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THE COMPLEX SYMPLECTIC STRUCTURE OF THE SPACE OF QUASIFUCHSIAN DEFORMATIONS

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ABSTRACT

We study the (complex) symplectic geometry of the space $Q(S)$ of the quasifuchsian structures of a closed Riemannian surface S of genus $g>1$. We prove that this space is a flat complex symplectic manifold and we describe the hamiltonian nature of the quasifuchsian bending vector fields.

A *quasifuchsian group* Γ is a discrete subgroup of $PSL(2,C)$, which is quasiconformally congruent to a fuchsian group of the first kind. The limit set of Γ is a Jordan curve. If Ω_Γ is the domain of discontinuity of Γ in the complex plane C and $\mathcal{H}^3$ is the upper hyperbolic semispace, then the Kleinian 3-manifold $M_\Gamma=(\mathcal{H}^3\cup\Omega_\Gamma)/\Gamma$ is diffeomorphic to $S\times[0,1]$. The space $Q(S)$ of the *quasifuchsian structures* on a closed Riemann surface S of genus $g>1$ can be regarded as the space of *marked quasifuchsian manifolds,* where a marking of M_Γ is a choice of isomorphism between $\pi_1(M_\Gamma)$ and $\pi_1(S)$. This work is divided in three sections:

In section 1 we mostly refer to previous results concerning the geometry and the deformations of $Q(S)$. For more details about the notions of *complex length* and *bending along a lamination* which are widely used here, the reader is referred to the works of Kourouniotis and Epstein-Marden [K1],[K2],[K3],[E-M].

In section 2 we construct a closed holomorphic form Ω in $Q(S)$ using the geometry of M_Γ and the laws that describe the variations of the complex length under bending. In this way the couple $(Q(S),\Omega)$ turns out to be a *complex symplectic manifold*, a notion which is an extension in the complex case of the notion of the *symplectic manifold*. Spaces of the form $Hom(\pi_1(S)\to G)/\sim$ where G is a Lie group with an invariant inner product on its Lie algebra, are already known to be symplectic by a construction of W.Goldman based in strong algebraic topological tools [G1]. In the case where $G=PSL(2,C)$ the symplectic form is described explicitely in [G2](Theorem p.40). The interested reader can compare Goldman's formulas with the laws that our construction is based upon, to obtain that the two symplectic forms are the same. Our construction also reveals the hamiltonian nature of the bending vector fields. A duality formula is given at the end of this section.

In section 3 we use an analytic continuation method to prove that $(Q(S),\Omega)$ is globally a *flat* complex symplectic manifold. S.Wolpert has proved a similar result for the *Weil-Petersson form* in the *Teichmüller space* $T(S)$; namely that $T(S)$ is a flat real symplectic manifold [W2].

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1. The space of quasifuchsian structures

We fix a compact Riemann surface S of genus $g>1$ and identify its fundamental group $\pi_1(S)$ with a Fuchsian group Γ_0. The space $Q(S)$ is the quotient of the set of discrete faithful representations $\rho \in Hom(\Gamma_0 \to PSL(2,C))$ such that $\Gamma = \rho(\Gamma_0)$ is quasifuchsian, with a left action of $PSL(2,C)$ via inner automorphisms. This space is a $6g-6$ complex dimensional complex manifold [B], and complex coordinates for it are provided by the following two theorems [K3]:

THEOREM A. *Given a partition of the surface S by $3g-3$ simple closed geodesics γ_i, there exist complex length functions $\lambda_{\gamma_i}: Q(S) \to C$ and bending functions $\beta_i: Q(S) \to C$ $i=1,\ldots,3g-3$ which form a system of global complex coordinates for $Q(S)$.*

THEOREM B. *Given a partition of the surface S by $3g-3$ simple closed geodesics γ_i, then for every $\rho \in Q(S)$ there exist a neighborhood $V(\rho)$ and $3g-3$ simple closed curves α_i with $\gamma_i \cap \alpha_j = \emptyset$ if $i \neq j$ such that the complex length functions $\lambda_{\gamma_i}, \lambda_{\alpha_i}$: $i=1,\ldots,3g-3$ form a system of local complex coordinates for $Q(S)$.*

In [K1] C.Kourouniotis defined a holomorphic transformation of $Q(S)$ called the *bending* deformation, which is a generalisation on $Q(S)$ of the *quakebending* deformation defined on the Teichmüller space $T(S)$ [E-M]. We firstly recall the notion of a *measured lamination* on S.

A *geodesic lamination* Λ_S on S is a foliation of a closed subset of S such that the leaves are geodesics. The leaves together with the components of $S-\Lambda_S$ are called the *strata* of Λ_S. A geodesic lamination on S lifts to a lamination Λ in the hyperbolic plane.

A *transverse measure* μ is a finite complex Borel measure on each closed geodesic segment α in the hyperbolic plane, subject to certain conditions:
i) If α is entirely contained in a stratum of Λ, then the measure of α is zero.
ii) The measures of two geodesic segments with their respective endpoints lying in the same strata of Λ are equal.

A *measured lamination* (Λ,μ) is a geodesic lamination equiped with a transverse measure. From here on we shall denote a measured lamination (Λ,μ) only with the symbol of the measure μ

Let μ be a measured lamination on S, and ρ a point of $Q(S)$. Then in a neighborhood W of 0 in C we define a quasiconformal deformation of ρ called *bending along* μ

$$B_\mu(\rho): W \to Q(S): t \to B_\mu(t,\rho)$$

The mapping B_μ posseses all the properties of a (local) holomorphic flow [K1].

Let α be a simple geodesic on S and λ_α its complex length function. The *first variation of the complex length of* α under bending along μ is given by:

$$T_\mu \lambda_\alpha = \frac{d}{dt}(0)[\lambda_\alpha(B_\mu(t,\rho))]$$

If v is a measured lamination on S the *second variation of* λ_α under bending along μ and v is given by:

$$T_\mu T_v \lambda_\alpha = \frac{\partial^2}{\partial t \partial s}(0,0)[\lambda_\alpha(B_\mu(t, B_v(s,\rho))]$$

Suppose that μ, v are simple closed curves α, β respectively equipped with transverse measure the counting one. Then the variations of the complex length have the following geometric interpretation:

$$T_\beta \lambda_\alpha = \sum_{p\in\alpha\cap\beta} \cosh\sigma(\rho(\beta),\rho(\alpha))_p$$

$$T_\beta T_\gamma \lambda_\alpha =$$

$$= \frac{1}{2\sinh\frac{1}{2}\lambda_\alpha}\sum_{p\in\alpha\cap\beta}\sum_{q\in\alpha\cap\gamma}\sinh\sigma(\rho(\alpha),\rho(\beta))_p \sinh\sigma(\rho(\alpha),\rho(\gamma))_q \cosh(\frac{1}{2}\lambda_\alpha - \sigma_{pq}) +$$

$$+ \frac{1}{2\sinh\frac{1}{2}\lambda_\beta}\sum_{p\in\alpha\cap\beta}\sum_{r\in\beta\cap\gamma}\sinh\sigma(\rho(\alpha),\rho(\beta))_p \sinh\sigma(\rho(\beta),\rho(\gamma))_r \cosh(\frac{1}{2}\lambda_\beta - \sigma_{pr})$$

where by σ we denote the *complex distance* between geodesics $\rho(\cdot)$ in the upper semispace $\mathcal{H}^3$ (See [K3]).

The following laws are crucial for our construction in section 3:

$$\text{I.}\ T_\alpha\lambda_\beta + T_\beta\lambda_\alpha = 0$$

$$\text{II.}\ T_\alpha T_\beta \lambda_\gamma + T_\beta T_\alpha \lambda_\gamma + T_\gamma T_\beta \lambda_\alpha = 0$$

Laws I and II are the generalisation on $Q(S)$ of S.Wolpert's laws concerning the first and the second variation of the geodesic length under *twisting* in $T(S)$ [W1].

Proposition A. [K3]. *If α is a simple closed curve on S, then T_α is a holomorphic vector field on $Q(S)$.*

We shall call T_α a *Q-bending vector field*. Let now ρ be a point of $Q(S)$, α be a simple closed curve on S, and let also

$$T_\alpha = \frac{1}{2}(F_\alpha - iB_\alpha)$$

the holomorphic tangent Q-bending vector at ρ. F_α, B_α are real vectors and

$$B_\alpha = J_Q F_\alpha,\quad F_\alpha = -J_Q B_\alpha$$

where J_Q is the almost complex structure of $Q(S)$. Let β be another simple closed curve on S and $\lambda_\beta = l_\beta + i\vartheta_\beta$ its complex length function (l_β is the geodesic length function and ϑ_β is the angle function [K2], [K3]).

Making some elementary calculations we have:

$$T_\alpha \lambda_\beta = \frac{1}{2}(F_\alpha - iB_\alpha)(l_\beta + i\vartheta_\beta) = \frac{1}{2}(F_\alpha l_\beta + B_\alpha \vartheta_\beta) + \frac{i}{2}(F_\alpha \vartheta_\beta - B_\alpha l_\beta)$$

and also

$$T_\alpha \lambda_\beta = \sum_{p \in \alpha \cap \beta} \cosh\sigma(\rho(\alpha), \rho(\beta))_p =$$

$$= \sum_{p \in \alpha \cap \beta} \cosh d(\rho(\alpha), \rho(\beta))_p \cos\phi(\rho(\alpha), \rho(\beta))_p + i \sinh d(\rho(\alpha), \rho(\beta))_p \sin\phi(\rho(\alpha), \rho(\beta))_p$$

where by d we denote the hyperbolic distance between geodesics $\rho(\cdot)$ in the upper semispace $\mathcal{H}^3$, and by ϕ their angle ($\sigma = d + i\phi$).

Equating real and imaginary parts and applying the Cauchy-Riemann equations we obtain:

$$F_\alpha l_\beta = B_\alpha \vartheta_\beta = \sum_{p \in \alpha \cap \beta} \cosh d(\rho(\alpha), \rho(\beta))_p \cos\phi(\rho(\alpha), \rho(\beta))_p$$

$$F_\alpha \vartheta_\beta = -B_\alpha l_\beta = \sum_{p \in \alpha \cap \beta} \sinh d(\rho(\alpha), \rho(\beta))_p \sin\phi(\rho(\alpha), \rho(\beta))_p$$

When ρ is a fuchsian point then $d(\rho(\alpha), \rho(\beta))_p = 0$ since $\rho(\alpha), \rho(\beta)$ are intersecting geodesics in $\mathcal{H}^3$ and therefore:

$$T_\alpha \lambda_\beta = F_\alpha l_\beta = t_\alpha l_\beta = \sum_{p \in \alpha \cap \beta} \cos\phi(\rho(\alpha), \rho(\beta))_p$$

where t_α is the Fenchel-Nielsen twist vector field (See [W1] for the definition).

Lemma 1.1. *Let γ_i, α_i be as in theorem B. The Q-bending vector fields T_{γ_i} $(= \frac{\partial}{\partial \beta_i})$ and T_{α_i} form a basis of the holomorphic tangent space $T^{(1,0)}(Q(S))$.*

Proof: Let $V(\rho_0)$ be an open neighborhood of a point $\rho_0 \in Q(S)$ with local coordinates $(\lambda_{\gamma_1}, \dots, \lambda_{\gamma_{3g-3}}, \lambda_{\alpha_1}, \dots, \lambda_{\alpha_{3g-3}})$. The Q-bending vectors $T_{\gamma_i}, T_{\alpha_i}$ are linear combinations of the vectors $\frac{\partial}{\partial \lambda_{\gamma_i}}, \frac{\partial}{\partial \lambda_{\alpha_i}}$. Specifically at each $\rho \in V(\rho_0)$ we have:

$$[T] = \begin{bmatrix} 0 & A \\ -A & B \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial \lambda} \end{bmatrix}$$

where $[T]$ is the complex matrix $[T_{\gamma_1}, \dots, T_{\gamma_{3g-3}}, T_{\alpha_1}, \dots, T_{\alpha_{3g-3}}]^T$, and $\left[\frac{\partial}{\partial \lambda}\right]$ is the complex matrix $[\frac{\partial}{\partial \lambda_{\gamma_1}}, \dots, \frac{\partial}{\partial \lambda_{\gamma_{3g-3}}}, \frac{\partial}{\partial \lambda_{\alpha_1}}, \dots, \frac{\partial}{\partial \lambda_{\alpha_{3g-3}}}]^T$.

The matrix B is a non-zero $(3g-3) \times (3g-3)$ complex matrix with entries $T_x \lambda_{\alpha_i}$ or 0 and A is the $(3g-3) \times (3g-3)$ diagonal complex matrix with diagonal entries

$$[T_{ii}] = T_{\gamma_i} \lambda_{\alpha_i} = \sum_{p \in \gamma_i \cap \alpha_i} \cosh \sigma(\rho(\gamma_i), \rho(\alpha_i))_p$$

The determinant of the transformation is equal to $(-1)^{3g-3}(\det A)^2$ which is different from zero since at every $\rho \in V$ we have $T_{\gamma_i} \lambda_{\alpha_i} \neq 0$ for every $i = 1, \dots, 3g-3$.

2. The complex symplectic structure of Q(S) and the Hamiltonian nature of Q-bending fields.

With the aid of laws I and II we firstly define locally a closed holomorphic 2-form in the holomorphic tangent space $T^{(1,0)}(Q(S))$. Let ρ_0 be a point of $Q(S)$ and $V(\rho_0)$, T_{γ_i}, T_{α_i} as in the preceeding section. For indices α,β running through γ_i, α_i, $i=1,\dots,3g-3$ we set for each $\rho \in V$;

$$\Omega_{(\rho)}(T_\alpha, T_\beta) = T_{\alpha(\rho)}\lambda_\beta = \sum_{p\in\alpha\cap\beta} \cosh\sigma(\rho(\alpha),\rho(\beta))_p$$

By law I one obtains that Ω is *skew-symmetric* and by the very definition that Ω is *holomorphic* since the quantities $T_\alpha\lambda_\beta$ are all holomorphic functions in $V(\rho_0)$.

The form Ω does not depend on the choice of local coordinates for $V(\rho_0)$ or to be more specific on the choice of the curves α_i of theorem B. One way to see this is to consider α,β simple closed curves on S and to establish that the above formula is also valid in this case.

Indeed by lemma 1.1 there exist holomorphic functions f^i_α, g^i_α and f^i_β, g^i_β, $i=1,\dots,3g-3$ defined on $V(\rho_0)$ such that

$$T_\alpha = f^i_\alpha T_{\gamma_i} + g^i_\alpha T_{\alpha_i} \text{ and } T_\beta = f^i_\beta T_{\gamma_i} + g^i_\beta T_{\alpha_i}$$

where upper and lower indices denote summation. The calculations now are straight forward:

$$\Omega(T_\alpha, T_\beta) = \Omega(f^i_\alpha T_{\gamma_i} + g^i_\alpha T_{\alpha_i}, f^j_\beta T_{\gamma_j} + g^j_\beta T_{\alpha_j}) =$$
$$= f^i_\alpha f^j_\beta \Omega(T_{\gamma_i}, T_{\gamma_j}) + f^i_\alpha g^j_\beta \Omega(T_{\gamma_i}, T_{\alpha_j}) + g^i_\alpha f^j_\beta \Omega(T_{\alpha_i}, T_{\gamma_j}) + g^i_\alpha g^j_\beta \Omega(T_{\alpha_i}, T_{\alpha_j}) =$$
$$= f^i_\alpha f^j_\beta T_{\gamma_i}\lambda_{\gamma_j} + f^i_\alpha g^j_\beta T_{\gamma_i}\lambda_{\alpha_j} + g^i_\alpha f^j_\beta T_{\alpha_i}\lambda_{\gamma_j} + g^i_\alpha g^j_\beta T_{\alpha_i}\lambda_{\alpha_j} =$$
$$= f^j_\beta (f^i_\alpha T_{\gamma_i} + g^i_\alpha T_{\alpha_i})\lambda_{\gamma_j} + g^j_\beta (f^i_\alpha T_{\gamma_i} + g^i_\alpha T_{\alpha_i})\lambda_{\alpha_j} =$$
$$= f^j_\beta T_\alpha \lambda_{\gamma_j} + g^j_\beta T_\alpha \lambda_{\alpha_j} \quad \text{which by law I is equal to}$$
$$-f^j_\beta T_{\gamma_j}\lambda_\alpha - g^j_\beta T_{\alpha_j}\lambda_\alpha = -T_\beta\lambda_\alpha = T_\alpha\lambda_\beta.$$

The form Ω is a closed (2,0) *form*: Let d' be the holomorphic differential operator on $Q(S)$ and $T_\alpha, T_\beta, T_\gamma$, Q-bending vector fields on curves α,β,γ of S respectively. Then

$$d'\Omega(T_\alpha, T_\beta, T_\gamma) = T_\alpha\Omega(T_\beta, T_\gamma) - T_\beta\Omega(T_\alpha, T_\gamma) + T_\gamma\Omega(T_\alpha, T_\beta) -$$
$$-\Omega([T_\alpha, T_\beta], T_\gamma) + \Omega([T_\alpha, T_\gamma], T_\beta) - \Omega([T_\beta, T_\gamma], T_\alpha)$$

Using laws I and II we find out that this is equal to:

$$T_\alpha T_\beta\lambda_\gamma - T_\beta T_\alpha\lambda_\gamma + T_\gamma T_\alpha\lambda_\beta -$$
$$-[T_\alpha, T_\beta]\lambda_\gamma + [T_\alpha, T_\gamma]\lambda_\beta + [T_\beta, T_\gamma]\lambda_\alpha =$$
$$T_\alpha T_\beta\lambda_\gamma + T_\beta T_\gamma\lambda_\alpha - T_\gamma T_\alpha\lambda_\beta - T_\alpha T_\beta\lambda_\gamma + T_\beta T_\alpha\lambda_\gamma +$$
$$+T_\alpha T_\gamma\lambda_\beta - T_\gamma T_\alpha\lambda_\beta - T_\gamma T_\alpha\lambda_\beta - T_\beta T_\gamma\lambda_\alpha + T_\gamma T_\beta\lambda_\alpha = 0$$

We can also check that our form Ω *is non-degenerate* in the sense that if there exists a holomorphic vector field Z such that $\Omega(Z,W)=0$ for all holomorphic W then $Z=0$. Indeed, the equation $\Omega(Z,W)=0$ for all W is equivalent to the $6g-6$ equations

$$\Omega(Z,T_\alpha)=0$$

where α is an index moving on indices γ_i, α_i, $i=1,\ldots,3g-3$

Now since $Z=f^i T_{\gamma_i}+g^i T_{\alpha_i}$ for some holomorphic functions f^i, g^i we have from the equations

$$\Omega(Z,T_{\gamma_i})=0$$

that $g^i=0$ since $\Omega(T_{\gamma_i},T_{\gamma_j})=0$ and $\Omega(T_{\alpha_i},T_{\gamma_j})=\sum_{p\in\alpha_i\cap\gamma_j}\cosh\sigma(\rho(\alpha_i),\rho(\gamma_j))_p\delta^i_j$, where δ^i_j is the Kronecker delta.

Therefore $Z=f^i T_{\gamma_i}$ and by the same reasoning we get from the equations $\Omega(Z,T_{\alpha_i})=0$ that $f^i=0$. We conclude that $Z=0$.

Definition 2.1. *Let M be a $2n$ complex manifold. We say that M carries a complex symplectic structure if there exists a non-degenerate, closed (2,0)-form defined on M .*

A trivial example is $\mathbf{C}^{2n}$ with complex coordinates $z_1,\ldots,z_n,w_1,\ldots,w_n$ where we have the standard flat form

$$\Omega_0=\sum_{i=1}^{n} d'z_i\wedge d'w_i$$

The couple (M,Ω) is called a *complex symplectic manifold*. We recall that a $2n$ real manifold M is called *symplectic* if it carries a *real symplectic structure* i.e if there exists a non-degenerate closed
2-form defined on M. The following is rather obvious:

Proposition 2.2. *A complex symplectic manifold is also a real symplectic manifold.*

Proof: Let (M,Ω) a complex symplectic manifold. Since Ω is a complex form we have $\Omega=\omega+i\phi$ where ω,ϕ are non-degenerate real closed forms on M defining symplectic structures on M.

From the above discussion we obtain :

THEOREM 2.3 *The space of quasifuchsian structures is a complex symplectic as well as a real symplectic manifold.*

We turn now to a more general discussion. Let (M,Ω) be a complex symplectic manifold. The holomorphic form Ω defines an isomorphism between holomorphic tangent and cotangent bundles. For every point $p\in M$ and every holomorphic vector $Z\in T_p^{(1,0)}(M)$, a holomorphic 1-form $\Omega(Z)$ is describing this isomorphism:

$$(\Omega(Z))_p=\Omega_p(Z,\cdot)=i_Z\Omega_p$$

where i_z is the interior product with respect to Z.

Let f be a holomorphic function on M. We shall call a holomorphic vector field H_f^C the *complex hamiltonian* of f if for every holomorphic vector field Ξ defined on M we have:

$$\Omega(H_f^{\mathbf{C}})(\Xi) = \Omega(H_f^{\mathbf{C}},\Xi) = -d'f(\Xi) = -\Xi f$$

If φ_t is the (local) holomorphic flow of $H_f^{\mathbf{C}}$, then for every t belonging to a sufficiently small neighborhood of 0 in $\mathbf{C}$, we have $\varphi_t^*\Omega = \Omega$. Also the complex Lie derivative of Ω along $H_f^{\mathbf{C}}$ is zero.

Suppose that $f = u + iv$ where u, v are smooth functions of M satisfying the Cauchy-Riemann equations. Consider the real symplectic manifold (M,ω) where $\omega = \mathrm{Re}\,\Omega$. For the (real) hamiltonian vector fields H_u, H_v of u, v respectively, we have:

$$\omega(H_u,\xi) = -du(\xi) = -\xi u$$
$$\omega(H_v,\xi) = -dv(\xi) = -\xi v$$

for every real vector field ξ on M .

We can check easily that

$$J_M H_u = -H_v \text{ and } J_M H_v = H_u$$

where J_M is the almost complex operator of J_M, the real tangent space of M. Indeed, since ω is J_M anti-invariant being the real part of a holomorphic form, we have:

$$\omega(J_M H_u,\xi) = \omega(H_u, J_M\xi) = -du(J_M\xi) = -(J_M\xi)u$$

If $\Xi = \frac{1}{2}(\xi - iJ_M\xi)$ is the holomorphic vector field corresponding to ξ then

$$\xi = \Xi + \bar{\Xi} \text{ and } J_M\xi = i(\Xi - \bar{\Xi}) \text{ so}$$
$$(J_M\xi)u = \frac{i}{2}(\Xi - \bar{\Xi})(f + \bar{f}) = \frac{i}{2}(\Xi f - \bar{\Xi}\bar{f}) =$$
$$= -\mathrm{Im}(\Xi f) = \mathrm{Im}[\Omega(H_f^{\mathbf{C}}.\Xi)].$$

On the other hand

$$\omega(H_u,\xi) = -du(\xi) = -\xi u = \frac{i}{2}(\Xi + \bar{\Xi})(f - \bar{f}) = \frac{i}{2}(\Xi f - \bar{\Xi}\bar{f}) = -\mathrm{Im}(\Xi f)$$

therefore

$$\omega(J_M H_u,\xi) = \omega(-H_v,\xi)$$

and since ω is non-degenerate we have $J_M H_u = -H_v$ where applying J_M again we obtain $J_M H_v = H_u$

We also note that

$$\omega(H_u,\xi) = -\xi u = -\frac{1}{2}(\Xi + \bar{\Xi})(f + \bar{f}) = -\mathrm{Re}(\Xi f) = \mathrm{Re}[\Omega(H_f^{\mathbf{C}},\Xi)].$$

Thus

$$\Omega(H_f^{\mathbf{C}},\Xi) = \mathrm{Re}[\Omega(H_f^{\mathbf{C}},\Xi)] + i\,\mathrm{Im}[\Omega(H_f^{\mathbf{C}},\Xi)] = \omega(H_u,\xi) - i\omega(J_M H_u,\xi) =$$

$$\mathrm{Re}[\Omega(\frac{1}{2}(H_u - iJ_M H_u),\Xi)] + i\,\mathrm{Im}[\Omega(\frac{1}{2}(H_u - iJ_M H_u),\Xi)] = \Omega(\frac{1}{2}(H_u - iJ_M H_u),\Xi)$$

Since Ω is non-degenerate we have:

$$H_f^{\mathbf{C}} = \frac{1}{2}(H_u - iJ_M H_u) = \frac{1}{2}(H_u + iH_v)$$

Let a be a simple closed curve on S and $\lambda_a = l_a + i\vartheta_a$ its complex length function defined on $Q(S)$. The geodesic length function l_a and the angle function ϑ_a are smooth on $Q(S)$. Applying the results of the previous discussion we obtain the following:

Proposition 2.4. *The Q-bending vector field* $T_a = \frac{1}{2}(F_a - iB_a)$ *is the complex hamiltonian of the complex length function* λ_a: $T_a = H^{C}_{\lambda_a}$. *Furthermore*

$$H_{l_a} = F_a \textit{ and } H_{\vartheta_a} = -B_a$$

Proposition 2.5. *(Duality formula)*

$$\Omega(T_a, \cdot) = -d'\lambda_a$$

Observe that when we restrict ourselves in the Teichmüller space, then formula 3.5 is just S.Wolpert's duality formula:

$$\omega_{WP}(t_a, \cdot) = -dl_a$$

where ω_{WP} is the Weil-Petersson symplectic form (See also lemma 3.3).

3. Q(S) is a flat complex symplectic manifold.

In [W2] S. Wolpert proved the following formula concerning the expression of the Weil-Petersson symplectic form ω_{WP} of the Teichmüller space in terms of the global real analytic Fenchel-Nielsen coordinates $(l_{\gamma_1}, \ldots, l_{\gamma_{3g-3}}, \tau_1, \ldots, \tau_{3g-3})$:

$$\omega_{WP} = \sum_{i=1}^{3g-3} dl_{\gamma_i} \wedge d\tau_i$$

where l_{γ_i} $i = 1, \ldots, 3g-3$ are geodesic length functions of closed geodesics γ_i forming a partition of S and τ_i $i = 1, \ldots, 3g-3$ are the *twist functions* corresponding to these geodesics. According to this formula the form ω_{WP} is *flat symplectic* and the Teichmüller space is a (real) *flat symplectic manifold*. Note here that the term *flat* refers only to the *symplectic nature* of $T(S)$ and *not* to the *Kählerian* one (where it is known that $T(S)$ has negative holomorphic curvature). In this section we establish the natural extension of Wolpert's result in our case. Namely we prove our main:

THEOREM 3.1. $(Q(S), \Omega)$ *is a flat complex symplectic manifold. The expression of the form in the global coordinates* $\lambda_{\gamma_i}, \beta_i$ $i = 1, \ldots, 3g-3$ *of theorem A is*

$$\Omega = \sum_{i=1}^{3g-3} d'\lambda_{\gamma_i} \wedge d'\beta_i \quad \textit{Also}$$

$$\Omega\left(\frac{\partial}{\partial \beta_i}, \cdot\right) = -d'\lambda_{\gamma_i} \textit{ and } \Omega\left(\frac{\partial}{\partial \lambda_{\gamma_i}}, \cdot\right) = d'\beta_i$$

and therefore the holomorphic vector fields $\frac{\partial}{\partial \beta_i}, \frac{\partial}{\partial \lambda_{\gamma_i}}$ *are complex hamiltonian for the form* Ω.

Our proof is based on an analytic continuation argument. We start with

Lemma 3.2. (*Analytic continuation in several variables*)
Let $F: \mathbf{C}^n \to \mathbf{C}$, $F = F(z_1, \ldots, z_n), z_i = x_i + y_i$ *be a holomorphic function and suppose that* $F(x_1, \ldots, x_n, 0, \ldots, 0) = 0$. *Then* F *is identically zero in* $\mathbf{C}^n$.

Proof: Consider the function $w_1 = w_1(z_1) = F(x_1, \ldots, x_n, y_1, 0, \ldots, 0)$ which is holomorphic in the variable z_1. Now $w_1(x_1, 0) = F(x_1, \ldots, x_n, 0, \ldots, 0) = 0$ therefore by analytic continuation we have $w_1(z_1) = 0$. Consider then $w_2 = w_2(z_2) = F(x_1, x_2, \ldots, x_n, y_1, y_2, 0, \ldots, 0)$ which is holomorphic in the variable z_2 (we stabilise all the other variables). We have $w_2(x_2, 0) = F(x_1, \ldots, x_n, y_1, 0, \ldots, 0) = 0$ therefore $w_2(z_2) = 0$ and we continue with this procedure until we exhaust all variables.

By the duality formula 2.5 we easily see that the following hold:

$$\Omega(\frac{\partial}{\partial \beta_i}, \frac{\partial}{\partial \lambda_{\gamma_j}}) = -\delta_{ij} \text{ and } \Omega(\frac{\partial}{\partial \beta_i}, \frac{\partial}{\partial \beta_j}) = 0$$

for $i, j = 1, \ldots, 3g-3$ and thus

$$\Omega = \sum_{i=1}^{3g-3} d'\lambda_{\gamma_i} \wedge d'\beta_i + \sum_{1 \le i < j \le 3g-3} \Omega(\frac{\partial}{\partial \lambda_{\gamma_i}}, \frac{\partial}{\partial \lambda_{\gamma_j}}) d'\lambda_{\gamma_i} \wedge d'\lambda_{\gamma_j}$$

where

$$\lambda_{\gamma_i} = l_{\gamma_i} + i\vartheta_i,\ \beta_i = \tau_i + i\psi_i \text{ and } \frac{\partial}{\partial \lambda_{\gamma_i}} = \frac{1}{2}(\frac{\partial}{\partial l_{\gamma_i}} - i\frac{\partial}{\partial \vartheta_{\gamma_i}}),\ \frac{\partial}{\partial \beta_i} = \frac{1}{2}(\frac{\partial}{\partial \tau_i} - i\frac{\partial}{\partial \psi_i}) \text{ [K3].}$$

After straightforward calculations we obtain

$$\omega = \mathrm{Re}\,\Omega = \sum_{i=1}^{3g-3} (dl_{\gamma_i} \wedge d\tau_i - d\vartheta_i \wedge d\psi_i) +$$
$$+ \sum_{1 \le i < j \le 3g-3} \mathrm{Re}[\Omega(\frac{\partial}{\partial \lambda_{\gamma_i}}, \frac{\partial}{\partial \lambda_{\gamma_j}})](dl_{\gamma_i} \wedge dl_{\gamma_j} - d\vartheta_i \wedge d\vartheta_j) -$$
$$- \sum_{1 \le i < j \le 3g-3} \mathrm{Im}[\Omega(\frac{\partial}{\partial \lambda_{\gamma_i}}, \frac{\partial}{\partial \lambda_{\gamma_j}})](d\vartheta_i \wedge dl_{\gamma_j} + dl_{\gamma_i} \wedge d\vartheta_j)$$

and

$$\phi = \mathrm{Im}\,\Omega = \sum_{i=1}^{3g-3} (d\vartheta_i \wedge d\tau_i + dl_{\gamma_i} \wedge d\psi_i) +$$
$$+ \sum_{1 \le i < j \le 3g-3} \mathrm{Re}[\Omega(\frac{\partial}{\partial \lambda_{\gamma_i}}, \frac{\partial}{\partial \lambda_{\gamma_j}})](d\vartheta_i \wedge dl_{\gamma_j} + dl_{\gamma_i} \wedge d\vartheta_j) +$$
$$+ \sum_{1 \le i < j \le 3g-3} \mathrm{Im}[\Omega(\frac{\partial}{\partial \lambda_{\gamma_i}}, \frac{\partial}{\partial \lambda_{\gamma_j}})](dl_{\gamma_i} \wedge dl_{\gamma_j} - d\vartheta_i \wedge d\vartheta_j)$$

where of course

$$\mathrm{Re}[\Omega(\frac{\partial}{\partial \lambda_{\gamma_i}}, \frac{\partial}{\partial \lambda_{\gamma_j}})] = \omega(\frac{\partial}{\partial l_{\gamma_i}}, \frac{\partial}{\partial l_{\gamma_j}}) \text{ and } \mathrm{Im}[\Omega(\frac{\partial}{\partial \lambda_{\gamma_i}}, \frac{\partial}{\partial \lambda_{\gamma_j}})] = \phi(\frac{\partial}{\partial l_{\gamma_i}}, \frac{\partial}{\partial l_{\gamma_j}})$$

When restricting ourselves on the tangent bundle of the fuchsian points of $Q(S)$ then the functions ϑ_i, ψ_i, $i = 1, \ldots, 3g-3$ are all zero, therefore in this case:

$$\Omega = \omega + i\phi = \sum_{i=1}^{3g-3} dl_{\gamma_i} \wedge d\tau_i + \sum_{1\le i<j\le 3g-3} \mathrm{Re}[\Omega(\frac{\partial}{\partial\lambda_{\gamma_i}}, \frac{\partial}{\partial\lambda_{\gamma_j}})] dl_{\gamma_i} \wedge dl_{\gamma_j} +$$

$$+ i \sum_{1\le i<j\le 3g-3} \mathrm{Im}[\Omega(\frac{\partial}{\partial\lambda_{\gamma_i}}, \frac{\partial}{\partial\lambda_{\gamma_j}})] dl_{\gamma_i} \wedge dl_{\gamma_j} =$$

$$= \sum_{i=1}^{3g-3} dl_{\gamma_i} \wedge d\tau_i + \sum_{1\le i<j\le 3g-3} \Omega(\frac{\partial}{\partial\lambda_{\gamma_i}}, \frac{\partial}{\partial\lambda_{\gamma_j}}) dl_{\gamma_i} \wedge dl_{\gamma_j}$$

We shall show that the holomorphic functions $\Omega_{ij} = \Omega(\frac{\partial}{\partial\lambda_{\gamma_i}}, \frac{\partial}{\partial\lambda_{\gamma_j}})$ are real valued at fuchsian points: Indeed if ρ is such a point then in a neighborhood of ρ we have from lemma 1.1 that

$$\frac{\partial}{\partial\lambda_{\gamma_i}} = f^j_{\gamma_i} T_{\gamma_j} + g^j_{\gamma_i} T_{\alpha_j}$$

where $f^j_{\gamma_i}, g^j_{\gamma_i}$ are holomorphic functions defined on this neighborhood. According to the matrix equation in lemma 1.1 we have by multiplication with the inverse matrix that

$$\left[\frac{\partial}{\partial\lambda}\right] = \begin{bmatrix} A^{-1}BA^{-1} & -A^{-1} \\ A^{-1} & 0 \end{bmatrix} [T]$$

Therefore

$$\left[\frac{\partial}{\partial\lambda_{\gamma}}\right] = \left[A^{-1}BA^{-1} \quad \vert \quad -A^{-1}\right][T]$$

where

$$\left[\frac{\partial}{\partial\lambda_{\gamma}}\right] = [\frac{\partial}{\partial\lambda_{\gamma_1}}, \ldots, \frac{\partial}{\partial\lambda_{\gamma_{3g-3}}}]^T$$

Recall that the entries of the (diagonal) matrix A are of the form $T_{\gamma_i}\lambda_{\alpha_i}$ or 0 where the entries of the matrix B are of the form $T_{\alpha_i}\lambda_{\alpha_j}$ or 0. All these quantities (and eventually the quantities $f^j_{\gamma_i}, g^j_{\gamma_i}$ which are entries of the matrices $A^{-1}BA^{-1}$ and $-A^{-1}$ respectively) evaluated on a fuchsian point are real, for if ρ is such a point then we have

$$T_\alpha\lambda_\beta = \sum_{p\in\alpha\cap\beta} \cos\phi(\rho(\alpha),\rho(\beta))_p$$

where α,β indices running through indices γ_i, α_i, $i=1,\ldots,3g-3$.

Now it is easy to see that

$$\Omega_{ij}(\rho) = \Omega_{(\rho)}(\frac{\partial}{\partial\lambda_{\gamma_i}}, \frac{\partial}{\partial\lambda_{\gamma_j}}) = \Omega_{(\rho)}(f^l_{\gamma_i} T_{\gamma_l} + g^l_{\gamma_i} T_{\alpha_l}, f^k_{\gamma_j} T_{\gamma_k} + g^k_{\gamma_j} T_{\alpha_k}) =$$

$$= f^l_{\gamma_i} f^k_{\gamma_j} T_{\gamma_l}\lambda_{\gamma_k} + f^l_{\gamma_i} g^k_{\gamma_j} T_{\gamma_l}\lambda_{\alpha_k} + g^l_{\gamma_i} f^k_{\gamma_j} T_{\alpha_l}\lambda_{\gamma_k} + g^l_{\gamma_i} g^k_{\gamma_j} T_{\alpha_l}\lambda_{\alpha_k}$$

which is clearly real when evaluated at ρ.

Lemma 3.3. *At fuchsian points* $\omega = \omega_{WP}$

Proof: Let α,β be simple closed curves on S and t_α, t_β the corresponding twist vector fields. Then at a fuchsian ρ we have by the holomorphicity of bending:

$$\omega_{WP_{(\rho)}}(t_\alpha, t_\beta) = \frac{d}{dx}(x=0)(l_\beta(B_\alpha(x,\rho))) =$$

$$\mathrm{Re}[\frac{d}{dz}(z=0)(\lambda_\beta(B_\alpha(z,\rho))] = \mathrm{Re}[\mathrm{T}_{\alpha_{(\rho)}} \lambda_\beta] = \mathrm{Re}[\Omega_{(\rho)}(\mathrm{T}_\alpha, \mathrm{T}_\beta)] = \omega_{(\rho)}(t_\alpha, t_\beta)$$

From Lemma 3.3 and S.Wolpert's formula for ω_{WP} we have that

$$\mathrm{Re}[\Omega(\frac{\partial}{\partial\lambda_{\gamma_i}}, \frac{\partial}{\partial\lambda_{\gamma_j}})] = \omega(\frac{\partial}{\partial l_{\gamma_i}}, \frac{\partial}{\partial l_{\gamma_j}}) = \omega_{WP}(\frac{\partial}{\partial l_{\gamma_i}}, \frac{\partial}{\partial l_{\gamma_j}}) = 0$$

and also by the previous discussion

$$\mathrm{Im}[\Omega(\frac{\partial}{\partial\lambda_{\gamma_i}}, \frac{\partial}{\partial\lambda_{\gamma_j}})] = 0$$

at fuchsians.

Now since each $\Omega_{ij} = \Omega(\frac{\partial}{\partial\lambda_{\gamma_i}}, \frac{\partial}{\partial\lambda_{\gamma_j}})$ is a holomorphic function on $Q(S)$,that is

$$\Omega_{ij} = \Omega_{ij}(\lambda_{\gamma_1}, \ldots, \lambda_{\gamma_{3g-3}}, \beta_1, \ldots, \beta_{3g-3})$$

and also at fuchsian points

$$\Omega_{ij} = \Omega_{ij}(l_{\gamma_1}, \ldots, l_{\gamma_{3g-3}}, \tau_1, \ldots, \tau_{3g-3}, 0, \ldots, 0, 0, \ldots, 0) = 0,$$

we obtain by lemma 3.2 that $\Omega_{ij} = 0$ in $Q(S)$ and thus the proof is complete.

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