tools merely increase the gulf between the able and weak students (differential skill enhancement).

Another issue is the saturation level of options. On one hand the conventional graphics calculators – especially with dedicated programs installed – provide their own level of algebraic assistance at new levels. On the other hand algebraic software in computers, particularly Mathematica, Maple and Derive, provide powerful assistance in algebra for the able students in real terms. Ball (2001) notes that different algebraic packages provide quite different solution options to simple tasks like equation solution so even the type of CAS can be an issue. However, these software packages can all certainly be used to obviate much unnecessary algebraic manipulation and calculus in the same way as the classical tables of formulae and integrals did in previous decades.

Students in Swinburne engineering courses already use Mathematica from second year and a pilot program is introducing algebraic calculators into first year. In this context, it is interesting to consider what the real impact these tools have on existing assessment and how might first year engineering students benefit from access to these algebraic calculators. In the test examples supplied here we see clearly how the algebraic processes are radically simplified by use of a CAS. This is a report of work in progress and necessarily raises more questions than it answers at present.

There exist several brands of algebraic calculators on the market suitable for use in mathematics courses. Suitable models are available from Hewlett – Packard, Casio and Texas Instruments. These include the TI 92 and its relative TI 89 from Texas and the samples in this paper are drawn from this latter model.

Test Examples and Discussion

Engineering Mathematics 1 is a common subject for all engineering students in first year at Swinburne University of Technology. The students enter with a background in calculus and algebra from a base level subject, Mathematical Methods, in final school year (or its equivalent) and most also have studied an advanced level subject, Specialist Mathematics. They have been previously been allowed normal graphics calculators and given a brief formula sheet.

Topics include some error analysis and multi-base arithmetic, but the focus is on functions using algebra, graphs and calculus. There is a brief unit on statistics. In second semester discrete mathematics, matrices and vectors are studied along with differential equations, curves and calculus of functions of several variables.

The 2001 semester 1 paper contained 23 questions for a total of 180 marks. Questions include many involving traditional algebra and calculus. The calculator could help substantially in many questions on this examination – especially in the calculus area. The detail of some questions, contrasting the suggested traditional answer with the calculator is given as follows.

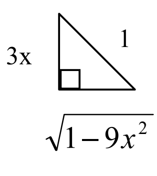
These questions totalled 120 marks – about 66% of the marks on the paper. The calculator could also have been used in other questions either to complete or check calculus or in traditional graphing and statistics. We begin by looking at sample questions from the previous examination with their classical and new solutions.

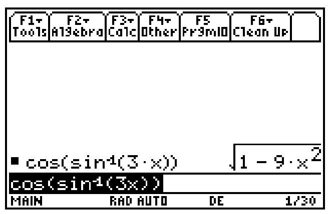
Topic: Trigonometric Algebra

Sample: Express $\cos(\sin^{-1}3x)$ as an algebraic expression in terms of x.

Let $\theta = \sin^{-1}3x$. so that $\sin\theta = 3x$.

We can draw a right triangle with $\sin\theta = 3x$





From the triangle, $\cos\theta = \cos(\sin^{-1}3x) = \sqrt{1-9x^2}$.

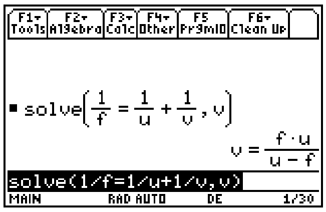
Discussion:

These questions have been designed to relate various trigonometric and inverse trigonometric functions. There are actual applications of course – a classical model of the daylight hours at a given latitude is a case in point – but its arguable whether this traditional approach is critical. Notice a diagram is the main aspect of solution here – the calculator cuts to the algebraic form directly. Students generally find these types of problems hard, possibly because they draw on various skills at once.

Transposition of Formulae

Sample: According to the lens formula, used in optics, $\frac{1}{f}=\frac{1}{u}+\frac{1}{v}$ where *f* is the focal length of a lens, *u* is the distance of an object from the lens, and *v* is the distance of the object's image from the lens. Rearrange the formula to make *v* the subject of the formula.

$$\frac{1}{f}=\frac{1}{u}+\frac{1}{v} \Rightarrow \frac{1}{v}=\frac{1}{f}-\frac{1}{u}$$
$$\Rightarrow \frac{1}{v}=\frac{u-f}{fu}$$
$$\Rightarrow v=\frac{fu}{u-f}$$



Discussion:

This is a necessary activity for formula use as it is too inconvenient to give every version of a formula. Algebraic manipulation in transposing formulae like this is legendary for generating errors as discussed in Tobin, (1998b).

If formula use is critical as well as knowledge of the meaning of formula elements then the algebra here has likely hindered weaker students in the past. In this optics example the solution is very direct. However use of a calculator for algebraic manipulation in practice can be a daunting effort at times as well. Consider this real example (drawn from the course notes) using the Darcy-Weisbach equation for turbulent flow. The students are asked to rearrange the formula to find the head loss due to friction, h_F.

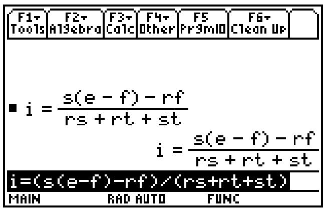
$$Q=-2{,}222D^{\frac{5}{2}}\sqrt{g\frac{h_F}{L}}\log_e\left(\frac{k}{3.7D}+\frac{4.1365}{\left(\frac{Q}{vD}\right)^{0.89}}\right)$$

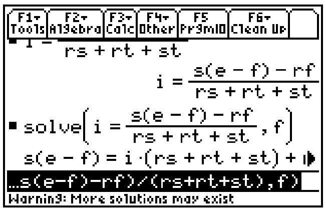
To enter such a formula takes time and substantial care on using real multiplication symbols not just expecting adjacent symbols to be read as multiplying. In such a case it is easier to do the transposition by hand! Another example on algebra gives the following equation from an electric circuit.

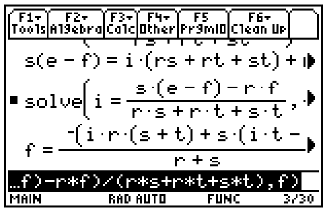
$$i_2 = \frac{R_2(\varepsilon_1 - \varepsilon_2) - R_1\varepsilon_2}{R_1R_2 + R_1R_3 + R_2R_3}$$

The students were asked to rearrange this to find ε_2.

Clearly we need to use new symbols and keep track of their one-to-one correspondence here. Suppose we use *r* for R_1, *s* for R_2 and *t* for R_3. We can use *e* for ε_1 and *f* for ε_2. Then using *i* for i_2 we may enter the equation in the calculator as shown. It all looks fine for use so we may try to solve for *f*. The result is shown in the middle screendump – the calculator has not picked out the appropriate multiplications. Expressly including the multiplication gives the correct result shown far right. This reminds us that relapsing into 'natural' writing can produce serious errors especially for a casual user!





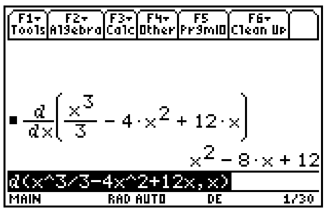


These examples certainly remind us of Ball's remarks that understanding is needed to use algebraic computing technology efficiently! Students need to know the essence of 'undoing' operations, and their place in equation solving. They also must add a new layer of understanding on the fine detail of the CAS itself and how variables may be defined and used.

Differential Calculus

Sample: Find the gradient and second derivative at any point on the graph of the function y = $\frac{x^3}{3} - 4x^2 + 12x$ and hence find maximum and minimum points of inflection on the graph.. Sketch the graph, showing all maximum and minimum points, point of inflection and intercepts.

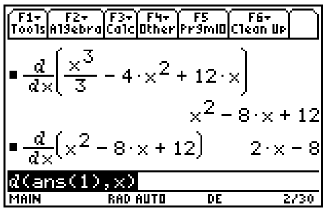
The components of interest here are the derivatives.



$$y = \frac{x^3}{3} - 4x^2 + 12x$$

$$\Rightarrow \frac{dy}{dx} = x^2 - 8x + 12 \text{ and } \frac{d^2y}{dx^2} = 2x - 8.$$

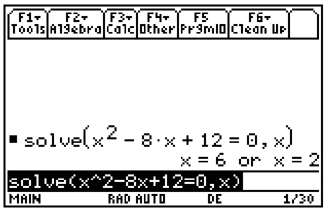
To locate maximum and minimum we set $\frac{dy}{dx} = 0$.



We can also use the calculator for the algebra.

$x^2 - 8x + 12 = 0 \Rightarrow (x - 6)(x - 2) = 0$ so x = 2 or 6.

(or solve by formula)



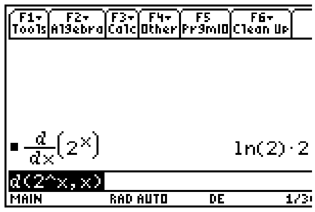
The rest of this question could also be solved by assistance of a conventional graphics calculator, although all those features exist on the CAS as well.

Discussion:

This problem requires students understand notions like gradient and point of inflexion and how these link to derivatives. This is not a simple black box - a calculator can only solve the components of the problem, with a good student using these to put together the elements in a solution. This is a suitable illustration of the calculator as tool and assistant rather than a replacement for thinking.

Sample: Find $\frac{dy}{dx}$ given that $y = 2^x$.

$y = 2^x$. Take natural logs both sides.



$\ln y = \ln 2x = x\ln 2$.

Differentiate with respect to x:

$$\frac{d\ln y}{dx} = \ln 2 \text{ so } \frac{d\ln y}{dy}\frac{dy}{dx} = \ln 2$$

$$\frac{1}{y}\frac{dy}{dx} = \ln 2 \text{ so } \frac{dy}{dx} = y\ln 2 = 2^x \ln 2.$$

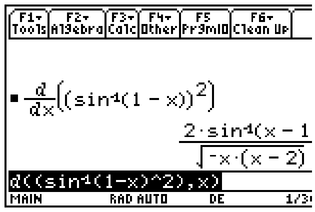
Sample: Given that $y = [\sin^{-1}(1-x)]^2$, find $\frac{dy}{dx}$.

$y = [\sin^{-1}(1-x)]^2$.

Let $u = \sin^{-1}(1-x)$.

Then $y = u^2$ and $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 2u\frac{du}{dx}$

Let $v = 1 - x$ so $u = \sin^{-1} v$.



$$\frac{du}{dx} = \frac{du}{dv}\frac{dv}{dx} = \frac{1}{\sqrt{1-v^2}}\,(\text{-}1) = \frac{-1}{\sqrt{1-v^2}}.$$

Hence $\dfrac{dy}{dx} = 2u\ \dfrac{du}{dx} = 2(\sin^{-1}(1\text{-}x))\left(\dfrac{-1}{\sqrt{1-(1-x)^2}}\right)$

$$= \frac{-2\sin^{-1}(1-x)}{\sqrt{1-(1-x)^2}} \text{ or } \frac{-2\sin^{-1}(1-x)}{\sqrt{x(2-x)}}$$

Discussion:

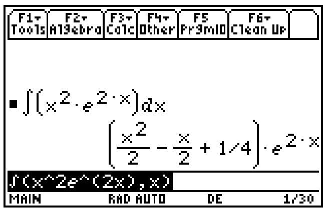
These provide a black box solution – one for a problem usually solved using tables and one for a chain rule derivative where sign problems can also occur. Problems such as this have often been set but there will likely be a decreased need for elaborate differentiation by hand in future. This is even more true of students in 'applied' rather than 'advanced' courses and reminds us that the algebra needs of all students need not be the same, bearing on the first issue raised in this paper.

Antidifferentiation

Sample: Find I = $\int x^2 e^{2x}\,dx$

Let $u = x^2$. Then $\dfrac{du}{dx} = 2x$. Also let $\dfrac{dv}{dx} = e^{2x}$ so $v = \dfrac{e^{2x}}{2}$. By integration by parts we have

I = $\dfrac{x^2e^{2x}}{2} - \int 2x\dfrac{e^{2x}}{2}dx$. Repeat integration by parts with $u = x$ and $\dfrac{dv}{dx} = e^{2x}$ so $v = \dfrac{e^{2x}}{2}$. Now

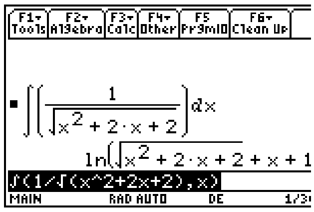


$$\text{I} = \frac{x^2e^{2x}}{2} - \left\{\frac{xe^{2x}}{2} - \int\frac{e^{2x}}{2}dx\right\}$$

$$= \frac{x^2e^{2x}}{2} - \frac{xe^{2x}}{2} + \frac{e^{2x}}{4} + c$$

Sample: Find I = $\displaystyle\int \frac{dx}{\sqrt{x^2+2x+2}}$

$$\mathrm{I} = \int \frac{dx}{\sqrt{x^2+2x+1+1}} = \int \frac{dx}{\sqrt{(x+1)^2+1}}$$



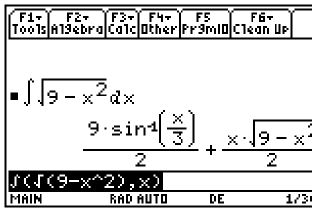
Let $u = x + 1$. Then $\frac{du}{dx} = 1$ or $du = dx$.

So $\mathrm{I} = \int \frac{du}{\sqrt{u^2+1}} = \sinh^{-1}u + \mathrm{c}$

$= \sinh^{-1}(x+1) + \mathrm{c}$ or $\ln\,[(x+1)+\sqrt{(x+1)^2+1}\,] + \mathrm{c}$

Sample: Find $\int \sqrt{9-x^2}\,dx$, and hence or otherwise find $\int_{-3}^{3} \sqrt{9-x^2}\,dx$

I = $\int \sqrt{9-x^2}\,dx$. Let x = 3sinθ so that dx = 3cosθdθ.



Then I = $\int \sqrt{9-9\sin^2\theta}\,3\cos\theta d\theta$

$= \int \sqrt{9(1-\sin^2\theta)}\,3\cos\theta d\theta$

$= \int 3\sqrt{(1-\sin^2\theta)}\,3\cos\theta d\theta$

$= \int 3\sqrt{(\cos^2\theta)}\,3\cos\theta d\theta$

$= \int 3\cos\theta\, 3\cos\theta d\theta$

$= 9\int \cos^2\theta d\theta$

$=9\int \frac{1}{2}(1+\cos 2\theta)d\theta$

$= \frac{9}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] + c$ for c a constant

$= \frac{9}{2}\left[\sin^{-1}\frac{x}{3} + \frac{2\sin\theta\cos\theta}{2}\right] + c$

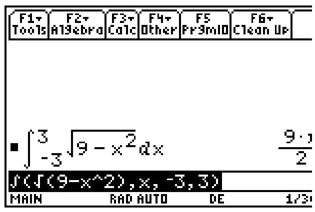
$= \frac{9}{2}\left[\sin^{-1}\frac{x}{3} + \sin\theta\cos\theta\right] + c$

But sinθ = $\frac{x}{3}$ so cos θ = $\frac{\sqrt{9-x^2}}{3}$

Hence I = $\frac{9}{2}\left[\sin^{-1}\frac{x}{3} + \frac{x}{3}\frac{\sqrt{9-x^2}}{3}\right] + c$

Then $\int_{-3}^{3}\sqrt{9-x^2}\,dx = \frac{9}{2}\left[\sin^{-1}\frac{x}{3} + \frac{x}{3}\frac{\sqrt{9-x^2}}{3}\right]_{-3}^{3}$

$= \frac{9}{2}\left[\sin^{-1}(1)+0\right] - \frac{9}{2}\left[\sin^{-1}(-1)+0\right] = \frac{9}{2}\left(\frac{\pi}{2} - (\frac{-\pi}{2})\right) = \frac{9\pi}{2}$



Discussion:

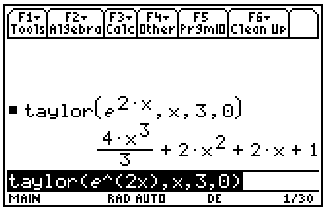
These problems illustrate more difficult antiderivatives. Students find integration by parts particularly hard. In both cases there is a black box approach afforded by the algebraic calculator. As with the previous derivative problems, it is arguable how much time we need to have students learn these manipulation techniques in more general level courses. This is particularly true when students have been commonly given extensive tables to assist them in the past - the calculator merely extends this help.

Polynomial Approximation

Find the Taylor polynomial of order 3 which approximates the function $f(x) = e^{2x}$ about $x = 0$.

$f(x) = e^{2x}$ so $f(0) = e^0 = 1$

$f'(x) = 2e^{2x}$ so $f'(0) = 2e^0 = 2$



$f''(x) = 4e^{2x}$ so $f''(0) = 4e^0 = 4.$

$f'''(x) = 8e^{2x}$ so $f'''(0) = 8e^0 = 8.$

Then for a third order Taylor expansion,

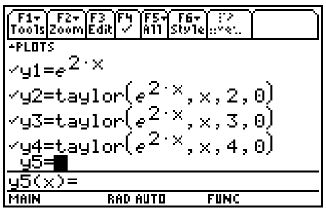
$$f(x) = f(0) + x f'(0) + \frac{x^2}{2} f''(0) + \frac{x^3}{3!} f'''(0) \text{ approx.}$$

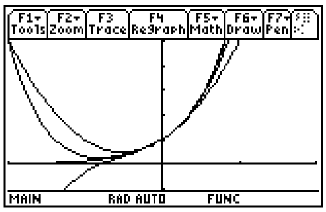
or $p_3(x) = 1 + 2x + 4\frac{x^2}{2} + 8\frac{x^3}{3!} = 1 + 2x + 2x^2 + \frac{4}{3}x^3$

Discussion

Polynomial approximations give some insight into function behaviour, which can be useful. The actual need for approximations is now diminishing as technology enable the original (possibly transcendental) functions to be used more readily. Consideration of when and how we apply approximations could actually be extended of course. Current courses usually focus on polynomial approximations but other approaches such as rational function approximations (Pade approximations) have been useful in the past but given little syllabus time. These could be included in a future syllabus.

It is also possible to use a number of easily generated Taylor polynomials to illustrate the power of higher order models in extending the range of use. This is illustrated following where Taylor polynomials of degree 2, 3 and 4 are used to approximate the function and graphs are provided.





Partial Fractions

Express $\dfrac{5x^2-9x+2}{(x-1)^2(x-2)}$ as partial fractions.

Let $\dfrac{5x^2-9x+2}{(x-1)^2(x-2)} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x-2}$ (1)

Multiplying (1) by $(x-1)^2(x-2)$ we get

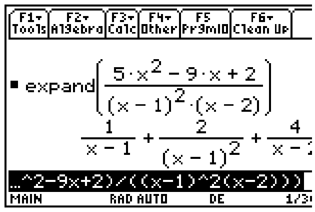
$5x^2 - 9x + 2 = A(x-1)(x-2) + B(x-2) + C(x-1)^2$ (2)

Substituting $x = 1$ in (2):

$5 - 9 + 2 = -B$ so $B = 2$ (3)

Substituting $x = 2$ in (2):

$20 - 18 + 2 = C$ so $C = 4$ (4)



By (3) and (4) in (2)

$5x^2 - 9x + 2 = A(x-1)(x-2) + 2(x-2) + 4(x-1)^2$

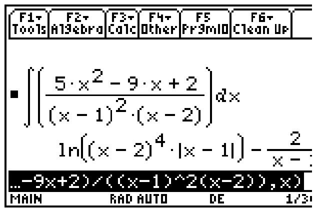
Substituting for $x = 0$: we get $2 = A(-1)(-2)+2(-2)+4(-1)^2$

So $2A - 4 + 4 = 2$ and $A = 1$.

Hence

$$\frac{5x^2-9x+2}{(x-1)^2(x-2)} = \frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{4}{x-2}$$

Hence find $\displaystyle\int \frac{5x^2-9x+2}{(x-1)^2(x-2)}\,dx$



Although the question asks that this be done as a consequence of part (a) the integral can be solved directly and our usual recourse to partial fractions becomes redundant.

Discussion:

Partial fractions have been commonly taught to provide for integration of rational functions …this technology radically simplifies the integration. There do not seem to be valuable modelling issues lost by this simplification - at least at this level. Few problems call for summing rational expressions and fewer still for re-expressing in such sum forms.

Curriculum Issues

The preceding examples demonstrate that tests on facts and skills in mathematics will be substantially affected by access to algebraic calculators. The use of these tools in calculus is much more significant than prior access to tables or conventional graphics calculators could achieve. The issue of how much understanding a student needs to have becomes important as many traditional problems take on ‘black box’ solutions. A student needing a three-year mathematics course and using this in other subjects has different needs from a student who has a one-year course, which does not directly feed too many higher year subjects thereafter.

The examples suggest that the calculators can make the most direct impact in calculus. There is a strong case for reducing the time spent on generating every type of derivative and antiderivative from rules and focussing more on the meaning and application of these notions. This is not very different from making tables of formulae available, and is particularly justifiable for students in more general courses. On the other hand, the algebraic work in manipulating equations can be so convoluted that the calculator may be a source of difficulty for the student rather than an assistant as the examples reveal. Of course, this requires that the students can perform algebra efficiently by hand. It is likely that an able mathematician will commonly do better than a CAS machine user on a range of algebraic and calculus features but this may not be true of the average user.

Thomas (2001) has discriminated the use of CAS in two forms - as a conceptual process representation tool (CPRT) and a conceptual object representation tool (CORT). The examples given in this paper illustrate the CPRT form, which uses the CAS to ease a process. This is common for assessment problems. Use of the CAS in CORT form requires more focus on the object. In this context it can be more a learning tool than assessment tool. We can for example, demonstrate features such as symmetry in a graph, or show how successive Taylor approximations perform on modelling a function locally, rather than directly obtaining such an approximation, as illustrated in the discussion on the Taylor polynomial example.

Collectively these conceptual representation tools or CRTs may provide the engine for a new approach to learning higher mathematics. In practice what we aim to achieve will remain the same – we examine common function forms in graphical, algebraic and numerical terms and use the results for modelling real situations and solving abstracted problems. A CAS tool has access to all these three aspects of course, so students can examine all aspects of a problem at once. Conventional graphics calculators have limited analysis directly to graphical and numerical aspects, although these have often fed algebraic interpretations too of course. However we can now see that some calculus procedures at least may be better automated which impinges on curriculum issues as a substantial time is currently spent on learning techniques a CAS machine can do instantly and accurately and with likely no loss of understanding of the basic notion. At this level of operation we are doing no worse for calculus than removing the square root process does for arithmetic!

This curriculum goal has always been hard to achieve, but given more time released from routine work it may be made easier. Modelling of real problems can be enhanced by CAS of course. For example, traditional exam questions which asked a student to set up but not solve a DE to model a situation could now require that solution be obtained and outputs discussed for numerical validity – thus (say) asymptotic behaviour could be discussed when the solution is at hand.

The issue of different CAS systems raised in the introduction, has not been addressed here but can be important. Consider the simple problem of solving a cubic equation. A CAS procedure like Mathematica will generate exact solutions, often with such a convoluted form that the real solution(s) are not very visible directly. The Derive package in the TI-89 actually looks for simpler exact solutions but if the problem does not admit these conveniently, it reverts to giving numerical approximates directly when coefficients are numbers. This is quite a reasonable option for the user of course but it places the algebraic solver on these calculators on the same level as the numerical solvers of the traditional graphics calculators.

Far more mathematics including engineering mathematics is now data driven. Statistics are more important to use now because there are ready tools for number crunching and modelling. As the models can be automated, it becomes important to dwell on the meaning of the models, their range of use and limitations. In practice this may mean that a test question on regression should not be simply asking for a linear model to be found and maybe used in a given context. Far better to require students to decide if the model is suitable and how it can be used. They may be required to consider residuals, nonlinearity and other matters, which are not mere numerical outputs from a calculator. In this way we ensure that no calculators can take away conceptual understanding.

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