CONVEX SETS AND HEXAGONS

Ji GAO
Department of Mathematics, Community College of Philadelphia
1700 Spring Garden Street, Philadelphia, PA 19130-3991
e-mail: jgao@ccp.cc.pa.us

ABSTRACT

Euclid presented his fundamental results about 300 B.C., but Euclidean Geometry is still alive today. We studied the new properties of convex sets and its inscribed hexagons in a two dimensional Euclidean space. As an application, these results solved a question in Geometry of Banach Spaces. From my teaching experience at Community College of Philadelphia, I think the material is reasonable and suitable to be added to the Linear Algebra course and/or Functional Analysis course. It may encourage others to know that the tools we give our students remain useful in modern research.

1 Introduction

In [1], we used elementary geometry to discuss the properties of the rhombi inscribed in the unit circle C of a two dimensional normed vector space, and proved that the well-known property from Euclidean geometry, namely that every rhombus inscribed in unit circle C has sides of C-length $\sqrt{2}$, does not characterize the Euclidean space. The result is that if the curve C of unit vectors is invariant under rotation by 45^o, then every rhombus inscribed in C has sides of C-length $\sqrt{2}$. In the first part of this paper we still use elementary geometry to discuss the properties of so-called normal hexagons inscribed in the unit circle C of a two dimensional normed vector space, and we consider another well-known property from Euclidean geometry, namely that every normal hexagon inscribed in an unit circle C has side-medians of C-length $\frac{\sqrt{3}}{2}$. However, we also prove that this property does not characterize the Euclidean space either. By using the term side-median for a polygon inscribed in the unit circle C of a normed vector space, we mean the median of the triangle with the origin as a vertex and a side of the polygon as base. In the second part of this paper, which is an appendix, we present more properties of rhombi inscribed in the unit circle C we discussed in [1].

2 Inscribed Hexagons

As we have already shown in [1]: we can use any bounded convex set which is symmetric with respect to the origin and contains the origin as an interior point in a two dimensional Euclidean space to define a new norm. On the other hand, the unit disk of any normed vector space is a bounded convex set which is symmetric with respect to the origin and contains the origin as an interior point.

Definition: A hexagon in a normed vector space with unit circle C is called a normal hexagon if it has six sides of same C-length, and each pair of opposite sides are parallel. The normal hexagon is called a unit normal hexagon if it has six sides of C-length 1.

The unit circle of the standard Euclidean space E^2 is a standard circle, and there is unique regular hexagon inscribed in the standard circle with a given point on the standard circle as the one of its vertices. From [2], for any invertible matrix A we can define an inner product on E^2 by $< x, y >= Ax \cdot Ay$, and every inner product arises in this way. Under the linear isometry $x \longrightarrow A^{-1}x$, the image of the standard Euclidean unit circle is the unit circle C of unit vectors with respect to the inner product, which is an ellipse, and the image of any regular hexagon inscribed in the Euclidean circle is a normal hexagon inscribed in this ellipse C. Since the unique regular hexagon in the standard Euclidean circle has sides of Euclidean length 1, and six side-medians of Euclidean length $\frac{\sqrt{3}}{2}$, it follows that the unique normal hexagon inscribed in an ellipse C with a given point as one of its vertices has sides of C-length 1, and side-medians of C-length $\frac{\sqrt{3}}{2}$.

The question is: does the property above characterize the Euclidean space? That is, if a normed vector space has the property that every normal hexagon inscribed in C of unit vectors has side-medians of C-length $\frac{\sqrt{3}}{2}$, does the norm arise by an inner product?

Observe that the two dimensional standard Euclidean space E^2 and a two dimensional normed vector space with C as its unit circle are set up in the same plane. In the following, for a given vector x in the plane we use $|x|$ to denote the general Euclidean length (in the Euclidean space) and $\|x\|_C$ to denote the C-length (in the normed vector space). Let K and $C = \partial K$ be the unit disk and unit circle of the two dimensional normed vector space respectively, then both K and C are symmetric with respect to the origin, in addition K is a convex set with the origin as an interior point. So, geometrically the question above is equivalent to the following question: if a convex set K, which is symmetric with respect to the origin and contains the origin as an interior point (and therefore $C = \partial K$ could be the unit sphere of some normed vector space), has the property that every normal hexagon inscribed in $C = \partial K$ has side-medians of C-length $\frac{\sqrt{3}}{2}$, must C be an ellipse in E^2 (Therefore $C = \partial K$ should be the unit sphere of an Euclidean space)?

To answer this question we need the following results.

Let T be a tangent line of K, then $T \cap K = T \cap C$ is either a single point or a line segment with $||T \cap K||_C = ||T \cap C||_C \leq 2$.

Lemma 1: Let $x \in C$, T be the tangent line parallel to the vector x, and L be a line parallel to x too. Then when L moves parallel from the position passing through the origin towards T, the $||L \cap K||_C$ is non-increasing from 2 to $||T \cap K||_C = ||T \cap C||_C$. Furthermore, for any a, where $||T \cap K||_C = ||T \cap C||_C \leq a < 2$, there is unique $u \in C$ and corresponding $v \in C$ such that vector $u - v$ is parallel to x, and $||u - v||_C = a$.

Proof: Let L_1 moves parallel to L_2 towards T, and $u_1, v_1 \in L_1 \cap C, u_2, v_2 \in L_2 \cap C$ (see Figure 1). If $||u_2 - v_2||_C > ||u_1 - v_1||_C$, or $||u_2 - v_2||_C \geq ||u_1 - v_1||_C < 2$, then at least one of u_1, v_1 falls inside the trapezoid with vertices $-x, x, u_2$, and v_2.

This contradicts the convexity of K. Therefore $||u_2 - v_2||_C \leq ||u_1 - v_1||_C$, or when $||u_1 - v_1||_C < 2, ||u_2 - v_2||_C < ||u_1 - v_1||_C$.

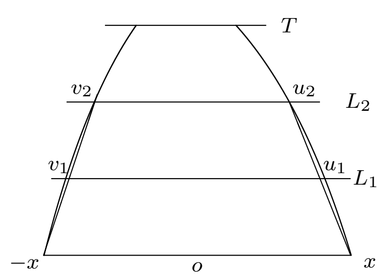

Figure 1:

Lemma 2: Let $x \in C$, then there exists at least one normal hexagon inscribed in C with x as one of vertices.

Proof: Let T be the tangent line to C, and parallel to x. If $||T \cap K||_C = ||T \cap C||_C \leq 1$ (see Figure 2), from lemma 1 we can take $u, v \in C$, such that $u - v$ is parallel to x, and $||u - v||_C = 1$. From parallelograms with vertices u, v, o, and x, and vertices $u, v, -x$, and o, we have $||u - x||_C = ||v||_C = 1$, and $||v - (-x)||_C = ||u||_C = 1$. So, the hexagon with vertices $x, u, v, -x, -u$, and $-v$ is an inscribed normal hexagon.

¿From lemma 1 again there is unique u and corresponding $v \in C$ such that $u - v$ is parallel to x, and $||u - v||_C = 1$. So, in this case the inscribed hexagon with x as one of vertices is unique.

If $||T \cap K||_C = ||T \cap C||_C > 1$ (see Figure 3), we can take infinite many pairs of $u, v \in T \cap K = T \cap C$ such that $u - v$ is parallel to x, and $||u - v||_C = 1$. So, in this

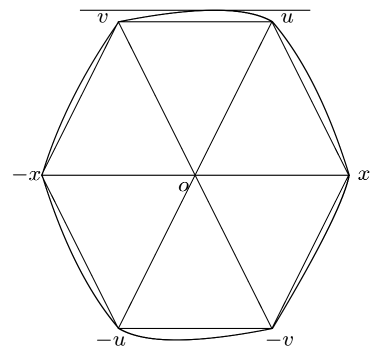

Figure 2:

case there are infinite many normal hexagons inscribed in C.

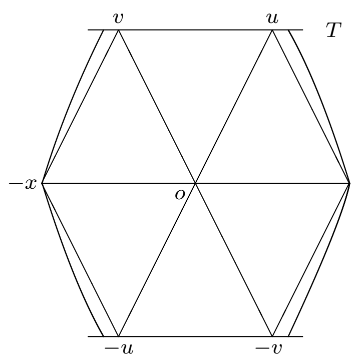

Figure 3:

Consider a normed vector space in a plane with a normal hexagon as the unit circle C, then the normal hexagon itself is an inscribed normal hexagon in C. It has side-medians of C-length 1, but it is not a inner product space.

Lemma 3: Let x be a vector in C, and $x, u, v,$ and $-x$ in C are counterclockwise located, then $||v - x||_C \geq ||u - x||_C$, and $||v - (-x)||_C \leq ||u - (-x)||_C$.

Proof: Let u' and v' be the normalizations of $u - x$, and $v - x$ respectively. Then u' and $v' \in C$. If $u' = v'$, then $u, v,$ and x are colinear. So $||v - x||_C \geq ||u - x||_C$. Otherwise x, u', v', and $-x$ are counterclockwise located too (see Figure 4).

Case 1: If the line L_v passing through v and v' intersects the line L_x through $-x$ and x at a point Q, and Q is on left side of $-x$, then $||v - x||_C > 1$ (see Figure 5). The L_u passing through u and u' is either parallel to the line L_x (in this case $||u - x||_C = 1$, therefore $||v - x||_C \geq ||u - x||_C = 1$), or intersects L_x at a point P. If P is on the right side of x, then $||u - x||_C < 1$ (therefore $||v - x||_C \geq ||u - x||_C$). If P is on the left side of $-x$, then from the convexity of K, P must be on the left side of Q. By considering similar triangles with vertices u, x, P and vertices u', o, P, we have $||u - x||_C = \frac{|Px|}{|Po|}$. Similarly from similar triangles with vertices v, x, Q and vertices v', o, Q, we have $||v - x||_C = \frac{|Qx|}{|Qo|}$. Since $\frac{|Px|}{|Po|} \leq \frac{|Qx|}{|Qo|}$, we have $||v - x||_C \geq ||u - x||_C$.

Case 2: If line L_v is parallel to line L_x , then $||v - x||_C = 1$ (see Figure 6). ¿From convexity of K, the line L_u either intersects line L_x on the right side of x, or L_u is

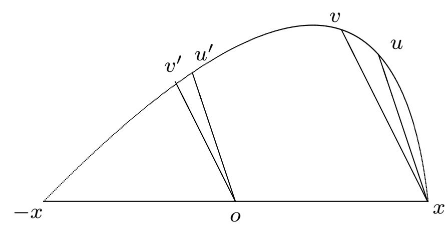

Figure 4:

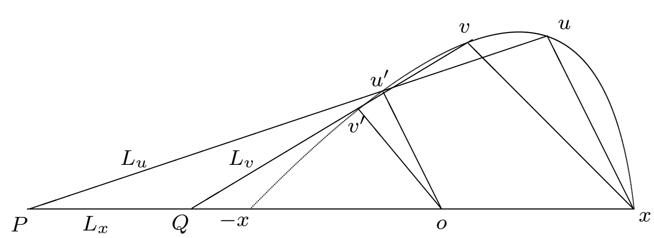

Figure 5:

parallel to L_x so $||u-x||_C \leq 1$. We still have $||v-x||_C \geq ||u-x||_C$.

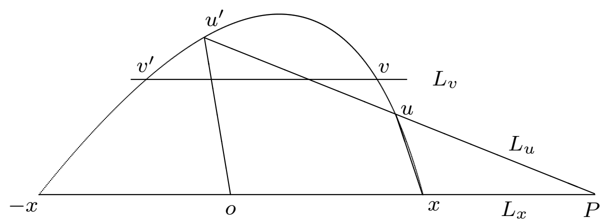

Figure 6:

Case 3: If the point Q, the intersection of line L_v and line L_x is on the right side of x, then $||v-x||_C \leq 1$ (see Figure 7). From convexity of K again, the point P, the intersection of L_u and line L_x, is either on the left side of Q, or coincides with Q. Similar to case 1, by considering the similar triangles we have $||v-x||_C = \frac{|Qx|}{|Qo|} \geq \frac{|Px|}{|Po|} = ||u-x||_C$.

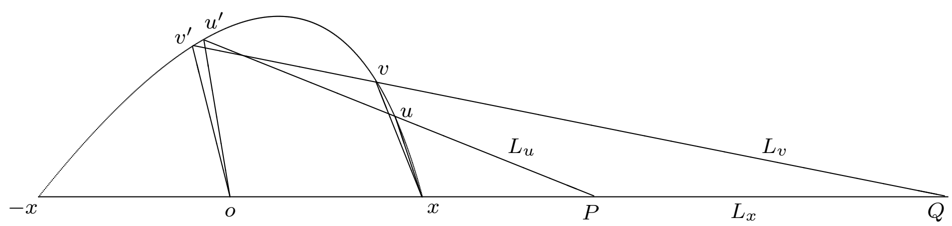

Figure 7:

Similarly, we can prove $||v-(-x)||_C \leq ||u-(-x)||_C$. The proof of lemma 3 is completed.

Lemma 4: If the curve C of unit vectors is invariant under rotation by 30^o, then C does not contain any line segment with C-length greater then or equal to 1.

Proof: Suppose $u, v' \in C$ such that $||v'-u||_C \geq 1$, and the line segment L connecting u and $v' \subseteq C$. If $\angle v'ou < 30^o$, take a vector v such that $\angle vou = 30^o$, and $|ov| = |ou|$, then $v \in C$, and by lemma 3 $\|v-u\|_C \geq \|v'-u\|_C \geq 1$. If $\angle v'ou \geq 30^o$, take $v \in L$ such that $\angle vou = 30^o$. From the hypothesis, $|ov| = |ou|$, and the line segment $[u, v]$ connecting u and v coincides with L. So we have $v' = v$ and therefore $\|v-u\|_C \geq 1$. Let $w = \frac{u+v}{2}$, then $w \in K$, and $||w||_C \leq 1$ (see Figure 8). Let $t = v-u$, then $||t||_C = ||v-u||_C \geq 1, |t| = |u-v| = 2|u|\sin 15^o$, and the angle between t and w is 90^o. Let s be the image of rotating w couterclockwise by 90^o, then $||s||_C \leq 1$, and $|s| = |w| = |u|\cos 15^o$. Since $\cos 15^o > 2\sin 15^o$, we have $|s| > |t|$. But $||t||_C \geq ||s||_C$. This is a contradiction. The proof is completed.

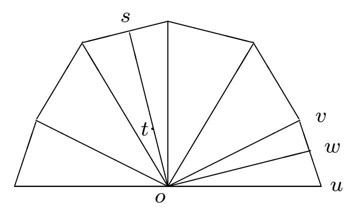

Figure 8:

Theorem 1: If the curve C of unit vectors is invariant under rotation by 30^o, then every normal hexagon inscribed in C has side-medians of C-length $\frac{\sqrt{3}}{2}$.

Proof: Let u be a vector in C. Since C is invariant under rotation by 30^o, C does not contain any line segment with C-length greater than or equal to 1. Therefore the normal hexagon inscribed in C with u as one of its vertices is unique (lemma 2). Let u_1, u_2, u_3, and u_4 be the vectors obtained by turning u counterclockwise by successive steps of 30^o (see Figure 9). Then the hexagon with vertices $u, u_2, u_4, -u, -u_2$, and $-u_4$ is the unique normal hexagon inscribed in C.

We have $||\frac{u+u_2}{2}||_C = \frac{|\frac{u+u_2}{2}|}{|u_1|} = \frac{|\frac{u+u_2}{2}|}{|u|} = \frac{\sqrt{3}}{2}$. Similarly, we have $||\frac{u_2+u_4}{2}||_C = ||\frac{u_4+(-u)}{2}||_C = ||\frac{(-u)+(-u_2)}{2}||_C = ||\frac{(-u_2+(-u_4)}{2}||_C = ||\frac{(-u_4)+u}{2}||_C = \frac{\sqrt{3}}{2}$. The proof is completed.

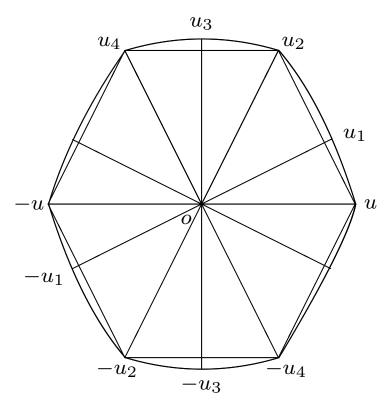

Figure 9:

So ellipses are not only curves C with the property that every inscribed normal hexagon in C has side-medians of $\frac{\sqrt{3}}{2}$. A regular polygon with 12n sides in particular a regular twelvegon will satisfy the condition. Therefore the image of any regular polygon with 12n sides under any invertible linear map has this property too. Equivalently we have proved that the property that every normal hexagon inscribed in C of unit vectors has side-medians of C-length $\frac{\sqrt{3}}{2}$ does not characterize the Euclidean space.

3 Appendix

In the second part of this paper we study more properties of inscribed normal parallelograms, rhombi, in the unit circle C.

We have already proved the uniqueness of rhombus inscribed in the curve C of unit vectors with a given point of C as a vertex in [1]. Now we prove the existence of this kind of rhombus.

Theorem 2: There is a rhombus inscribed in C of unit vectors with a given point of C as a vertex.

Proof: Let $x \in C$, from lemma 3 when u moves from x to $-x$ counterclockwise, $||u - x||_C$ continuously increases from 0 to 2, and $||u - (-x)||_C$ continuously decreases from 2 to 0. So there exists $y \in C$, such that $||y-x||_C = ||y-(-x)||_C$. The Parallelogram with vertices $x, y, -x$, and $-y$ is a rhombus inscribed in C, with a given point x as a vertex.

Finally, by combining theorem 1 of [1] and the theorem 2 above, we have the following theorem.

Theorem: There is one and only one rhombus inscribed in C, with a given point in C as a vertex.

4 Discussion

In this paper, the question we posed: a conjecture about the characteristic of Euclidean spaces belongs to the subject of the Geometric Functional Analysis. All figures which appeared: hexagons, circles, ellipses, symmetric convex sets belong to Elementary Geometry, the course students studied at high schools and/or in a freshman level at colleges. The concepts and methods which we need to prove the lemmas and main theorem: linear vector spaces, norm and normed vector spaces, Euclidean spaces, and linear transformations belong to Linear Algebra, the course we are teaching. Based on the knowledge in Elementary Geometry, all the concepts and methods about linear spaces and linear transformations, which make one of the most important parts of the Linear Algebra course are needed to prove the lemmas and main theorem. After my lectures students learned that the basic figures in Elementary Geometry have meaning in the Geometry of Banach spaces they never imagined: different Ellipses are unit spheres of different Euclidean spaces, and different symmetric convex sets are unit spheres of different normed spaces and so on. And students also learned that the concepts, methods and results in Linear Algebra course are useful and powerful in proving results in more advanced mathematical courses. The students told me that they understood better and deeper what the definitions of the abstract spaces really mean, relations among topics in the different chapters of the course, and learned how to think mathematically, and how to use their knowledge in practice. They also told me that they were inspired by my lectures to do research, and they recognized the tools they acquired in the classroom remain useful in modern research.

So lectures on this subject in my Linear Algebra course help students to review the Elementary Geometry, to enhance the understanding of the Linear Algebra course, and encourage them to study Real and Functional Analysis in the future. I think the material of this paper is suitable and reasonable to be added to current Linear Algebra course and/or Functional Analysis course.

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