A QUASI-QUALITATIVE APPROACH TO LIMITS

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ABSTRACT

The classical definition of limit of a function involving 'epsilon and delta' is not readily understood by students studying calculus for first time. Though teaching/learning calculus from Non-standard models of number system and infinitesimals is relatively easier , it is not widely practised. Under these circumstances increased use of Landau symbols is suggested. This will promote a greater qualitative understanding of limits and the rate of growth of functions.

1. Introduction

Every teacher of basic calculus experiences some difficulty in communicating the concept of limit of a function / sequence to students learning the subject for the first time. This is not surprising, for the precise formulation of the concept of limit of a function or sequence eluded the best of mathematical minds for centuries. More specifically, Cauchy's treatment of the concept of limit in his *Analyse algebrique* (1821) and subsequent treatment of calculus in his *Lecons sur le calcul infinitesimal* (1823 and 1829) was considered an enormous advance over the exposition of Newton and Leibnitz. However, Cauchy's definition of limit (and continuous function) was discontinued after 1880 when Heine and Weierstrass formulated the modern ε - δ definition. Cauchy was not always rigorous and he did not distinguish between continuity and uniform continuity. His discussion of power series reveals that Cauchy treated pointwise and uniform convergence of the functional series without distinction. Later in 1872 Weierstrass shocked the mathematical world with an example of a nowhere differentiable everywhere continuous function when his predecessors and contemporaries thought otherwise. In the light of these historical developments it is clear that the concepts of limit, continuity and differentiability of functions and their interrelationship are intrinsically abstruse. It is but natural that students find these concepts hard to comprehend. For an enlightening discussion of the difficulties involved in providing formal approaches to these concepts, Artigue [1] may be referred.

2. Diverse (equivalent) definitions of limit and continuity

The well-known Weierstrassian 'ε - δ definition' of continuity of a function at a point is only one of the several options for formulating this idea precisely. A slight variation, based on the concept of a neighbourhood of a point (real number) makes the definition qualitative (and topological). In terms of neighbourhoods the continuity of f at x_0 can be restated as follows :

for each neighbourhood V of $f(x_0)$ there is a neighbourhood U of x_0 such that $f(U) \subseteq V$.

Based on the concept of convergence of real sequences, continuity of a function f at x_0 can be viewed as a property of regularity that requires the convergence of $f(x_n)$ to $f(x_0)$ whenever the sequence (x_n) converges to x_0.

Around 1960, Abraham Robinson validated the use of infinitesimals in calculus by means of his Non-standard Analysis. Since then, there have been numerous attempts to simplify Nonstandard Analysis and make it more accessible to undergraduate and high school students. The purpose of such attempts is to retain the intuitive approach of Newton and Leibnitz based on infinitesimals without compromising on rigor. At the same time these soften Robinson's original metanumerical foundations of the real number system. Notable among such contribution are those due to Kinsler, Schwarzenberger and Tall. Such a non-standard formulation leads to simpler algebraic treatment of problems of calculus.

3. A quasi-qualitative approach

Concepts of nets and filters suffice to investigate questions of convergence in a general topological context. However, from a pedagogical point of view, it is impracticable to introduce these abstract concepts at the undergraduate level. Besides, Non-standard methods of calculus, despite their merits are not widely practised. However, the' $\varepsilon - \delta$ approach' is still in vogue at the undergraduate level. Nevertheless it is worthwhile to expose students to qualitative methods of studying limits, even if the concept of limit is defined after Weierstrass using epsilons and deltas. In the sequel, a procedure embodying such a quasi-qualitative approach is outlined. It is quasi-qualitative as it is based on classical 'ε - δ definition' of a limit. This is exemplified by the systematic use of the three Landau symbols defined below.

Definition 3.1

Let (x_n) and (y_n) be sequences of real numbers, where $y_n > 0$.

(i) If there is a constant K such that $|x_n| \leq K y_n$ for all n one writes $x_n = \boldsymbol{O}(y_n)$;

(ii) If $\lim_{n\to\infty} \frac{x_n}{y_n} = 0$, one writes $x_n = \boldsymbol{o}(y_n)$;

(iii) If $\lim_{n\to\infty} \frac{x_n}{y_n} = 1$, one writes $x_n \sim y_n$

The symbols ***o*** , ***O*** and ~ above are usually called Landau symbols as the German mathematician Landau (1877 – 1938) was the first to use the ***o, O*** notations systematically. But Landau himself attributes this notation to Paul Bachman (1837-1920). P.Du Bois-Reymond (1831-89) had earlier used a notation that included the symbol ~ defined above for comparing the rates of growth of two increasing functions tending to infinity.

For functions defined in a neighbourhood of zero, one has the following

Definition 3.2 Let f,g be two real-valued functions defined in a neighbourhood of zero and suppose g is non-zero in that neighbourhood.

(i) If $|f(x)| \leq K g(x)$ for some K> 0 for all x with $|x| < \delta$, then one writes $f = \boldsymbol{O}(g)$ as $x \to 0$;

(ii) If $\lim_{x\to 0} \frac{f(x)}{g(x)} = 0$, then one writes $f = \boldsymbol{o}(g)$ as $x \to 0$;

(iii) If $\lim_{x\to 0} \frac{f(x)}{g(x)} = 1$, then one writes $f \sim g$ as $x \to 0$;

These growth conditions can also be considered when the independent variable x tends to infinity. Landau symbols have been discussed in the text-books of Burkill and Burkill [2] and Hardy [3] and are used extensively in analytic number theory. Early training in the use of these symbols helps students acquire of qualitative understanding of the relative growth of functions. Table 1 displays some well-known limits and their quasi-qualitative versions (in terms of the Landau symbols) :

Students can be encouraged to reformulate problems on limits quasi-qualitatively using Landau symbols, in the spirit of the above table of limits.

Some basic properties of Landau symbols are presented below in the form of a theorem (see Hardy [3]).

Theorem 3.1_ Let f and g be real-valued functions defined in a neighbourhood of zero. Then as $x \to 0$,

(a) $\boldsymbol{O}(f) + \boldsymbol{O}(g) = \boldsymbol{O}(f+g)$;

(b) $\boldsymbol{O}(f)\,\boldsymbol{O}(g) = \boldsymbol{O}(fg)$;

(c) $\boldsymbol{O}(f)\,\boldsymbol{o}(g) = \boldsymbol{o}(fg)$;

(d) If $h \sim g$, then $h + \boldsymbol{o}(f) \sim f$.

The above result is true even when f , g and h are considered real sequences. Students must be cautioned to note that $\boldsymbol{o}(1) = \boldsymbol{O}(1)$ will not, in general, imply $\boldsymbol{O}(1) = \boldsymbol{o}(1)$. For instance, as $x \to 0$, $\sin x = \boldsymbol{O}(\cos x)$, while $\cos x \neq \boldsymbol{o}(\sin x)$, though $\sin x=\boldsymbol{o}(1)$ and $\cos x= \boldsymbol{O}(1)$.

These symbols can be profitably employed to define differentiability of functions, as in the following

Definition 3.3 Let $f : D \subseteq \mathbf{R} \to \mathbf{R}$ be a function and x_0 an interior point of D. f is said to be differentiable at x_0 if there is a number $f'(x_0)$ such that

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \boldsymbol{o}(h) \text{ as } h\to 0 \qquad (\text{I})$$

Formula (I) , essentially due to Weierstrass , is often called first order Taylor's formula and can be readily extended to real-valued functions of n-variables and to vector-valued functions of n - variables. For $f : D \subseteq \mathbf{R}^n \to \mathbf{R}$ and $x_0 = (x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(n)})$ an interior point of D and $h = (h_1,h_2, \ldots,h_n)$ (I) can be modified as

$$f(x_0+h) = f(x_0) + \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(x_o) + o(\|h\|) \text{ as } h \to 0 \qquad (\text{II})$$

Here $\frac{\partial f}{\partial x_i}$ are first-order partial- derivatives of f and $\| h \| = \sqrt{\sum_{i=1}^{n} h_i^2}$.

Use of first order Taylor's formula and Landau symbols leads to a quick proof of the chain rule (see Rudin [4]). It also clarifies the ideas underlying the proof of L' Hospital's rule . As a sample we have

Theorem 3.2 Let $f , g : D \subseteq \mathbf{R} \to \mathbf{R}$ **,** where x_0 is an interior point of D. Suppose f , g , f ′ and g ′ are defined at x_0 and $f(x_0) = g(x_0) = f'(x_0) = g'(x_0) = 0$. If $g''(x_0) \neq 0$, then

$$\lim_{h\to 0}\frac{f(x_0+h)}{g(x_0+h)} = \frac{f''(x_0)}{g''(x_0)}$$, h being a real number sufficiently small in absolute value.

Proof : Using (I), as $h \to 0$

$$\frac{f(x_0+h)}{g(x_0+h)} = \frac{f(x_0+\frac{h}{2})+\frac{h}{2}f'(x_0+\frac{h}{2})+o(\frac{h}{2})}{g(x_0+\frac{h}{2})+\frac{h}{2}g'(x_0+\frac{h}{2})+o(\frac{h}{2})} \qquad (\text{III})$$

$$= \frac{f(x_0)+f'(x_0)\frac{h}{2}+o(\frac{h}{2})+\frac{h}{2}(f'(x_0)+\frac{h}{2}f''(x_0)+o(\frac{h}{2}))}{g(x_0)+g'(x_0)\frac{h}{2}+o(\frac{h}{2})+\frac{h}{2}(g'(x_0)+\frac{h}{2}g''(x_0)+o(\frac{h}{2}))}$$

$$= \frac{\frac{h^2}{4}f''(x_0)+o(\frac{h}{2})+\frac{h}{2}o(\frac{h}{2})}{\frac{h^2}{4}g''(x_0)+o(\frac{h}{2})+\frac{h}{2}o(\frac{h}{2})} \qquad \text{as } f'(x_0) = g'(x_0) = 0 = f(x_0) = g(x_0)$$

As $g''(x_0) \neq 0$, Proceeding to the limit in (III) as $h \to 0$ we get

$$\lim_{x\to 0}\frac{f(x_0+h)}{g(x_0+h)} = \frac{f''(x_0)}{g''(x_0)} .$$

Theorem 3.2 can be readily formulated for the case when higher order derivatives of f and g also vanish at x_0. The proof is a direct application of first order Taylor's formula without recourse to Taylor's mean-value theorem and the use of Landau symbols makes the proof direct and transparent.

4. Conclusion

The use of Landau symbols affords a qualitative approach to many problems involving limits and derivatives. It also serves to mitigate the punctilious use of epsilons and deltas. Clearly an increased use of Landau symbols in a basic calculus course will improve the learner's understanding of the concepts of limit and derivative.

REFERENCES

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Table 1

Some well-known limits	Their formulations with Landau symbols
1. $\lim_{x\to 0}\frac{\text{Sin } x}{x} = 1$	$\text{Sin } x \sim x$ as $x \to 0$
2. $\lim_{x\to 0}\frac{\text{Cos } x}{1-\frac{x^2}{2}} = 1$	$\text{Cos } x \sim 1 - \frac{x^2}{2}$ as $x \to 0$
3. $\lim_{x\to\infty}\frac{x^n}{e^x} = 0$, n ε **N**	$x^n = o(e^x)$ as $x \to \infty$, n ε **N**
4. $\lim_{x\to\infty}\frac{\log x}{x} = 0$	$\log x = \boldsymbol{o}(x)$ as $x \to \infty$
5. $\lim_{x\to\infty} e^{-x} = 0$	$e^{-x} = \boldsymbol{o}(1)$ as $x \to \infty$
6. $\lvert \text{Cos } x + \text{Sin } x \rvert \leq 2$, x ε $\mathbf{R}^+$	$\text{Cos } x + \text{Sin } x = \boldsymbol{O}(1)$ as $x \to \infty$