

**CALCULATION OF AREAS:
The discussion of a mathematical-historical problem that exposes
students' conceptions of Proofs**

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ABSTRACT

The present study constitutes an attempt to check students' conceptions about the nature and the significance of mathematical proofs. The setting of this study was a mathematical-historical discussion within the framework of a course dealing with the development of mathematics. The students - elementary school pre-service mathematics teachers - were exposed to some problems taken from the Egyptian mathematics. After the lesson – that included the presentation of a formal proof of the main statement discussed - the students were asked to answer individually and in writing questions concerning the Egyptian method to calculate the area of a quadrilateral. The analysis of their answers reinforces the conception that pre-service teachers may know how to perform the “ceremony” of proof but in general, they do not appropriately conceive its meaning or its role establishing truth in mathematics.

1. Introduction

Already in the eighties, researches have shown that students do not quite understand the essence and significance of mathematical proof although they are generally capable of performing the “ceremony” of proof. This result fits Arthur Eddington’s image, that as far as they are concerned “*Proof is the idol before whom the pure mathematician tortures himself*”.

In the study conducted by Fishbein & Kedem (1992) students received a proof of a mathematics statement and then were asked to state whether further concrete examples were required in order to establish the truth of the same statement. Its main finding showed that, although most students claimed that they had understood the proof, they felt that they should examine further examples in order to consider whether it is true or not.

The present study constitutes another attempt to check students’ conceptions about the nature and the significance of mathematical proofs.

2. The Study

Following Hanna (1996), I believe that “... proof deserves a prominent place in the curriculum because it continues to be a central feature of mathematics itself, as the preferred method of verification, and because it is a valuable tool for promoting mathematical understanding.” (Hanna, 1996, p.22). And although mathematics teachers in elementary schools do not generally deal directly with proofs of mathematical statements, they must know and understand the legitimate mathematical methods to establish the validity of a statement. Moreover, as mathematics teachers they must teach their students to justify their assertions and how to present these justifications in a manner that would convince the others that their claims are valid. It is therefore important to examine mathematics students’ and teachers’ conceptions of *proof* and to what extent they are aware of the various functions of proof as a mathematical activity.

The setting of this study was a mathematical-historical discussion within the framework of a course dealing with the development of mathematics. The students - elementary school pre-service mathematics teachers - were exposed to some problems taken from the Egyptian mathematics. The population included 25 students majoring in mathematics teaching for elementary schools. This population consisted of 18 females and 7 males. The students belonged to the Jewish sector ($n_1=13$) and to the Arab sector ($n_2=12$). During the meeting the students communicated in Hebrew.

In general, the framework of this college course on the development of mathematics enables:

- a. The review of contents with which the students are familiar – i.e.: calculation of areas and the review of contents they will teach (area, quadrilaterals and their properties);
- b. The exposure of students to the need for more substantial tools in mathematics than measurement, observation and experimentation;
- c. The exposure of students to the idea of proof, to different kinds of proofs and to discuss “what giving a proof means”;
- d. The exploration of students’ conceptions of what constitutes evidence in mathematics and what the roles of proof are.

In a two-hours meeting they were shown a way to calculate the area of a quadrilateral as it appears in the Rhind Papyrus (Eves, 1982, p. 14)¹. The meeting was designed as follows:

- Calculation of the area of a *square* when the length of the sides is known.
- Calculation of the area of *rectangle* when the length of each side is known.
- For each one of these quadrilaterals, the teacher presented the calculation of its area according to the Egyptian method.
- An attempt to calculate the area of a *rhombus* where the length of its sides is known and the teacher's presentation of the calculation according to the Egyptian method. (See Figure 1)

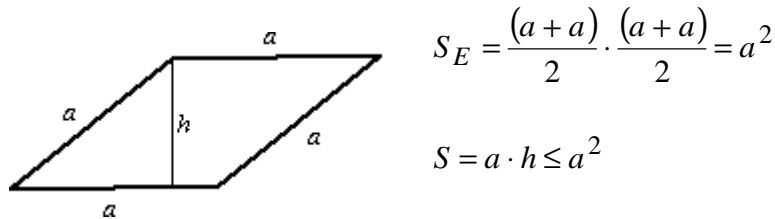


Figure 1

- Calculation of the area of an *isosceles trapezoid* where the length of its sides is known and the teacher's presentation of the calculation according to the Egyptian method (figure 2).

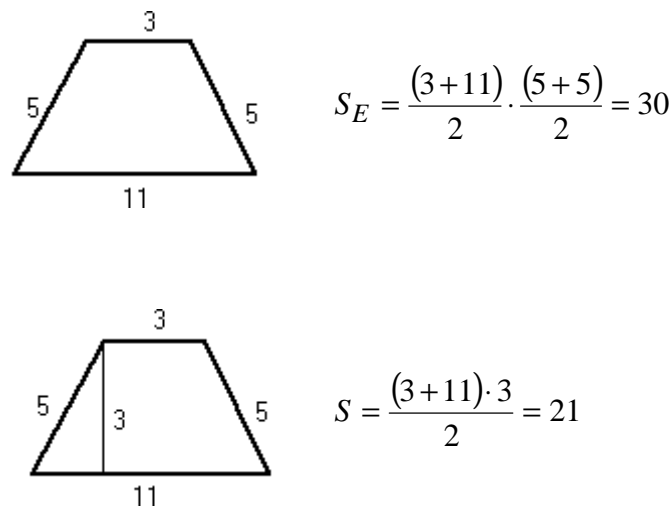


Figure 2

- Short discussion on the differences between the two values obtained in section (v)
- The teacher asked the participants to build a quadrilateral – assisted by a suitable software – where the area calculated according to the Egyptian method (S_E) is smaller than its area calculated according to “our” method (S).

¹ The Egyptian method for finding the area of the general quadrilateral is to take the product of the arithmetic means of the opposite sides.

h. After such an example was not found, some students suspected that such a quadrilateral does not exist. In other words, the following conjecture was formulated: *In every quadrilateral, the number obtained from the Egyptian method (S_E) is bigger or equal to the area of the quadrilateral (S).*

i. The teacher presented a proof for that statement and the proof was discussed in class (see Appendix).

Immediately after the lesson, the students were asked to answer individually and in writing the following questions:

1. Describe the method according to which the ancient Egyptians calculated the area of a quadrilateral.
2. Is their method correct? Explain.
3. Have we the right to judge the method's correctness? Can't we be mistaken? Explain.

The *first* question was asked in order to make sure that the students understood the method to calculate the area of a quadrilateral according to the length of its sides. It also enables to find different ways of formulating this method.

The *second* question reveals to what extent the presentation of a proof in class influences the students' consideration of the validity of the statement proved.

The *third* question is the main one in this study, and it enables to disclose the degree of students' understanding that any result contrary to a proved one must be false.

3. Findings

All the students were able to describe correctly the Egyptian method of calculating the area of a quadrilateral. They established that with regard to rectangles, the method was precise, while in other cases it resulted in findings, which differed, from those obtained by "our" methods.

The distribution of students' answers to the other two questions is presented in Figure 3.

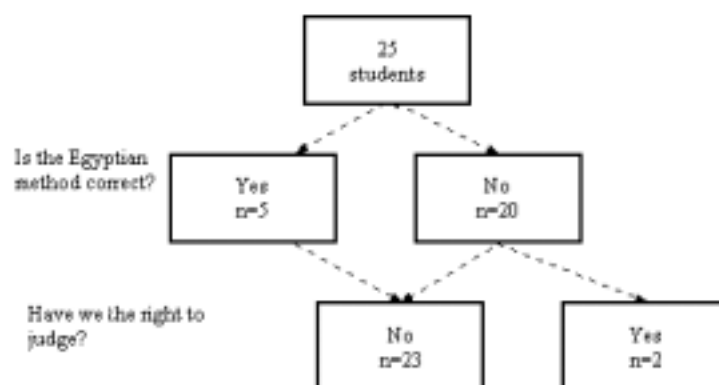


Figure 3

When designing the meeting, the rhombus and the trapezoid were chosen having in mind that these quadrilaterals are known to the students - i.e. they know how to calculate their area. The rhombus was introduced in a *generic* way by presenting the length of its sides by means of the

parameter a and without giving any information about its angles. Students were supposed to notice that in this case, the number provided by the Egyptian method constitutes in fact the area of the square with side a and all the other rhombuses of the family have smaller area. According to Peled and Zaslavsky (1997), this should have been a generic counterexample, but there is no evidence that the students grasped it.

The trapezoid was introduced as a *specific* quadrilateral and it was chosen purposefully to constitute an example that leads to the conclusion that the Egyptian method is wrong since there is at least one quadrilateral for which the area given by the Egyptian method is different from the area given by “our” methods.

Five students maintained that the Egyptian method is correct. Although the results obtained from the calculation of the area of the trapezoid described in Figure 2 were still on the blackboard, these students were not aware of the fact that at least one of these results must be wrong since they contradict each other. These students recognized that the results are different but did not recognize the contradiction between them. One of them - Orly - was extremely skeptical:

“Who says *they* were wrong and *we* are right? The results are different, I agree. But maybe our method is the wrong one.”

It seems that Orly forgot that “our” method to calculate the area of a trapezoid -for example- can be proved, making it true. From her comments, I learn that it is important to explicitly stress the fact that any result different from a proved one, must be false. This was not clear to the five students that contended that the Egyptian method is correct.

The eighteen other students claimed that the Egyptian calculation is indeed wrong. They related the error to the following categories:

The Egyptian method is incorrect because

- i. “it is not proved” (8 students);
- ii. “it is not clear how did they get it” (4 students);
- iii. “we found a counterexample” (3 students);
- iv. “we proved that their result is, in general, larger than the real area” (3 students).

The eight responses in category (a) illustrate another aspect of what De Villiers referred to as a “fundamental axiom” upon which mathematics is assumed to be based: “Something is true if and only if it can be (deductively) proved” (De Villiers, 1997, p.20). In their case, the statement is formulated in a variant that logically follows from the axiom: “If something is not proved, then is false.” It may be interesting to investigate further this view since it may be interpreted at least in two ways: “If something *cannot* be proved, then it is false” (a statement that may be refuted if we think about statements like axioms or definitions) or “If *I* did not prove something, then I cannot accept it as true” (a statement that is not appropriate to elementary school teachers for whom the deductive nature of the mathematical contents they teach is not always very clear).

Orly’s response is an illustrative one from this category:

“I think that they [the Egyptians] *were wrong, because they did not prove* that their way is the right one, and did not demonstrate how they arrived to these methods and why. I consider it possible that we are wrong, but that someone must refute our methods and prove that our calculation methods are wrong.”

Category (b) includes responses of students that seem to refer to an aspect of the role of *explanation* played proofs: in general, the main role of proof is considered to be the *verification* that a mathematics statement is true but not always we are able to provide proofs that show *why* this statement is true. Although they were exposed to an *explanatory* proof that the Egyptian

method is correct if and only if the quadrilateral is a rectangle, the students were not exposed to the reasoning that led the Egyptians to the formulation of their method of calculation. From this lack of information they concluded that – as Rinat expressed it – “the Egyptians were not rigorous enough, hence they were mistaken.”

Out of all the students (n=25), only two of them – Daniel and Gabriel - pointed out that we do have the right to claim that the Egyptian method is wrong.

While Daniel argued:

“The proof at the blackboard *tells us* that our claim is true and it gives us the right to say *we are right*... The moment I saw the example of the trapezoid, I *knew* that the Egyptians were wrong...”,

Gabriel reasoned as follows:

“I think that everything is relative: The error is relative, because when one says ‘error’ one must add ‘error with regard to...’ or ‘error in the framework of...’. The Geometry with which we are familiar is Euclidean, but I won’t judge according to it. According to the Euclidean Geometry, the Egyptians were wrong for the following reasons: 1) Our calculations today prove to be more precise than the Egyptians. 2) *Our calculations are based on proof*, but the Egyptians calculated according to estimations or maybe as a result of trial and error considerations. Maybe one day someone will say that we were wrong, because all our calculations were built on the Euclidean Geometry, which was based on axioms formulated by Euclid and these axioms, are irrelevant. The terms ‘right’ or ‘wrong’ are therefore relative and depend on the rules they were subjected to...”

These two students exposed two important aspects of proof that, in my opinion, deserve more attention among teachers educators: a) the role of examples and counter-examples while proving or disproving a conjecture - while a million of examples are not enough to establish the truth of an universal statement, one sole counter-example is enough to disprove it; b) the appreciation of the deductive methods used in geometry, specially the fundamental role played by axioms and definitions. Out of 25 students, only two were able to identify the notion of proof with the notion of certainty, meaning that - under the conditions the statement is proved – one can be sure that no counter-example exists and nobody will ever be able to construct or find such a counter-example. Moreover, Gabriel reminds us of the fact that no theorems or formal proofs are known in Egyptian mathematics and that there is no clear distinction between calculations that are exact and those that are only approximations.

The other 23 students advocated that we have no right to claim that the Egyptian calculation is wrong. For example, Ariel, Bruria and Chris claimed:

Ariel:

“We have no right to judge because it is a different culture and the knowledge depends on the culture. The achievements of every culture should be respected.”

Bruria:

“We have no right to judge and claim that they were wrong, because their time was different from ours. They developed their methods according to the ways and means they had available ... We cannot say that their calculations were wrong as long as we do not have a proof of the error. *It is important to keep in mind, that everything is right until you prove the opposite.* This also applies to the Egyptians: In their time their calculation was correct.”

Chris:

“I think, that the Egyptians arrived at their formula from a more ancient one. They proved it in their own way and the Egyptians accepted this formula at that time, until our formula was established. *I don't think that we have the right to claim that the Egyptians were wrong, because our formula is based on theirs.* The formula used by us is an improvement and development of the ancient one and *does not contradict it.* It is possible that in the future the calculations of the area of a general quadrilateral will be investigated and a better and more correct formula than ours will be discovered. Then our formula will not be good and right, or even not precise enough.”

These three excerpts illustrate the students' lack of understanding of what the meaning of giving a proof really is. They were indeed able to recognize that the Egyptian method is easy to use since it only involves elementary arithmetical operations and that it was used for *every* quadrilateral. On the other hand, they admitted that nowadays they do not know of a method to calculate the area of a general quadrilateral but a special method for each kind of quadrilateral. These advantages of the Egyptian method are mere illusions if we consider the fact that the Egyptian method *was proved* not to be accurate. I agree with these students that it is important to respect the achievements of every culture but this respect does not imply that we cannot compare and point out that some results are contradictory and some methods are not accurate. Pre-service teachers need to be exposed to the developmental aspect of mathematics, to different paradigms of proof, to the meaning of truth in mathematics and to the ways truth is achieved in mathematics. This exposure may foster their conceptual understanding of proof beyond their algorithmic knowledge of how to prove.

4. Concluding Remark

These preliminary results ask for further analysis but from the exposed above it appears that Hanna's recommendation is relevant more than ever:

“With today's stress on teaching *meaningful* mathematics, teachers are being encouraged to focus on the explanation of mathematical concepts and students are being asked to justify their findings and assertions. This would seem to be the right climate to make the most of proof as an explanatory tool, as well as to exercise it in its role as the ultimate form of mathematical justification. But for this to succeed, students must be made familiar with the standards of mathematical argumentation; in other words, *they must be taught proof*” (Hanna, 1996, p.33).

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APPENDIX

Given:

Quadrilateral $ABCD$

$BC=a, CD=b, DA=c, AB=d$

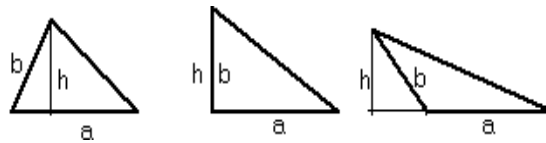
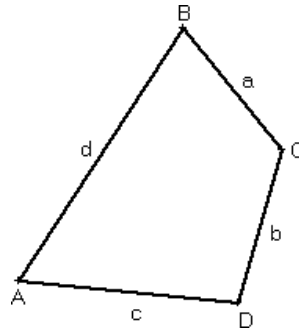
Prove that:

$$S(ABCD) \leq \left(\frac{a+c}{2}\right) \cdot \left(\frac{b+d}{2}\right)$$

Proof:

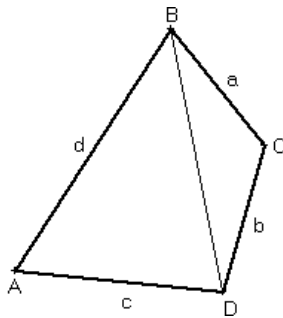
Lemma The area of a triangle with sides a and b , is no larger than

$$\frac{ab}{2}.$$



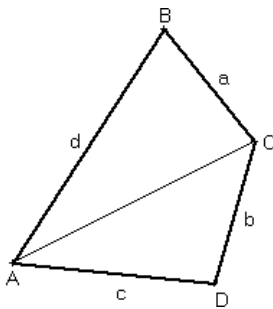
In the quadrilateral $ABCD$, built diagonal BD . Then, according to the Lemma above:

$$S(ABCD) = S(\triangle BAD) + S(\triangle BCD) \leq \frac{cd}{2} + \frac{ab}{2}$$



In the quadrilateral $ABCD$, built diagonal AC . Then, according to the Lemma above,

$$S(ABCD) = S(\triangle ABC) + S(\triangle ADC) \leq \frac{ad}{2} + \frac{bc}{2}$$



Adding, we get that

$$2 \cdot S(ABCD) \leq \left(\frac{cd}{2} + \frac{ab}{2}\right) + \left(\frac{ad}{2} + \frac{bc}{2}\right) = \frac{d(c+a) + b(a+c)}{2} = \frac{(a+c) \cdot (b+d)}{2}$$

Therefore, $S(ABCD) \leq \frac{(a+c) \cdot (b+d)}{4}$ in every quadrilateral $ABCD$.

The equality holds *if and only if* the quadrilateral is a rectangle.