6.5

(1) a) $\int_D \frac{1}{\sqrt{xy}}\, dA \,,\; D = [0,1]\times[0,1]$

$D_{\eta,\delta} = [\eta, 1]\times[\delta, 1] \qquad 0 \le \eta \le 1$

$\qquad 0 \le \delta \le 1$

$I_{\eta,\delta} = \left(\int_\eta^1 \frac{dx}{\sqrt{x}}\right)\left(\int_\delta^1 \frac{dy}{\sqrt{y}}\right) =$

$= 4\left[\sqrt{x}\right]_\eta^1 \left[\sqrt{y}\right]_\delta^1 =$

$= 4(1-\sqrt{\eta})(1-\sqrt{\delta}) \xrightarrow[(\eta,\delta)\to(0,0)]{} 4$

d) $\int_0^1 \int_0^{e^y} \log x \, dx\, dy$

$x = e^y,\; y = \ln x$

$D_{\eta,\delta} = \{\eta \le y \le 1 \,,\; \delta \le x \le e^y\}$

$I_{\eta,\delta} = \int_\eta^1 \left(\int_\delta^{e^y} \log x \, dx\right) dy =$

$= \int_\eta^1 \left[x\log x - x\right]_\delta^{e^y} dy = \int_\eta^1 e^y y - e^y - \delta\log\delta + \delta \, dy$

$= e - 2e + \delta(1-\log\delta) - n e^{n} + 2e^{n} - \delta(1-\log\delta)n =$

$\ast \lim_{\delta\to 0} \delta\log\delta = \lim_{\delta\to 0} \dfrac{\log\delta}{\frac{1}{\delta}} \overset{D'L}{=} =$

$= \lim \dfrac{\frac{1}{\delta}}{-\left(\frac{1}{\delta}\right)^{2}} = 0$

Άρα $I = -e + 2$

② α) $D = \{(x,y) \mid x \geq \alpha,\ \varphi_1(x) \leq y \leq \varphi_2(x)\}$

$D_n = \{(x,y) \mid \alpha \leq x \leq n,\ \varphi_1 \leq y \leq \varphi_2\}$

$I = \iint_D f(x,y)\,dx\,dy = \lim_{n\to\infty} \iint_{D_n} f(x,y)\,dx\,dy$

b) $\int_D xy e^{-(x^2+y^2)}\,dx\,dy$, $x \geq 0$, $0 \leq y \leq 1$

$I_n = \iint_{D_n} xy e^{-(x^2+y^2)}\,dx\,dy =$

$= \left(\int_0^n x e^{-x^2}\,dx\right)\left(\int_0^1 y e^{-y^2}\right)dy =$

$= \left[-\dfrac{e^{-x^2}}{2}\right]_0^n \cdot \left[-\dfrac{e^{-y^2}}{2}\right]_0^1 =$

$$= \frac{1}{4}\left(-e^{-n^2}+1\right)\left(-e^{-1}+1\right) =$$

$$\xrightarrow[n\to\infty]{} \frac{1}{4}\left(1-\frac{1}{e}\right)$$

(5) $\displaystyle\int_D \frac{x+y}{x^2+2xy+y^2}\, dx\, dy \quad , D=[0,1]^2$

$D_{\eta,\delta} = [\eta,1]\times[\delta,1]$

$$I_{\eta,\delta} = \int_\eta^1 \int_\delta^1 \frac{1}{x+y}\, dy\, dx =$$

$$= \int_\eta^1 \left[\ln(x+y)\right]_{y=\delta}^{y=1} dx =$$

$$= \int_\eta^1 \ln(1+x) - \ln(\delta+x)\, dx =$$

$$= \int_\eta^1 \ln(1+x)\,dx - \int_\eta^1 \ln(\delta+x)\,dx =$$

$$= \left[(x+1)\ln(x+1)\right] - (x+1) - (\delta+x)\ln(\delta+x) + (x+\delta)\Big]_n^1$$

$$= 2\ln 2 - 2 - (\delta+1)\ln(\delta+1) + (\delta+1) -$$

$$\to 2\ln 2.$$

(6) $f \geq 0$ και ολοκληρώσιμη και συνεχής στο D

$$f(x,y) \leq g(x,y)$$

Αν $\iint_D g \, dx dy$ υπάρχει τότε $\iint_D f$ υπάρχει

$$\iint_D g \, dx dy = \lim_{(\eta,\delta)\to(0,0)} \iint_{D_{\eta,\delta}} g \, dx dy$$

$$f \leq g$$

Άρα $\iint_{D_{\eta,\delta}} f \, dx dy \leq \int_{D_{\eta,\delta}} g \, dx dy$

Άρα $\lim_{(\eta,\delta)\to(0,0)} \iint_{D_{\eta,\delta}} f(x,y) \, dx dy \leq \iint_D g \, dx dy$

(7) $I = \iint_D \frac{\sin^2(x-y)}{\sqrt{1-x^2-y^2}} \, dx dy$, $D = \{ x^2+y^2 \leq 1 \}$

Έστω $I_\delta = D(0,\delta)$ $\forall (x,y) \in D(0,\delta)$

$$\frac{\sin^2(x-y)}{\sqrt{1-x^2-y^2}} \leq \frac{1}{\sqrt{1-x^2-y^2}}$$

$$\iint_{D(0,\delta)} \frac{\sin^2(x-y)}{\sqrt{1-x^2-y^2}}\,dx\,dy \le \iint_{D(0,\delta)} \frac{dx\,dy}{\sqrt{1-x^2-y^2}} \,dx\,dy$$

$$= \int_0^{\delta}\int_0^{2\pi} \frac{r}{\sqrt{1-r^2}}\,d\theta\,dr = 2\pi\left[\frac{\sqrt{1-r^2}}{-2}\right]_0^{\delta} =$$

$$= -\pi\left(\sqrt{1-\delta^2} - 1\right) =$$

$$= \pi\left(1-\sqrt{1-\delta^2}\right) \xrightarrow[\delta\to 1]{} \pi$$

(11) $$I = \iiint_{\omega} \frac{(x^2+y^2+z^2)^{1/4}}{\sqrt{z+(x^2+y^2+z^2)^2}}\,dx\,dy\,dz$$

$x^2+y^2+z^2=\alpha^2$, $x\geq 0$, $y\geq 0$, $z\geq 0$

(προβλημα στο $(0,0,0)$))

παιρνω $W_\varepsilon = \{(x,y,z) \mid \varepsilon^2 \le x^2+y^2+z^2 \le \alpha^2\}$

Σφαιρικές

$$I_\varepsilon = \int_\varepsilon^{\alpha}\int_0^{\pi/2}\int_0^{\pi/2} \frac{\rho^{5/2}\sin\varphi}{\sqrt{\rho\cos\varphi+\rho^4}}\,d\theta\,d\varphi\,d\rho =$$

$$= \frac{\pi}{2}\int_\varepsilon^{\alpha}\int_0^{\pi/2} \frac{\rho^{5/2}\sin\varphi}{\sqrt{\rho\cos\varphi+\rho^4}}\,d\varphi\,d\rho =$$

$$= \frac{\pi}{2} \int_{\varepsilon}^{\alpha} \Big[-2\rho^{3/2} \sqrt{\rho\cos\varphi + \rho^4} \Big]_{\varphi=0}^{\varphi=\pi/2} =$$

$$= -\pi \int_{\varepsilon}^{\alpha} -\rho^{3/2}\sqrt{\rho^4} + \rho^{3/2}\sqrt{\rho+\rho^4}\, d\rho =$$

$$= \pi \int_{\varepsilon}^{\alpha} \rho^{7/2} - \rho^{3/2}\sqrt{\rho+\rho^4}\, d\rho =$$

$$= \pi \left[\frac{\rho^{\frac{9}{2}}}{\frac{9}{2}} \right]_{\varepsilon}^{\alpha} - \pi \int_{\varepsilon}^{\alpha} \rho^2 \sqrt{1+\rho^3}\, d\rho =$$

$$= \pi \frac{2}{9} \alpha^{9/2} - \frac{4\pi}{9} \varepsilon^{9/2} - \pi \left[\frac{2}{9} (1+\rho^3)^{3/2} \right]_{\rho=\varepsilon}^{\rho=\alpha}$$

$$= \frac{2\pi}{9} \left(\alpha^{9/2} - \varepsilon^{9/2} \right) - \frac{\pi 2}{9} \left[(1+\alpha^3)^{3/2} - (1+\varepsilon^3)^{3/2} \right]$$

$$\xrightarrow{\varepsilon \to 0} \frac{2\pi}{9} \alpha^{9/2} - \frac{2\pi}{9} (1+\alpha^3)^{3/2} + \frac{2\pi}{9}$$

(x,y,z) , $x^2+y^2+z^2 \geq 1$

$$\int_D \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^2}$$

$$I_M = \int_{D_M} \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^2}$$

όπου $D_M = \{(x,y,z) \mid 1 \leq x^2+y^2+z^2 \leq M^2\}$

$$= \int_0^{\pi}\int_0^{2\pi}\int_1^{M} \frac{\rho^2 \sin\varphi}{(\rho^2)^2}\, d\rho\, d\theta\, d\varphi =$$

$$= \int_0^{\pi}\int_1^{M} 2\pi \frac{\sin\varphi}{\rho^2}\, d\rho\, d\varphi =$$

$$= 2\pi \left(\int_1^{M} \frac{d\rho}{\rho^2}\right)\left(\int_0^{\pi} \sin\varphi\, d\varphi\right) =$$

$$= 2\pi \left[-\frac{1}{\rho}\right]_{\rho=1}^{\rho=M} \left[-\cos\varphi\right]_{\varphi=0}^{\varphi=\pi} =$$

$$= 2\pi\left(-\frac{1}{M}+1\right)(1+1) =$$

$$= 4\pi\left(1-\frac{1}{M}\right) \xrightarrow{M\to\infty} 4\pi$$