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Master thesis

Straight ruled surfaces in the Roto-translational group

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Foreword

This thesis was written for the purpose of obtaining the M.Sc. diploma in Pure Mathematics, as this is specified in the regulation of the postgraduate programme of the Department of Mathematics and Applied Mathematics, University of Crete. The thesis was supervised by Associate Professor I.D . Platis.

Contents

Chapter 1

Introduction

The roto-translational group $\mathcal{RT}$ is the group comprising rotations and translations of the Euclidean plane. It is 3-dimensional Lie group, isomorphic to $\mathbb{R}^2 \times S^1$ with multiplication given by

$$(x, y, \theta) \star (x', y', \theta') = (x + x' \cos\theta - y' \sin\theta, y + x' \sin\theta + y' \cos\theta, \theta + \theta'),$$

for all $(x, y, \theta), (x', y', \theta') \in \mathbb{R}^2 \times S^1$. The group $\mathcal{RT}$ constitutes a model of sub-Riemannian Geometry that does not come from a nilpotent (Carnot) group. And this is in contrast to the well known case of the *Heisenberg group*; that is the Lie group whose underlying manifold is $\mathbb{R}^3$ and the multiplication is

$$(x, y, t) \cdot (x', y', t') = (x + x', y + y', t + t' - 2(xy' - x'y)),$$

for all $(x, y, t), (x', y', t') \in \mathbb{R}^3$.

Embedding two-dimensional surfaces into 3-dimensional sub-Riemannian manifolds is an extremely interesting topic; for the case of the Heisenberg group there is a quite extensive bibliography, illustratively, see [10], [9], [4], as well as the references therein. Suppose that a surface $\mathscr{S}$ is embedded in a 3-dimensional sub-Riemannian manifold M. If $p \in \mathscr{S}$, then consider the horizontal tangent space of $\mathscr{S}$ at p to be the horizontal tangent space of M at p. Under some conditrions, one can then define a horizontal normal vector to $\mathscr{S}$ at p, which in the concept of sub-Riemannian Geometry of surfaces plays the role of the normal vector in the Euclidean case. A horizontal mean curvature H_p is then defined by means of the horizontal normal vector field and the natural question is then which of these surfaces are horizontally minimal, i.e., they have everywhere vanishing horizontal mean curvature.

In [11], Platis proved that when a $\mathscr{C}^2$ surface is embedded in the Heienberg group which is a straight ruled surface, then it is horizontally minimal and locally contactomorphic to the complex plane. The term contactomorphic here is in the broad sense: The Heisenberg group carries a contact structure compatible with iis sub-Riemannian one, and therefore an embedded surface inherits a 1-form by pulling back the contact form of the Heisenberg group. A straight ruled surface in this context is the counterpart of a developable surface in the Euclidean case; it is well known that those surfaces have everywhere vanishing Gaussian curvature and are locally isometric to the complex plane. The analogies are quite transparent.

In this thesis we take upon the question if a straight ruled surface in the roto-translational group is horizontally minimal. We prove that indeed this is the case, working in the universal cover $\mathscr{G}$ of $\mathcal{RT}$. The techniques we use are mainly those that have been used in [11]. Thus we present a quite large class of surfaces which solve the minimal surface problem in $\mathscr{G}$ (and in $\mathcal{RT}$) in the case of $\mathscr{C}^2$ surfaces.

The roto-translation group has some interesting applications; in the following we mention a few. There is an application to the neuro-biological image completion, in fact in the prediction of injuries and in geometrical model in mechanics and robotics. In the first case it has been proved that the solutions to the minimal surface problem in the roto-translational group are linked with the method that the brain completes missing visual data in the first layer of the visual cortex (V1), see [2]. The virtual cortex (V1) consists of the simple cells that are sensitive to brightness gradients with particular orientation. Simple cells are arranged in columns with the same orientantion and the columns have a hypercolumn structure which represent all possible orientantions.The model of this hypercolumn structure is $\mathbb{R}^2 \times S^1$ which is precisely the roto-translational group $\mathcal{RT}$. In [5] it is invastigated the above solutions to the minimal surface problem with Dirichlet boundary conditions and is given a method by characterising the smooth minimal surfaces as ruled surfaces to compute a minimal spanning surface with fixed boundary data. In addition, it is described a number of obstacles in existence and uniqueness; however, under suitable conditions, smooth minimal spanning surfaces with good properties exists. These results have great application to the neuro-biological image completion model but also to contracting disocclusion algorithms in digital image processing.

In the second case the roto-translational group plays a major role in the prediction and prevention of musculo-skeletal injuries is an important aspect of preventive health science. For instance, the motion of human's knees and ankles are translated by the roto-translational group, see [6].
In mechanics an interesting exaple is the motion of the hovercraft. The hovercraft is a promising future vehicle as it can move though water and any kind of ground require accurate route marking and positioning systems combined with suitable obstacle avoidance capabilities. The hovercraft is modeled as a planar rigid body subject to an external force. Its configuration space is again the roto-translation group. The hovercaft dynamics contrabillity and control algorithm for steerings is investigated in [8].
Finally the $\mathcal{RT}$ has an interesting application in robotics. For instance, the motion of a robot in the plane and the links of the robot such as its mechanical arms are usually modelled as rigid bodies; hence they can be investigated with the help of $\mathcal{RT}$, see [12].

This thesis is organised as follows: In Chapter 2 we introduce the roto-translation group $\mathcal{RT}$ and its universal cover $\mathscr{G}$ and study their geometric features. Chapter 3 is devoted to the study of the horizontal geometry of $\mathscr{C}^2$ surfaces embedded in $\mathscr{G}$. In Chapter 4 we describe straight ruled surfaces and prove our main theorem: A straight ruled surface in $\mathscr{G}$ is horizontally minimal.

Chapter 2

The roto-translational group and its universal cover

In this preliminary chapter we discuss the general geometric features our ambient spaces, that is, the roto-translational group $\mathcal{RT}$ and its universal cover $\mathscr{G} = \widetilde{\mathcal{RT}}$ in Sections 2.1, 2.2, respectively.

2.1 The roto-translational group

The roto-translational group $\mathcal{RT}$ is defined as the set comprising rotations and translations of the Euclidean space $\mathbb{R}^2$; this set has a non-Abelian group structure see Section 2.1.1. It is also isomorphic with $\mathbb{R}^2 \times S^1$ with the multiplication given by (2.1), see Section 2.1.2. Its Lie Algebra is discussed in Section 2.1.3 and the proof that $\mathcal{RT}$ is not a nilpotent group is in Section 2.1.4.

2.1.1 Definition and group structure

We consider the space $\mathbb{R}^2$ and for fixed $(x, y) \in \mathbb{R}^2$ we denote by $T_{(x,y)}$ the translation in the direction of (x, y):

$$T_{(x,y)}(a, b) = (x + a, y + b), \quad (a, b) \in \mathbb{R}^2.$$

Now for fixed $\theta \in S^1$ denote by R_θ the rotation in an angle θ:

$$R_\theta(a, b) = (a \cos\theta - b \sin\theta,\ a \sin\theta + b \cos\theta), \quad (a, b) \in \mathbb{R}^2.$$

We now fix both a (x, y) and a θ in $\mathbb{R}^2$ and S^1, respectively.

Definition 2.1. The *roto-translation* $RT_{(x,y,\theta)}$ is the composition $T_{(x,y)} \circ R_\theta$. Explicitly,

$$RT_{(x,y,\theta)}(a, b) = (x + a \cos\theta - b \sin\theta,\ y + a \sin\theta + b \cos\theta).$$

The *roto-translation group* is the set $\mathcal{RT}$ comprising roto-translations.

The next elementary proposition shows that $\mathcal{RT}$ is actually a group.

Proposition 2.2. The set $\mathcal{RT}$ has a (non abelian) group structure.

Proof. Let $RT_{(x',y',\theta')}$ and $RT_{(x,y,\theta)}$ be two elements of $\mathcal{RT}$. Then

$$RT_{(x',y',\theta')} \circ RT_{(x,y,\theta)} = (T_{(x',y')} \circ R_{\theta'}) \circ (T_{(x,y)} \circ R_\theta).$$

Since

$$R_{\theta'} \circ T_{(x,y)} = T_{R'_\theta(x,y)} \circ R_{\theta'},$$

we have

$$\begin{aligned} RT_{(x',y',\theta')} \circ RT_{(x,y,\theta)} &= T_{(x',y')} \circ T_{R_{\theta'}(x,y)} \circ R_{\theta'} \circ R_\theta \\ &= T_{(x',y')+R_{\theta'}(x,y)} \circ R_{\theta+\theta'}. \end{aligned}$$

The following hold:

$$RT_{(x,y,\theta)} \circ RT_{(0,0,0)} = RT_{(x,y,\theta)}, \quad RT^{-1}_{(x,y,\theta)} = RT_{(x',y',\theta')},$$

where

$$x' = -x\cos\theta - y\sin\theta, \quad y' = x\sin\theta - y\cos\theta, \quad \theta' - \theta.$$

To prove the above relations, let first (a, b, c) be such that $RT_{(a,b,c)} \circ RT_{(x,y,\theta)} = RT_{(x,y,\theta)}$ for all (x, y, θ). Then we must have

$$(a + x\cos c - y\sin c, b + x\sin c + y\cos c, \theta + c) = (x, y, \theta),$$

that is, the equations

$$\begin{aligned} &a + x\cos c - y\sin c = x, \\ &b + x\sin c + y\cos c = y, \\ &\theta + c = \theta, \end{aligned}$$

have to have solution for all (x, y, θ). Solving the system we obtain $a = b = c = 0$. To prove the second relation, suppose that for fixed (x, y, θ) there exists a (x', y', θ') such that

$$RT_{(x,y,\theta)} \circ RT_{(x',y',\theta')} = RT_{(0,0,0)}.$$

This is equivalent to

$$(x + x'\cos\theta - y'\sin\theta, y + x'\sin\theta + y'\cos\theta, \theta + \theta') = (0, 0, 0),$$

hence to the system of equations

$$\begin{aligned} &x + x'\cos\theta - y'\sin\theta = 0, \\ &y + x'\sin\theta + y'\cos\theta = 0, \\ &\theta + \theta' = 0. \end{aligned}$$

Solving the system in x', y', θ' we obtain the result. $\square$

2.1.2 The $\mathbb{R}^2 \times S^1$ model

The $\mathbb{R}^2 \times S^1$ model for $\mathcal{RT}$ is obtained in the following manner: Define a map $\mathcal{RT} \to \mathbb{R}^2 \times S^1$ given by

$$RT_{(x,y,\theta)} \mapsto (x, y, \theta).$$

It is clear that this map is a bijection. Moreover, $(\mathcal{RT}, \circ)$ is isomorphic to the set $(\mathbb{R}^2 \times S^1, \star)$, where $\star$ is the multiplication given by

$$(x, y, \theta) \star (x', y', \theta') = ((x, y) + R_\theta(x', y'),\ \theta + \theta')\,, \tag{2.1}$$

i.e.,

$$(x, y, \theta) \star (x', y', \theta') = (x + x' \cos\theta - y' \sin\theta, y + x' \sin\theta + y' \cos\theta, \theta + \theta'). \tag{2.2}$$

In this model, the neutral element of RT is $(0, 0, 0)$, and the inverse element of (x, y, θ) is $(-x\cos\theta - y\sin\theta, x\sin\theta - y\cos\theta, -\theta)$.

Remark 2.3. (Matrix model for $\mathcal{RT}$). There exists a matrix model for $\mathcal{RT}$: The matrix representation of elements in $\mathcal{RT}$ is

$$\mathcal{RT} = \left\{ \begin{pmatrix} R & b \\ 0 & 1 \end{pmatrix} \mid R \in \mathrm{SO}(2),\ \ b \in \mathbb{R}^2 \right\}.$$

This suggests clearly that $\mathcal{RT}$ may also be viewed as the group $\mathbb{R}^2 \times \mathrm{SO}(2)$ with multiplication the matrix multiplication.

2.1.3 Lie group structure

The roto-translational group is a Lie group with underlying manifold $\mathbb{R}^2 \times S^1$. We describe now a basis for its Lie algebra. We consider the following vector fields:

$$X_1 = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}, \qquad X_2 = \frac{\partial}{\partial \theta}, \qquad X_3 = \sin\theta \frac{\partial}{\partial x} - \cos\theta \frac{\partial}{\partial y}. \tag{2.3}$$

Proposition 2.4. The vector fields X_1, X_2, X_3 as in (2.3) are left-invariant vector fields.

Proof. Consider an arbitrary $a = (a_1, a_2, a_3) \in \mathcal{RT}$ and the left multiplication map L_a: for any $(x, y, \theta) \in \mathcal{RT}$ we have

$$L_a(x, y, \theta) = (a_1 + x\cos a_3 - y\sin a_3, a_2 + x\sin a_3 + y\cos a_3, \theta + a_3).$$

Clearly, L_a is differentiable. The matrix of the differential of L_a is

$$dL_a = \begin{pmatrix} \cos a_3 & -\sin a_3 & 0 \\ \sin a_3 & \cos a_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.4}$$

Thus we have

$$\begin{aligned} dL_a(X_1) &= \cos\theta dL_a(\frac{\partial}{\partial x}) + \sin\theta dL_a(\frac{\partial}{\partial y}) \\ &= \cos\theta(\cos a_3 \frac{\partial}{\partial x} + \sin a_3 \frac{\partial}{\partial y}) + \sin\theta(-\sin a_3 \frac{\partial}{\partial x} + \cos a_3 \frac{\partial}{\partial y}) \\ &= (\cos\theta\cos a_3 - \sin\theta\sin a_3)\frac{\partial}{\partial x} + (\cos\theta\sin a_3 + \sin\theta\cos a_3)\frac{\partial}{\partial y} \\ &= \cos(\theta + a_3)\frac{\partial}{\partial x} + \sin(\theta + a_3)\frac{\partial}{\partial y} \\ &= X_1(L_a). \end{aligned}$$

Also,

$$dL_a(X_2) = dL_a(\frac{\partial}{\partial\theta}) = \frac{\partial}{\partial\theta} = X_2(L_a),$$

and finally,

$$\begin{aligned} dL_a(X_3) &= \sin\theta dL_a(\frac{\partial}{\partial x}) - \cos\theta dL_a(\frac{\partial}{\partial y}) \\ &= \sin\theta(\cos a_3 \frac{\partial}{\partial x} + \sin a_3 \frac{\partial}{\partial y}) - \cos\theta(-\sin a_3 \frac{\partial}{\partial x} + \cos a_3 \frac{\partial}{\partial y}) \\ &= (\sin\theta\cos a_3 + \cos\theta\sin a_3)\frac{\partial}{\partial x} + (\sin\theta\sin a_3 - \cos\theta\cos a_3)\frac{\partial}{\partial y} \\ &= \sin(\theta + a_3)\frac{\partial}{\partial x} - \cos(\theta + a_3)\frac{\partial}{\partial y} \\ &= X_3(L_a). \end{aligned}$$

Therefore X_1, X_2, X_3 are all left-invariant. □

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Corollary 2.5. The vector fields X_1, X_2, X_2 span the 3-dimensional tangent space (Lie algebra) $T(\mathcal{RT}) = \mathfrak{rt}$ at each point, that is,

$$\mathfrak{rt} = \operatorname{span}\{X_1, X_2, X_3\}.$$

Corollary 2.6. Since the determinant of the Jacobian of L_a equals to 1, the Haar measure for $\mathcal{RT}$ is the Lebesgue measure in $\mathbb{R}^2 \times S^1$.

Proposition 2.7. We have the following Lie bracket relations for the left-invariant vector fields X_1, X_2, X_3 of $\mathcal{RT}$:

$$[X_1, X_2] = X_3, \quad [X_2, X_3] = X_1, \quad [X_1, X_3] = 0. \tag{2.5}$$

Proof. We calculate straightforwardly: First,

$$\begin{aligned}
[X_1, X_2] &= \left[\cos\theta\frac{\partial}{\partial x}, \frac{\partial}{\partial\theta}\right] + \left[\sin\theta\frac{\partial}{\partial y}, \frac{\partial}{\partial\theta}\right] \\
&= \sin\theta\frac{\partial}{\partial x} - \cos\theta\frac{\partial}{\partial y} \\
&= X_3.
\end{aligned}$$

Next,

$$\begin{aligned}
[X_2, X_3] &= X_2X_3 - X_3X_2 \\
&= \frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial}{\partial x} - \cos\theta\frac{\partial}{\partial y}\Big) \\
&= \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y} \\
&= X_1.
\end{aligned}$$

Finally,

$$\begin{aligned}
[X_1, X_3] &= \Big(\cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}\Big)\Big(\sin\theta\frac{\partial}{\partial x} - \cos\theta\frac{\partial}{\partial y}\Big) \\
&\quad -\Big(\sin\theta\frac{\partial}{\partial x} - \cos\theta\frac{\partial}{\partial y}\Big)\Big(\cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}\Big) \\
&= 0.
\end{aligned}$$

□

Corollary 2.8. If $\mathfrak{v}_0 = \operatorname{span}\{X_1, X_2\}$ and $\mathfrak{v}_1 = \operatorname{span}\{X_3\}$, then the Lie algebra $\mathfrak{rt}$ may be written as

$$\mathfrak{rt} = \mathfrak{v}_0 \oplus \mathfrak{v}_1.$$

2.1.4 $\mathcal{RT}$ is not nilpotent

In contrast to the Heisenberg group case, the roto-translational group $\mathcal{RT}$ is not nilpotent, see Proposition 2.10 below. We first recall the definition of a nilpotent group: We start by defining the *descending central series* of a Lie algebra $\mathfrak{g}$:

$$\mathfrak{g}^{(1)} = \mathfrak{g}, \quad \mathfrak{g}^{(k+1)} = [\mathfrak{g}, \mathfrak{g}^{(k)}].$$

Recall here that if $\mathfrak{a}$ and $\mathfrak{b}$ be two subspaces of a Lie algebra $\mathfrak{g}$, then we write

$$[\mathfrak{a}, \mathfrak{b}] = \mathrm{span}\{[X, Y]\} \text{ with } X \in \mathfrak{a}, \ Y \in \mathfrak{b}.$$

The following hold; their proof is straightforward:

(a) $\mathfrak{g}^{(k+1)} \subset \mathfrak{g}^{(k)}$ for every k;

(b) $\mathfrak{g}^{(n)} = \mathfrak{g}^{(n+1)}$ for some n implies $\mathfrak{g}^{(m)} = \mathfrak{g}^{(n)}$ for every $m \geq n$.

Definition 2.9. A Lie algebra $\mathfrak{g}$ is nilpotent if there exists a positive integer i such that $\mathfrak{g}^{(i)} \neq \{0\}$ and $\mathfrak{g}^{(i+1)} = \{0\}$.

Proposition 2.10. For every $n \geq 2$ we have

$$\mathfrak{rt}^{(n)} = \mathrm{span}\{X_1, X_3\}.$$

Thus $\mathcal{RT}$ is not nilpotent.

Proof. For $n = 2$ we have

$$\begin{aligned} \mathfrak{rt}^{(2)} = [\mathfrak{rt}, \mathfrak{rt}] &= \{\lambda_1[X_1, X_2] + \lambda_2[X_2, X_3] + \lambda_3[X_1, X_3]\} \\ &= \mathrm{span}\{X_1, X_3\}. \end{aligned}$$

Now we suppose that the statement it is true for some $n > 2$, that is,

$$\mathfrak{rt}^{(n)} = \mathrm{span}\{X_1, X_3\}.$$

Then for $(n + 1)$ we have

$$\mathfrak{rt}^{(n+1)} = [\mathfrak{rt}, \mathfrak{rt}^{(n)}] = [\mathfrak{rt}, \mathrm{span}\{X_1, X_3\}] = \mathrm{span}\{X_1, X_3\}.$$

$\square$

We notice that $\mathcal{RT}$ is *not* a Carnot group as it is not nilpotent.

2.2 The universal cover $\mathscr{G}$ of $\mathcal{RT}$

Instead of working with $\mathcal{RT}$, we choose from now on to work with its universal cover which we shall denote by $\mathscr{G}$. This group is $\mathbb{R}^3$ with coordinates (x, y, θ) and with multiplication given in the same manner as in $\mathcal{RT}$. Actually, it also has the same left-invariant vector fields, its Haar measure is just the Lebesgue measure in $\mathbb{R}^3$ and of course, $\mathscr{G}$ is not nilpotent. The advantage of working with $\mathscr{G}$ is that $\mathscr{G}$ is simply connected whereas $\mathcal{RT}$ is not. In this way, we shall always have a bijective correspondence between the group automorphisms of $\mathscr{G}$ and its Lie algebra, which is $\mathfrak{rt}$. The contact form (Section 2.2.1), the horizontal tangent bundle $\mathrm{H}(\mathscr{G}) = \langle X_1, X_2 \rangle$ (Section 2.2.2), and the sub-Riemannian distance in $\mathscr{G}$ (Section 2.2.3) are discussed in this section.

2.2.1 Invariant contact form

We may construct a left-invariant contact form ω for $\mathscr{G}$ so that ker $\omega = V_0$ and its Reeb field is X_3 as follows: Let $\omega = \alpha dx + \beta dy + \gamma d\theta$ be the wished form of $\mathscr{G}$. Solving the system:

$$\omega(X_1) = \omega(X_2) = 0 \quad \omega(X_3) = 1,$$

we get

$$\omega = \sin\theta dx - \cos\theta dy. \tag{2.6}$$

On the other hand,

$$d\omega = \cos\theta d\theta \wedge dx + \sin\theta d\theta \wedge dy$$

and for each X we have $d\omega(X, X_3) = 0$; this shows that X_3 is the Reeb field of ω. Finally, we show that $\omega \wedge d\omega$ equals to minus the Lebesque measure on $\mathbb{R}^3$: Indeed,

$$\begin{aligned} \omega \wedge d\omega &= (\sin\theta dx - \cos\theta dy) \wedge (\cos\theta d\theta \wedge dx + \sin\theta d\theta \wedge dy) \\ &= \sin^2\theta dx \wedge d\theta \wedge dy - \cos^2\theta dy \wedge d\theta \wedge dx \\ &= -\sin^2\theta dx \wedge dy \wedge d\theta - \cos^2\theta dx \wedge dy \wedge d\theta = -dx \wedge dy \wedge d\theta. \end{aligned}$$

Now we have:

Proposition 2.11. The contact form ω is left-invariant.

Proof. Let L_a, $a = (a_1, a_2, a_3)$ be a left-translation. We calculate straightforwardly:

$$\begin{aligned} L_a^*\omega &= \sin(\theta + a_3)d(a_1 + x\cos a_3 - y\sin a_3) \\ &\quad - \cos(\theta + a_3)d(a_2 + x\sin a_3 + y\cos a_3) \\ &= \sin(\theta + a_3)(\cos a_3 dx - \sin a_3 dy) \\ &\quad - \cos(\theta + a_3)(\sin a_3 dx + \cos a_3 dy) \\ &= (\cos a_3 \sin(\theta + a_3) - \sin a_3 \cos(\theta + a_3))dx \\ &\quad -(\cos a_3 \cos(\theta + a_3) + \sin a_3 \sin(\theta + a_3))dy \\ &= \sin\theta dx - \cos\theta dy = \omega. \end{aligned}$$

□

2.2.2 The horizontal tangent space of $\mathscr{G}$

Consider now the distribution H, defined as the subbundle $\mathfrak{v}_0$ of the tangent bundle spanned at every point by the left invariant frame X_1, X_2. From the construction of ω we have that $\mathrm{H} = \mathrm{span}\{X_1, X_2\} = \ker\omega$. We call H the *horizontal distribution* of $\mathscr{G}$. It is clearly non-integrable, since $[X_1, X_2] = X_3$.

Definition 2.12. A $\mathscr{C}^2$ curve $\gamma : [a, b] \to \mathscr{G}$ with

$$\gamma(t) = (x(t), y(t), \theta(t)) \in \mathscr{G},$$

is called *horizontal* if

$$\dot\gamma(t) \in \mathrm{H}_{\gamma(t)}(\mathscr{G}) \quad \text{for every } t \in [a, b].$$

In other words, horizontal curves are curves for which the component of the speed vector $\dot\gamma$ in the direction of X_3 vanishes.

Proposition 2.13. The $\mathscr{C}^2$ curve $\gamma(t) = (x(t), y(t), \theta(t))$ is horizontal if and only if it satisfies the differential equation

$$\dot x(t)\sin\theta(t) - \dot y(t)\cos\theta(t) = 0, \tag{2.7}$$

and the following cases may occur:

i) γ is a straight line of the form

$$x = \text{const.}, \quad \theta(t) = 2\kappa\pi \pm \pi/2, \quad \kappa \in \mathbb{Z}, \tag{2.8}$$

or,

$$y = \text{const.}, \quad \theta(t) = 2\kappa\pi \pm \pi, \quad \kappa \in \mathbb{Z}. \tag{2.9}$$

In the first case, the horizontal tangent is

$$\dot{\gamma}(t) = \pm \dot{y}(t) X_1,$$

and in the second case, the horizpntal tangent is

$$\dot{\gamma}(t) = \pm \dot{x}(t) X_1.$$

ii) $\gamma(t) = (x(t), y(t), \theta(t))$ where

$$\theta(t) = \tan^{-1}\left(\frac{\dot{y}(t)}{\dot{x}(t)}\right).$$

The horizontal tangent is then

$$\dot{\gamma}(t) = |\tilde{\gamma}(t)| X_1 + \kappa_s(t) X_2.$$

Here, $\tilde{\gamma}(t) = (x(t), y(t))$ is the projection of γ in $\mathbb{R}^2$, $|\tilde{\gamma}(t)|$ is its Euclidean norm and $\kappa_s(t)$ is the signed curvature of $\tilde{\gamma}(t)$.

Proof. From relations (2.3) we may write

$$\begin{aligned}
\frac{\partial}{\partial x} &= \cos\theta X_1 + \sin\theta X_3, \\
\frac{\partial}{\partial \theta} &= X_2 \\
\frac{\partial}{\partial y} &= \sin\theta X_1 - \cos\theta X_3.
\end{aligned}$$

Therefore

$$\begin{aligned}
\dot{\gamma}(t) &= \dot{x}(t)\frac{\partial}{\partial x} + \dot{y}(t)\frac{\partial}{\partial y} + \dot{\theta}(t)\frac{\partial}{\partial \theta} \\
&= (\dot{x}(t)\cos\theta(t) + \dot{y}(t)\sin\theta(t)) X_1 \\
&\quad + \dot{\theta} X_2 \\
&\quad + (\dot{x}(t)\sin\theta(t) - \dot{y}(t)\cos\theta(t)) X_3.
\end{aligned}$$

Hence it follows by definition that γ is horizontal if and only if Equation (2.7) holds. To track down all horizontal curves, we consider the trivial cases first, that is,

$$\dot{x}(t) = 0 \quad \text{and} \quad \dot{y}(t) = 0,$$

for $t \in \mathrm{I}$. In the first case, $\dot{y}(t) \neq 0$ since γ is regular, therefore $\theta(t) = 2\kappa\pi \pm \pi/2$, $\kappa \in \mathbb{Z}$ and the horizontal line is the straight line given by Equation (2.8). Similarly, for the second case we obtain the straight lines given by Equation (2.9). The formulae for their respective horizontal tangents follow.

Suppose now that $\dot{x}(t) \neq 0$ for $t \in \mathrm{I}$. Then we may write (2.7) as

$$\theta(t) = \tan^{-1}\left(\frac{\dot{y}(t)}{\dot{x}(t)}\right).$$

Taking the derivative in both sides we obtain

$$\dot{\theta}(t) = \frac{\ddot{y}(t)\dot{x}(t) - \ddot{x}(t)\dot{y}(t)}{(\dot{x}(t))^2 + (\dot{y}(t))^2},$$

i.e., $\dot{\theta}(t)$ is the signed curvature $\kappa_s(t)$ of the projection $\tilde{\gamma}(t) = (x(t), y(t))$ of $\gamma(t)$ in the plane. Since

$$\begin{aligned}
(\dot{x}(t)\cos\theta(t) + \dot{y}(t)\sin\theta(t))^2 &= \cos^2\theta(t)(\dot{x}(t) + \dot{y}(t)\tan\theta(t))^2 \\
&= \frac{1}{1 + \tan^2\theta(t)}\left(\dot{x}(t) + \frac{\dot{y}(t)}{\dot{x}(t)}\right)^2 \\
&= \frac{1}{1 + \frac{(\dot{y}(t))^2}{(\dot{x}(t))^2}} \frac{((\dot{x}(t))^2 + (\dot{x}(t))^2)^2}{(\dot{x}(t))^2} \\
&= (\dot{x}(t))^2 + (\dot{y}(t))^2,
\end{aligned}$$

the formula for the horizontal tangent follows.

□

2.2.3 Sub-Riemannian product, Carnot-Carathéodory distance

The sub-Riemannian structure of $\mathscr{G}$ which is compatible with its contact structure is given by defining an inner product $\langle\cdot,\cdot\rangle_p$ at each point $p \in \mathscr{G}$ so that X_1 and X_2 form an orthonormal basis of the horizontal tangent space $\mathrm{H_p}$. That is,

$$\langle X_1, X_1\rangle_p = \langle X_2, X_2\rangle_p = 1, \quad \langle X_1, X_2\rangle_p = \langle X_2, X_1\rangle_p = 0,$$

and the induced norm shall be denoted by $\|\cdot\|$. We define the horizontal length of a curve γ to be

$$\ell(\gamma) = \int_a^b \|\dot\gamma(t)\| dt = \int_a^b \sqrt{\langle \dot\gamma(t), (X_1)_{\gamma(t)}\rangle^2 + \langle \dot\gamma(t), (X_2)_{\gamma(t)}\rangle^2} dt,$$

which is of course

$$\begin{aligned} ||\gamma(t)|| &= \left(\langle \dot\gamma(t), (X_1)_{\gamma(t)}\rangle^2 + \langle \dot\gamma(t), (X_2)_{\gamma(t)}\rangle^2\right)^{1/2} \\ &= \left|(\dot x(t))^2 + (\dot y(t))^2 + \kappa_s^2(t)\right|. \end{aligned}$$

The *Carnot-Caratheodory* distance of two arbitrary points $p, q \in \mathscr{G}$ is

$$d_{cc}(p,q) = \inf_\gamma \ell(\gamma),$$

where γ is a horizontal curve joining p and q.

2.2.4 Euclidean distance

Let (x, y, θ) be the real coordinates in $\mathscr{G}$. We define a distance $d_{\mathscr{G}}$ by

$$d_{\mathscr{G}}(p,p') = |(p')^{-1} \star p|$$

for each $p = (x, y, \theta)$, $p' = (x', y', \theta') \in \mathscr{G}$. Here $|\cdot|$ is the Euclidean norm of $\mathbb{R}^3$. By calculating straightforwardly we have

$$\begin{aligned} d_{\mathscr{G}}(p,p') &= |(-x'\cos\theta' - y'\sin\theta', x'\sin\theta' - y'\cos\theta', -\theta') \star (x,y,\theta)| \\ &= |(x-x')\cos\theta' + (y-y')\sin\theta', (y-y')\cos\theta' - (x-x')\sin\theta', \theta-\theta'| \\ &= \sqrt{(x-x')^2 + (y-y')^2 + (\theta-\theta')^2}. \end{aligned}$$

That is, $d_{\mathscr{G}}$ is just the Euclidean distance in $\mathbb{R}^3$. Note that by definition

$$d_{\mathscr{G}}(p,p') = d_{\mathscr{G}}((p')^{-1} * p, (0,0,0)). \tag{2.10}$$

The distance $d_{\mathscr{G}}$ is invariant by left-translations, that is by the left action of $\mathscr{G}$ onto itself: Let $a = (a_1, a_2, a_3) \in \mathscr{G}$ and L_a the corresponding left-translation. Thus since for each $p, p' \in \mathscr{G}$ we have that (2.10) holds, it suffices to show that

$$d_{\mathscr{G}}(L_a(p), L_a(0)) = d_{\mathscr{G}}(p, 0).$$

For this, we have

$$\begin{aligned} d_{\mathscr{G}}(L_a(p), L_a(0)) &= d_{\mathscr{G}}(L_a(p), a) = d_{\mathscr{G}}(a^{-1} \star L_a(p), 0) \\ &= d_{\mathscr{G}}(L_{a^{-1}}(L_a(p)), 0) = d_{\mathscr{G}}(p, 0). \end{aligned}$$

The metric $d_{\mathscr{G}}$ is invariant by the following action of SO(2) on $\mathscr{G}$:

$$\begin{array}{rcl} A_\phi : \mathrm{SO}(2) \times \mathscr{G} & \longrightarrow & \mathscr{G} \\ (\phi, (x, y, \theta)) & \mapsto & (x \cos\phi - y \sin\phi, x \sin\phi + y \cos\phi, \theta). \end{array}$$

Indeed,

$$\begin{aligned} d_{\mathscr{G}}(A_\phi(p), A_\phi(p')) &= |\cos\phi(x - x') - \sin\phi(y - y'), \sin\phi(x - x') + \cos\phi(y - y'), \theta - \theta'| \\ &= d_{\mathscr{G}}(p, p'). \end{aligned}$$

It is also invariant by the following action of $\mathbb{R}^2$:

$$\begin{array}{rcl} A_b : \mathbb{R}^2 \times \mathscr{G} & \longrightarrow & \mathscr{G} \\ ((b_1, b_2), (x, y, \theta)) & \mapsto & (x + b_1, y + b_2, \theta). \end{array}$$

It is here clear that

$$d_{\mathscr{G}}(A_b(p), A_b(p')) = d_{\mathscr{G}}(p, p').$$

We conclude that $d_{\mathscr{G}}$ remains invariant by the left action of $\mathscr{G}$.

Remark 2.14. One may prove that the only isometries of $d_{\mathscr{G}}$ are in fact coming from the left action of $\mathscr{G}$. Any other isometry of $\mathbb{R}^3$ does not induce a left-invariant isometry of $d_{\mathscr{G}}$, see [7].

Chapter 3

Horizontal geometry of $\mathscr{C}^2$ surfaces embedded in $\mathscr{G}$

In this chapter we deal with some basic features of the horizontal geometry of $\mathscr{C}^2$ surfaces embedded in the universal cover $\mathscr{G}$ of the roto-translational group $\mathcal{RT}$. We take upon the manner of resentation in [11]. In Section 3.1 we introduce our surfaces in study and define their horizontal tangent space and their horizontal normal. $\mathscr{C}^2$ surfaces admit an 1-form; this is just the pull-back of the contact form ω of $\mathscr{G}$. This induced 1-form is studied in Section 3.2. The notion of the horizontal mean curvature of a $\mathscr{C}^2$ surface embedded in $\mathscr{G}$ is found in Section 3.3 and finally, graphs are studied in Section 3.4.

3.1 $\mathscr{C}^2$ surfaces in $\mathscr{G}$

Our object of study is $\mathscr{C}^2$ surfaces emebedded in $\mathscr{G}$, that is, $\mathbb{R}^3$. We review the well known $\mathscr{C}^2$ surfaces in Section 3.1.1. Being embedded in $\mathscr{G}$, $\mathscr{C}^2$ surfaces have a horizontal tangent space at each point, that is, the horizontal tangent space of $\mathscr{G}$ at this point. To this horizontal space there can be defined a horizontal normal vector, pretty much in the same way that the usual euclidean normal is defined, see Section 2.2.2. In contrast to the euclidean case, this horizontal normal cannot be defined at all points of the $\mathscr{C}^2$ surface;

there is an exceptional set. But outside this set, a unit horizontal normal vector fiels is defined properly, see Section 3.1.3. Finally, we define horizontal area in Section 3.1.4.

3.1.1 Regular surfaces

By a $\mathscr{C}^2$ surface we shall primarily mean a *regular surface* in $\mathbb{R}^3$, that is, a countable collection of surface patches $\sigma_\alpha : U_\alpha \to V_\alpha$, where U_α and V_α are open sets of $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively, such that:

1. Each σ_α is a $\mathscr{C}^2$ homeomorphism.
2. The differential $(\sigma_\alpha)_*$ is everywhere of rank 2.

Let p be an arbitrary point of $\mathscr{S}$ and suppose that $\sigma : U \to \mathbb{R}^3$,

$$\sigma(u,v) = (x(u,v), y(u,v), \theta(u,v)),$$

is a surface patch of $\mathscr{S}$ such that $p = \sigma(\mathbf{u_0})$ for some $\mathbf{u_0} = (u_0, v_0) \in U$. The tangent plane $T_p(\mathscr{S})$ of $\mathscr{S}$ at p is spanned by the vectors

$$\begin{aligned}
(\sigma_u)_p &= x_u(\mathbf{u_0})\left(\frac{\partial}{\partial x}\right)_p + y_u(\mathbf{u_0})\left(\frac{\partial}{\partial y}\right)_p + \theta_u(\mathbf{u_0})\left(\frac{\partial}{\partial \theta}\right)_p, \\
(\sigma_v)_p &= x_v(\mathbf{u_0})\left(\frac{\partial}{\partial x}\right)_p + y_v(\mathbf{u_0})\left(\frac{\partial}{\partial y}\right)_p + \theta_v(\mathbf{u_0})\left(\frac{\partial}{\partial \theta}\right)_p
\end{aligned}$$

and the normal N_p of $\mathscr{S}$ at p is the exterior product $(\sigma_u)_p \wedge (\sigma_v)_p$:

$$N_p = \partial(y,\theta)_{\mathbf{u_0}}\left(\frac{\partial}{\partial x}\right)_p + \partial(\theta,x)_{\mathbf{u_0}}\left(\frac{\partial}{\partial y}\right)_p + \partial(x,y)_{\mathbf{u_0}}\left(\frac{\partial}{\partial \theta}\right)_p.$$

Here,

$$\partial(y,\theta)_{\mathbf{u_0}} = \left|\frac{\partial(y,\theta)}{\partial(u,v)}\right|_{\mathbf{u_0}} = y_u(\mathbf{u_0})\theta_v(\mathbf{u_0}) - y_v(\mathbf{u_0})\theta_u(\mathbf{u_0})$$

and similarly for the other determinants.

3.1.2 Horizontal tangent space, horizontal normal

Definition 3.1. Let $\mathscr{S}$ be a regular surface and $p \in \mathscr{S}$. The *horizontal tangent plane* $\mathrm{H_p}(\mathscr{S})$ of $\mathscr{S}$ at p is defined to be the horizontal plane $\mathrm{H_p}(\mathscr{G})$.

To determine exactly $\mathrm{H_p}(\mathscr{S})$, we first introduce an exterior product in $\mathscr{G}$:

Definition 3.2. If $a, b \in \mathrm{T}(\mathscr{G})$, the tangent bundle of $\mathscr{G}$, are such that

$$a = a_1X_1 + a_2X_2 + a_3X_3 \quad \text{and} \quad b = b_1X_1 + b_2X_2 + b_3X_3,$$

we define

$$a \wedge^{\mathscr{G}} b = \begin{vmatrix} X_1 & X_2 & X_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)X_1 + (a_3b_1 - a_1b_3)X_2 + (a_1b_2 - a_2b_1)X_3. \tag{3.1}$$

Definition 3.3. The vector field $a \wedge^{\mathscr{G}} b$ shall be called the exterior $\mathscr{G}$-product of a and b.

Some elementary properties of the exterior $\mathscr{G}$-product are found in the next proposition.

Proposition 3.4. The following hold.

1. $a \wedge^{\mathscr{G}} b = -b \wedge^{\mathscr{G}} a$ for each $a, b \in \mathrm{T}(\mathscr{G})$.
2. Clock rules:

$$X_1 \wedge^{\mathscr{G}} X_2 = X_3, \quad X_2 \wedge^{\mathscr{G}} X_3 = X_1, \quad X_3 \wedge^{\mathscr{G}} X_1 = X_2.$$

Proof. For the proof of (1) we have

$$a \wedge^{\mathscr{G}} b = \begin{vmatrix} X_1 & X_2 & X_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = - \begin{vmatrix} X_1 & X_2 & X_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = -b \wedge^{\mathscr{G}} a.$$

Now to prove (2) we write

$$X_1 \wedge^{\mathscr{G}} X_2 = \begin{vmatrix} X_1 & X_2 & X_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = X_3,$$

and analogously for $X_2 \wedge^{\mathscr{G}} X_3$ and $X_3 \wedge^{\mathscr{G}} X_1$. $\square$

We wish now to express the vector fields

$$\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}$$

in terms of the left-invariant vector fields X_1, X_3. Note that $X_2 = \partial/\partial\theta$. To do so, we write

$$\begin{aligned} X_1 &= \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}, \\ X_3 &= \sin\theta \frac{\partial}{\partial x} - \cos\theta \frac{\partial}{\partial y}. \end{aligned}$$

By using simple Linear Algebra, we solve in $\partial/\partial x$, $\partial/\partial y$ to obtain:

$$\begin{aligned} \frac{\partial}{\partial x} &= \cos\theta X_1 + \sin\theta X_3, \\ \frac{\partial}{\partial y} &= \sin\theta X_1 - \cos\theta X_3. \end{aligned}$$

Hence we have the following expressions for $(\sigma_u)_p$ and $(\sigma_v)_p$:

$$\begin{aligned} (\sigma_u)_p \quad = \quad & (x_u(\mathbf{u_0})\cos\theta(\mathbf{u_0}) + y_u(\mathbf{u_0})\sin\theta(\mathbf{u_0}))(X_1)_p + \theta_u(\mathbf{u_0})(X_2)_p \\ & + (x_u(\mathbf{u_0})\sin\theta(\mathbf{u_0}) - y_u(\mathbf{u_0})\cos\theta(\mathbf{u_0}))(X_3)_p, \end{aligned}$$

and

$$\begin{aligned} (\sigma_v)_p \quad = \quad & (x_v(\mathbf{u_0})\cos\theta(\mathbf{u_0}) + y_v(\mathbf{u_0})\sin\theta(\mathbf{u_0}))(X_1)_p + \theta_v(\mathbf{u_0})(X_2)_p \\ & + (x_v(\mathbf{u_0})\sin\theta(\mathbf{u_0}) - y_v(\mathbf{u_0})\cos\theta(\mathbf{u_0}))(X_3)_p. \end{aligned}$$

The exterior $\mathscr{G}$-product of $(\sigma_u)_p$ and $(\sigma_v)_p$ is thus

$$\begin{aligned} (\sigma_u)_p \wedge^{\mathscr{G}} (\sigma_v)_p = & \Big(\partial(y,\theta)_{\mathbf{u_0}}\cos\theta(\mathbf{u_0}) - \partial(x,\theta)_{\mathbf{u_0}}\sin\theta(\mathbf{u_0})\Big)(X_1)_p + \partial(x,y)_{\mathbf{u_0}}(X_2)_p \\ & + \Big(\partial(x,\theta)_{\mathbf{u_0}}\cos\theta(\mathbf{u_0}) + \partial(y,\theta)_{\mathbf{u_0}}\sin\theta(\mathbf{u_0})\Big)(X_3)_p. \end{aligned} \tag{3.2}$$

We state now the definition of the horizontal normal at a point of a $\mathscr{C}^2$ surface:

Definition 3.5. The *horizontal normal* N_p^h of $\mathscr{S}$ at p is the element of the horizontal tangent space $\mathrm{H}_p(\mathscr{G})$ which is the horizontal part of $(\sigma_u)_p \wedge^{\mathscr{G}} (\sigma_v)_p$:

$$N_p^h = \Big(\partial(y,\theta)_{\mathbf{u_0}} \cos\theta(\mathbf{u_0}) - \partial(x,\theta)_{\mathbf{u_0}} \sin\theta(\mathbf{u_0})\Big)(X_1)_p + \partial(x,y)_{\mathbf{u_0}}(X_2)_p. \quad (3.3)$$

When $\|N_p^h\| \neq 0$, we may define the unit horizontal normal ν_p^h to p by

$$\nu_p^h = \frac{N_p^h}{\|N_p^h\|}, \quad (3.4)$$

where $\|\cdot\|$ denotes the norm of the sub-Riemannian product $\langle\cdot,\cdot\rangle$ in $\mathscr{G}$.

Notice that we have

$$\nu_p^h = (\nu_1)_p(X_1)_p + (\nu_2)_p(X_2)_p, \quad (3.5)$$

where

$$(\nu_1)_p = \frac{\partial(y,\theta)_{\mathbf{u_0}} \cos\theta(\mathbf{u_0}) - \partial(x,\theta)_{\mathbf{u_0}} \sin\theta(\mathbf{u_0})}{\|N_p^h\|},$$
$$(\nu_2)_p = \frac{\partial(x,y)_{\mathbf{u_0}}}{\|N_p^h\|}.$$

The horizontal normal N_p^h at a point $p \in \mathscr{S}$ depends on the choice of the surface patch as this follows from the next proposition.

Proposition 3.6. Let (U,σ) and $(\tilde{U},\tilde{\sigma})$ are two overlapping patches at $p = \sigma(\mathbf{u_0}) = \tilde{\sigma}(\tilde{\mathbf{u}}_\mathbf{0})$. Then if $\Phi = \sigma^{-1} \circ \tilde{\sigma}$ is the transition mapping, around p we have

$$N^h_{\tilde{\sigma}(\tilde{\mathbf{u}}_0)} = \det(J(\Phi)_{\mathbf{u_0}}) N^h_{\sigma(\mathbf{u_0})}. \quad (3.6)$$

Proof. For convenience, we shall drop $\sigma(\mathbf{u_0})$ and $\tilde{\sigma}(\tilde{\mathbf{u}}_0)$ and shall write instead σ and $\tilde{\sigma}$, respectively. The transition mapping is defined as $\Phi : \tilde{U} \to U$ such that $\Phi(\tilde{u},\tilde{v}) = (u,v)$. Thus the Jacobian matrix of Φ is

$$J(\Phi) = \begin{pmatrix} \frac{\partial u}{\partial \tilde{u}} & \frac{\partial v}{\partial \tilde{u}} \\ \frac{\partial u}{\partial \tilde{v}} & \frac{\partial v}{\partial \tilde{v}} \end{pmatrix}.$$

According to the chain rule

$$\tilde{\sigma}_{\tilde{u}} = \frac{\partial u}{\partial \tilde{u}}\sigma_u + \frac{\partial v}{\partial \tilde{u}}\sigma_v, \quad \tilde{\sigma}_{\tilde{v}} = \frac{\partial u}{\partial \tilde{v}}\sigma_u + \frac{\partial v}{\partial \tilde{v}}\sigma_v.$$

Simple calculation then yields

$$\tilde{\sigma}_{\tilde{u}} \wedge^{\mathscr{G}} \tilde{\sigma}_{\tilde{v}} = \det(J(\Phi))\sigma_u \wedge^{\mathscr{G}} \sigma_v.$$

Thus

$$N^h_{\tilde{\sigma}} = \det(J(\Phi))(\sigma_u \wedge^{\mathcal{RT}} \sigma_v)^h,$$

that is,

$$N^h_{\tilde{\sigma}} = \det(J(\Phi)N^h_{\sigma}. \tag{3.7}$$

□

3.1.3 Characteristic locus, unit horizontal normal field

Definition 3.7. Let $\mathscr{S}$ a regular surface. A point $p \in \mathscr{S}$ is called non characteristic if $N^h_p \neq 0$. The set of characteristic points

$$\mathfrak{C}(\mathscr{S}) = \{p \in \mathscr{S} \mid N^h_p = 0\},$$

is called the characteristic locus of $\mathscr{S}$.

Remark 3.8. By definition, the points of $\mathfrak{C}(\mathscr{S})$ are given in a local chart (U, σ) by the equations

$$\partial(y,\theta)\cos\theta - \partial(x,\theta)\sin\theta = 0 \quad \text{and} \quad \partial(x,y) = 0.$$

Away from points where the horizontal normal vanishes, a $\mathscr{C}^1$ horizontal vector field $\nu^h_{\mathscr{S}}$ of $\mathscr{S}$ is defined by

$$(\nu^h_{\mathscr{S}})_p = \nu^h_p.$$

3.1.4 Horizontal area

There is a notion of *horizontal area* (or perimeter) for $\mathscr{G}$. For a surface patch σ of $\mathscr{S}$, $\sigma : U \to \mathbb{R}^3$, $\sigma = \sigma(u,v)$, the *horizontal area* $\mathscr{A}^h(\sigma)$ of σ is

$$\mathscr{A}^h(\sigma) = \iint_U \|N^h(u,v)\| dudv,$$

where $N^h(u,v) = N^h_{\sigma(u,v)}$. The measure $\|N^h(u,v)\| dudv$ is the local restriction of the 3-dimensional Carnot-Carathéodory measure to $\mathscr{S}$.

3.2 The induced 1-form

There is a natural 1-form $\omega_{\mathscr{S}}$ defined on every $\mathscr{C}^2$ surface $\mathscr{S}$ embedded in $\mathscr{G}$; it is just the pull-back to $\mathscr{S}$ of the contact form ω of $\mathscr{G}$. We define $\omega_{\mathscr{S}}$ properly in Section 3.2.1 as well as contactomorphisms between embedded $\mathscr{C}^2$ surfaces. Of course the term *contactomorphism* here is in a broad sence; on $\mathscr{S}$ we always have $\omega_{\mathscr{S}} \wedge d\omega_{\mathscr{S}} = 0$, since $\dim(\mathscr{S}) = 2$. In Section 3.2.2 we prove Proposition 3.13: Away from characteristic points, $\omega_{\mathscr{S}}$ defines an integrable foliation whose integral curves are tangent to $\mathbb{J}\nu_{\mathscr{S}}$. Here, $\mathbb{J}$ is a natural complex operator defined on the horizontal tangent space of $\mathscr{S}$, see Equation (3.10), and $\nu_{\mathscr{S}}$ is the unit horizontal normal field to $\mathscr{S}$.

3.2.1 The induced 1-form. Contactomorphisms

Let $\mathscr{S}$ be a regular surface in $\mathscr{G}$ and denote by $\iota_{\mathscr{S}}$ the inclusion map $\iota_{\mathscr{S}} : \mathscr{S} \hookrightarrow \mathscr{G}$, given locally by the parametrisation $\sigma(u,v) = (x(u,v), y(u,v), \theta(u,v))$.

Definition 3.9. Let ω be the contact form of $\mathscr{G}$. The pullback $\omega_S = i^*_{\mathscr{S}}\omega$ defines a 1-form $\omega_{\mathscr{S}}$ on $\mathscr{S}$. We call $\omega_{\mathscr{S}}$ the *induced 1-form of $\mathscr{S}$*.

The form $\omega_{\mathscr{S}}$ in local parametrisation is given by

$$\begin{aligned}\omega_{\mathscr{S}} = \sigma^*\omega =& \sin\theta(x_u du + x_v dv) - \cos\theta(y_u du + y_v dv)\\ =&(\sin\theta x_u - \cos\theta y_u)du + (\sin\theta x_v - \cos\theta y_v)dv.\end{aligned}$$

Definition 3.10. Let $\mathscr{S}$ and $\tilde{\mathscr{S}}$ be regular surfaces and $f : \mathscr{S} \to \tilde{\mathscr{S}}$ be a smooth diffeomorphism. We may assume a weaker condition, that is we will require f to be a local diffeomorphism outside the characteristic locus of $\mathscr{S}$ and $\tilde{\mathscr{S}}$. The mapping f is called a *local contactomorphism* of $\mathscr{S}$ $\tilde{\mathscr{S}}$ if there exists a smooth function λ so that

$$f^*\omega_{\tilde{\mathscr{S}}} = \lambda\omega_{\mathscr{S}}. \tag{3.8}$$

We stress here that if $\sigma : U \to \mathbb{R}^3$ is a surface patch for $\mathscr{S}$, then since f is a local diffeomorphism, $\tilde{\sigma} = f \circ \sigma$ is a surface patch for $\tilde{\mathscr{S}}$ (with possible exception of characteristic points). It follows that $f : \mathscr{S} \to \tilde{\mathscr{S}}$ is a contactomorphism if and only if

$$\omega_{\tilde{\sigma}}(u,v) = \lambda(u,v)\omega_{\sigma}(u,v) \;\text{ for almost all } (u,v) \in U \tag{3.9}$$

Using the induced 1-form we may give a coordinate-independent characterisation of the characteristic locus:

Proposition 3.11. The characteristic locus $\mathfrak{C}(\mathscr{S})$ is the (closed) set of points of $\mathscr{S}$ at which $\omega_{\mathscr{S}} = 0$.

Proof. We have

$$\begin{aligned}
&\omega_{\mathscr{S}}(p) = 0 \;\text{ for some } p \in \mathscr{S}\\
&\iff \omega_p(\sigma_u) = \omega_p(\sigma_v) \;\text{ for each chart}(U,\sigma), \text{ containing } p\\
&\iff \sigma_u \text{ and } \sigma_v \in \mathrm{H}_p(\mathscr{S})\\
&\iff (\sigma_u \times \sigma_v)_p^h = 0\\
&\iff p \in \mathfrak{C}(\mathscr{S}).
\end{aligned}$$

□

3.2.2 Horizontal flow

In this section we wish to find conditions, so that a surface curve is horizontal. By a surface curve on a $\mathscr{C}^2$ surface $\mathscr{S}$ we mean a $\mathscr{C}^2$ mapping $\gamma : I \to \mathscr{S}$, where I is an open interval of $\mathbb{R}$. We start with the following proposition:

Proposition 3.12. Suppose that $\sigma : U \to \mathscr{G}$ is a surface patch. Let $\tilde{\gamma}(s) = (u(s), v(s))$ be a $\mathscr{C}^2$ curve on U such that $\gamma(t) = \sigma(u(t), v(t)), t \in I$, is a $\mathscr{C}^2$ surface curve. Then away from the characteristic locus $\mathfrak{C}(S)$, γ is horizontal if and only if

$$\dot{\tilde{\gamma}}(t) \in \ker(\omega_{\mathscr{S}})_{\gamma(t)}, \quad t \in I.$$

In other words,

$$(\sin\theta x_u - \cos\theta y_u)\dot{u} + (\sin\theta x_v - \cos\theta y_v)\dot{v} = 0,$$

where the dot denotes d/dt. In this case,

$$\dot{\gamma} = \big(\cos\theta x_u \dot{u} + \cos\theta x_v \dot{v} + \sin\theta y_u \dot{u} + \sin\theta y_v \dot{v}\big) X_1 + \big(\theta_u \dot{u} + \theta_v \dot{v}\big) X_2.$$

Proof. For

$$\gamma(t) = \sigma(u(t), v(t)) = (x(u(t), v(t)), y(u(t), v(t)), \theta(u(t), v(t)))$$

we have using the chain rule that

$$\begin{aligned} \dot{\gamma} &= (x_u\dot{u} + x_v\dot{v})\frac{\partial}{\partial x} + (y_u\dot{u} + y_v\dot{v})\frac{\partial}{\partial y} + (\theta_u\dot{u} + \theta_v\dot{v})\frac{\partial}{\partial \theta} \\ &= (\cos\theta x_u\dot{u} + \cos\theta x_v\dot{v} + \sin\theta y_u\dot{u} + \sin\theta y_v\dot{v})X_1 \\ &\quad +(\theta_u\dot{u} + \theta_v\dot{v})X_2 \\ &\quad +(\sin\theta x_u\dot{u} + \sin\theta x_v\dot{v} - \cos\theta y_u\dot{u} - \cos\theta y_v\dot{v})X_3. \end{aligned}$$

However, γ is horizontal if and only if the component of X_3 vanishes. That is,

$$(\sin\theta x_u - \cos\theta y_u)\dot{u} + (\sin\theta x_v - \cos\theta y_v)\dot{v} = 0.$$

Moreover,

$$\begin{aligned} \gamma \text{ horizontal} &\Leftrightarrow \omega(\dot{\tilde{\gamma}}) = 0 \\ &\Leftrightarrow \omega(\sigma_* \dot{\tilde{\gamma}}) = 0 \\ &\Leftrightarrow (\sigma^*\omega)(\dot{\tilde{\gamma}}) = 0 \\ &\Leftrightarrow \omega_{\mathscr{S}}(\dot{\tilde{\gamma}}) = 0 \\ &\Leftrightarrow \dot{\tilde{\gamma}} \in \ker \omega_{\mathscr{S}} \end{aligned}$$

□

We define a complex operator $\mathbb{J}$ on the horizontal tangent space of $\mathscr{S}$ by the relations

$$\mathbb{J}X_1 = X_2, \ \ \mathbb{J}X_2 = -X_1. \tag{3.10}$$

If $\nu_{\mathscr{S}} = \nu_1 X_1 + \nu_2 X_2$ is the unit horizontal normal field to $\mathscr{S}$, then

$$\mathbb{J}\nu_{\mathscr{S}}^h = -\nu_2 X_1 + \nu_1 X_2,$$

The main proposition of this section is the following:

Proposition 3.13. The 1-form $\omega_{\mathscr{S}}$ defines an integrable foliation of $\mathscr{S}$ (with singularities at characteristic points) by horizontal surface curves. These curves are tangent to $\mathbb{J}\nu_{\mathscr{S}}^h$.

Proof. Let

$$\omega_{\mathscr{S}} = \underbrace{(\sin\theta x_u - \cos\theta y_u)}_{A} du + \underbrace{(\sin\theta x_v - \cos\theta y_v)}_{B} dv$$

and $\alpha = \|N^h\|$. Consider

$$V = \frac{B}{\alpha}\frac{\partial}{\partial u} - \frac{A}{\alpha}\frac{\partial}{\partial v} \in \ker \omega_{\mathscr{S}} :$$

Indeed, applying V to $\omega_{\mathscr{S}}$ gives

$$\omega_{\mathscr{S}}(V) \ \ = \ \ (Adu + Bdv)\left(\frac{B}{\alpha}\frac{\partial}{\partial u} - \frac{A}{\alpha}\frac{\partial}{\partial v}\right) = 0.$$

Now,

$$
\begin{aligned}
\sigma_* V &= \frac{B}{\alpha}\sigma_u - \frac{A}{\alpha}\sigma_v \\
&= \frac{B}{\alpha}\Big((x_u\cos\theta + y_u\sin\theta)X_1 + \theta_u X_2 + (x_u\sin\theta - y_u\cos\theta)X_3\Big) \\
&\quad -\frac{A}{\alpha}\Big((x_v\cos\theta + y_v\sin\theta)X_1 + \theta_v X_2 + (x_v\sin\theta - y_v\cos\theta)X_3\Big) \\
&= \frac{\overbrace{\Big(B(x_u\cos\theta + y_u\sin\theta) - A(x_v\cos\theta + y_v\sin\theta\Big)}^{C}}{\alpha}X_1 \\
&\quad + \frac{\overbrace{B\theta_u - A\theta_v}^{D}}{\alpha}X_2 \\
&\quad + \frac{\overbrace{\Big(B(x_u\sin\theta - y_u\cos\theta) - A(x_v\sin\theta - y_v\cos\theta\Big)}^{E}}{\alpha}X_3.
\end{aligned}
$$

Here, straightforward calculations show that

$$
C = -\partial(x,y), \quad D = -\sin\theta\partial(x,\theta) + \cos\theta\partial(y,\theta)
$$

and

$$
E = (\sin\theta x_v - \cos\theta y_v)(x_u\sin\theta - y_u\cos\theta) - (\sin\theta x_u - \cos\theta y_u)(x_v\sin\theta - y_v\cos\theta) = 0.
$$

Thus, $\sigma_* V \in \mathrm{H}(\mathscr{S})$ and

$$
\sigma_* V = \frac{-\partial(x,y)}{\|N_\sigma^h\|}X_1 - \frac{\sin\theta\partial(x,\theta) + \cos\theta\partial(y,\theta)}{\|N_\sigma^h\|}X_2 = -\nu_2 X_1 + \nu_1 X_2 = \mathbb{J}\nu_{\mathscr{S}}^h. \tag{3.11}
$$

As $\omega_{\mathscr{S}}$ is a 1-form defined in a two dimensional manifold we have the integrability by Frobenius' Theorem. □

Definition 3.14. The foliation of $\mathscr{S}$ by the integrale curves of $\mathbb{J}\nu_{\mathscr{S}}$ is called the *horizontal flow* of $\mathscr{S}$.

3.3 Horizontal mean curvature

In horizontal geometry, the notion of horizontal mean curvature is the analogue of Gaussian curvature for $\mathscr{C}^2$ surfaces embedded in $\mathbb{R}^3$. We first state the definition.

Definition 3.15. Let p be a non characteristic point of a regular surface $\mathscr{S}$ and let also $\nu_p^h = \nu_1 X_1 + \nu_2 X_2$ be the horizontal normal of $\mathscr{S}$ at p. The *horizontal mean curvature* $H^h(p)$ *of* $\mathscr{S}$ *at* p is given by

$$H^h(p) = (X_1)_p \nu_1 + (X_2)_p \nu_2.$$

As in the Heisenberg group case, we have an equivalent geometric definition of horizontal mean curvature. According to this definition, the horizontal mean curvature at non characteristic points of a $\mathscr{C}$ surface in ?? is just the signed curvature of the projection to $\mathbb{R}^2$ of the leaf of the horizontal flow passing from p.

Proposition 3.16. Let $\mathscr{S}$ be a $\mathscr{C}$ surface in $\mathscr{G}$ and $p \in \mathscr{S}$ a non characteristic point. Let $\nu_{\mathscr{S}}^h = \nu_1 X_1 + \nu_2 X_2$ be the unit horizontal normal vector field of $\mathscr{S}$, and γ be the unique unit speed surface curve passing from p, which is tangent to $\mathbb{J}\nu_p^h$. If $\pi = pr_{\mathbb{R}^2}\gamma$ is the projection of γ on $\mathbb{R}^2$, p' is the projection of p and $\kappa_s(p')$ is the signed curvature of π at p', then

$$\kappa_s(p') = H^h(p).$$

Proof. We apply the chain rule to obtain

$$\begin{aligned} \dot{\nu_1} &= (\nu_1)_x \dot{x} + (\nu_1)_y \dot{y} + (\nu_1)_\theta \dot{\theta} \\ &= (\cos\theta X_1 \nu_1 + \sin\theta X_3 \nu_1)(x_u \dot{u} + x_v \dot{v}) + (\sin\theta X_1 \nu_1 - \cos\theta X_3 \nu_1)(y_u \dot{u} + y_v \dot{v}) \\ &\quad + X_2 \nu_1 (\theta_u \dot{u} + \theta_v \dot{v}) \end{aligned}$$

Recall that

$$\dot{\gamma} = (\cos\theta(x_u\dot{u} + x_v\dot{v}) + \sin\theta(y_u\dot{u} + y_v\dot{v}))X_1 + (\theta_u\dot{u} + \theta_v\dot{v})X_2 = -\nu_2 X_1 + \nu_1 X_2.$$

Therefore

$$\begin{aligned} \dot{\nu_1} &= -\nu_2 X_1 \nu_1 + (\sin\theta(x_u\dot{u} + x_v\dot{v}) - \cos\theta(y_u\dot{u} + y_v\dot{v}))X_3\nu_1 + \nu_1 X_2 \nu_1 \\ &= -\nu_2 X_1 \nu_1 + \nu_1 X_2 \nu_1. \end{aligned}$$

Similarly,

$$\dot{\nu}_2 = -\nu_2 X_1 \nu_2 + \nu_1 X_2 \nu_2.$$

Now, since $\nu_1^2 + \nu_2^2$ we have $\nu_1 X_2 \nu_1 = -\nu_2 X_2 \nu_2$. Therefore

$$\begin{aligned}\dot{\nu}_1 &= -\nu_2(X_1\nu_1 + X_2\nu_2),\\ \dot{\nu}_2 &= \nu_1(X_1\nu_1 + X_2\nu_2).\end{aligned}$$

Thus

$$\begin{aligned}\ddot{\pi} &= (-\dot{\nu}_2, \dot{\nu}_1) = (-\nu_1(X_1\nu_1 + X_2\nu_2), -\nu_2(X_1\nu_1 + X_2\nu_2))\\ &= \kappa_s(-\nu_1, -\nu_2),\end{aligned}$$

where κ_s is the signed curvature of the curve π. This yields

$$\kappa_s = X_1\nu_1 + X_2\nu_2$$

and the proof is complete. □

A local description follows.

Proposition 3.17. Let $\mathscr{S}$ be a regular surface in $\mathscr{G}$. In every surface patch $\sigma = (x, y, \theta)$ with $\cos\theta\partial(x,\theta) + \sin\theta\partial(y,\theta) \neq 0$ and sufficiently away from the characteristic locus, the horizontal mean curvature is given by

$$H^h = \frac{\partial(\nu_1,\theta) + \cos\theta\partial(x,\nu_2) + \sin\theta\partial(y,\nu_2)}{\cos\theta\partial(x,\theta) + \sin\theta\partial(y,\theta)}, \tag{3.12}$$

where $\nu_i,\ i = 1,2$ are the components of the unit normal vector ν of $\mathscr{S}$.

Proof. We have

$$\begin{aligned}(\nu_1)_u &= (\nu_1)_x x_u + (\nu_1)_y y_u + (\nu_1)_\theta \theta_u\\ &= \frac{\partial\nu_1}{\partial x}x_u + \frac{\partial\nu_1}{\partial y}y_u + \frac{\partial\nu_1}{\partial\theta}\theta_u\\ &= (\cos\theta X_1\nu_1 + \sin\theta X_3\nu_1)x_u + (\sin\theta X_1\nu_1 - \cos\theta X_3\nu_1)y_u + X_2\nu_1\theta_u\\ &= (\cos\theta x_u + \sin\theta y_u)X_1\nu_1 + \theta_u X_2\nu_1 + (\sin\theta x_u - \cos\theta y_u)X_3\nu_1.\end{aligned} \tag{3.13}$$

Similarly,

$$(\nu_2)_u = (\cos\theta x_u + \sin\theta y_u)X_1\nu_2 + \theta_u X_2\nu_2 + (\sin\theta x_u - \cos\theta y_u)X_3\nu_2, \quad (3.14)$$
$$(\nu_1)_v = (\cos\theta x_v + \sin\theta y_v)X_1\nu_1 + \theta_v X_2\nu_1 + (\sin\theta x_v - \cos\theta y_v)X_3\nu_1, \quad (3.15)$$
$$(\nu_2)_v = (\cos\theta x_v + \sin\theta y_v)X_1\nu_2 + \theta_v X_2\nu_2 + (\sin\theta x_v - \cos\theta y_v)X_3\nu_2. \quad (3.16)$$

From Equations (3.13) and (3.15) we have the system in $X_1\nu_1$ and $X_2\nu_1$:

$$(\cos\theta x_u + \sin\theta y_u)X_1\nu_1 + \theta_u X_2\nu_1 = (\nu_1)_u + (\cos\theta y_u - \sin\theta x_u)X_3\nu_1$$
$$(\cos\theta x_v + \sin\theta y_v)X_1\nu_1 + \theta_v X_2\nu_1 = (\nu_1)_v + (\cos\theta y_v - \sin\theta x_v)X_3\nu_1.$$

Provided that the discriminant

$$D = \cos\theta\partial(x,\theta) + \sin\theta\partial(y,\theta) \neq 0,$$

we then have

$$X_1\nu_1 = \frac{\partial(\nu_1,\theta) + X_3\nu_1\big(\cos\theta\partial(y,\theta) - \sin\theta\partial(x,\theta)\big)}{D} = \frac{\partial(\nu_1,\theta) + ||N^h||\nu_1 X_3\nu_1}{D}.$$

Similarly, from Equations (3.14) and (3.16) we obtain

$$\begin{aligned} X_2\nu_2 &= \frac{\cos\theta\partial(x,\nu_2) + \sin\theta\partial(y,\nu_2) + \partial(x,y)X_3\nu_2}{D} \\ &= \frac{\cos\theta\partial(x,\nu_2) + \sin\theta\partial(y,\nu_2) + ||N^h||\nu_2 X_3\nu_2}{D}. \end{aligned}$$

Now since $X_3(\nu_1^2 + \nu_2^2) = \nu_1 X_3\nu_1 + \nu_2 X_3\nu_2 = 0$, by adding we obtain the formula (3.12) and the proof is complete. □

Remark 3.18. If $\cos\theta\partial(x,\theta) + \sin\theta\partial(y,\theta) = 0$, then we obtain that the vector X_3 is in the tangent plane of p. Since the integral cirves of the vector field X_3 are all straight lines, it follows that there exists a straight line surface segmant passing from p. Since $\mathscr{S}$ is $\mathscr{C}^2$, that means that around p $\mathscr{S}$ is a plane, therefore $H^h(p) = 0$.

3.4 Graphs

Let a function $\theta = \theta(x,y)$, $(x,y) \in U$, U open in $\mathbb{R}^2$. The graph G_θ of θ admits a parametrisation by a unique surface patch

$$\sigma(x,y) = (x,y,\theta(x,y))$$

as a $\mathscr{C}^2$ surface in G_θ. Applying the formula we have found in this section we have:

1. The horizontal normal at the arbitrary $p = (x, y, \theta(x, y)$ is
$$N_p^h = (-\theta_x \cos\theta(x, y) - \theta_y \sin\theta(x, y))(X_1)_p + (X_2)_p.$$
Note that $||N_p^h||^2 = 1 + (\theta_x \cos\theta + \theta_y \sin\theta)^2 \neq 0$. Therefore the characteristic locus of G_θ is empty and the unit horizontal normal vector field
$$\nu_p^h = \frac{N_p^h}{||N_p^h||}$$
is defined for all $p \in \mathrm{G}_\theta$.

2. The induced 1-form
$$\omega_{\mathrm{G}_\theta} = \sin\theta(x, y)dx - \cos\theta(x, y)dy.$$

3. Since ν_2 depends only on x, y we have
$$H^h = X_1\nu_1.$$

We also have the following.

Proposition 3.19. Suppose that G_θ and $\mathrm{G}_{\theta'}$ are the graphs of the functions $\theta(x, y)$ and $\theta'(x, y)$ respectively, with $(x, y) \in U$, U open in $\mathbb{R}^2$. Then G_θ and $\mathrm{G}_{\theta'}$ are contactomorphic, if and only if
$$\theta'(x, y) = \theta(x, y) + \kappa\pi, \quad \kappa \in \mathbb{Z}.$$

Proof. Suppose first that
$$\theta'(x, y) = \theta(x, y) + \kappa\pi, \quad \kappa \in \mathbb{Z}$$
Then $\tan\theta'(x, y) = \tan(x, y)$ and
$$\begin{aligned}
\omega_{\mathrm{G}_\theta} &= \sin\theta(x, y)dx - \cos\theta(x, y)dy \\
&= \cos\theta(x, y)(\tan\theta(x, y)dx - dy) \\
&= \cos\theta(x, y)(\tan\theta'(x, y)dx - dy) \\
&= \frac{\cos\theta(x, y)}{\cos\theta'(x, y)} \cos\theta'(x, y)(\tan\theta'(x, y)dx + dy) \\
&= \lambda\omega_{\mathrm{G}_{\theta'}}
\end{aligned}$$

where

$$\lambda(x,y) = \frac{\cos\theta(x,y)}{\cos\theta'(x,y)}.$$

Conversely, if G_θ, $\mathrm{G}_{\theta'}$ are contactomorphic, then there exists $\lambda(x,y)$ such that

$$\sin\theta(x,y) = \lambda(x,y)\sin\theta'(x,y),$$
$$\cos\theta(x,y) = \lambda(x,y)\cos\theta'(x,y).$$

By dividing we obtain

$$\tan\theta(x,y) = \tan\theta'(x,y).$$

Therefore $\theta'(x,y) = \theta(x,y) + \kappa\pi, \quad \kappa \in \mathbb{Z}$. □

3.4.1 Examples

The plane $\mathbb{R}^2$

As our first example we consider the simplest graph G_0, that is when $\theta(x,y) = 0$, $(x,y) \in \mathbb{R}^2$. This represents the embedding of $\mathbb{R}^2$ into $\mathscr{G}$. We have

$$N_p^h = \nu_p^h = (X_2)_p, \quad \omega_{\mathrm{G}_0} = -dy$$

and the horizontal mean curvature is $H_p^h = 0$. Note that in this case $N_p = N_p^h$.

The metric sphere

The second example is that of the metric sphere S^2. We choose to represent it as a graph G_θ, $\theta(x,y) = 1 - x^2 - y^2 > 0$. The corresponding surface patch is

$$\sigma(x,y) = (x, y, 1 - x^2 - y^2), \quad (x,y) \in D,$$

where $D = \{(x,y) \in \mathbb{R}^2 : \ x^2 + y^2 < 1\}$. We then have for $p = (x, y, \theta(x,y))$ that

$$N_p^h = (2x\cos(1 - x^2 - y^2) + 2y\sin(1 - x^2 - y^2))(X_1)_p + (X_2)_p$$

and

$$\begin{aligned}\nu_1 &= \frac{2x\cos\theta(x,y) + 2y\sin\theta(x,y)}{||N^h||},\\ \nu_2 &= \frac{1}{||N^h||}.\end{aligned}$$

Calculations show that

$$H^h = \frac{2 + (4x^2 - 4y^2)\cos\theta\sin\theta - 4xy(\cos^2\theta - \sin^2\theta)}{((2x\cos\theta + 2y\sin\theta)^2 + 1)^{1/2}},$$

where $\theta(x,y) = 1 - x^2 - y^2$. Note that $H^h_p \sim 2$ for p close to the north pole.

Chapter 4

Straight ruled surfaces in $\mathscr{G}$

A straight ruled surface in $\mathscr{G}$ is a surface which is formed by a union of straight lines (the rulings of the surface). In this chapter we give the definition and examine the features of the horizontal geometry of a straight ruled surface in Section 4.1. Our main result lies in Section 4.2: Straight ruled surfaces are horizontally minimal.

4.1 Straight ruled surfaces

We construct straight ruled surfaces in $\mathscr{G}$ in Section 4.1.1 and examine their regularity. Their horizontal normal and characteristic locus are in Section 4.1.2 and their induced 1-form is in Section 4.1.3.

4.1.1 Construction and regularity

To construct a straight ruled surface we start from a $\mathscr{C}^2$ curve $\gamma = \gamma(s)$, where s lies in an open interval $I_s \subset \mathbb{R}$. The curve γ is a (not necessarily horizontal) smooth curve and $V = V(s)$ is a unit horizontal vector field along γ, i.e.,

$$V(s) \in \mathrm{H}_{\gamma(s)}(\mathscr{G}).$$

At any point $q \in \gamma$, say $q = \gamma(s)$, we consider the straight line passing from q in the direction of $V(s)$. Then a point p on the straight line satisfies

$$p = \gamma(s) + vV(s)$$

for some v. The *straight ruled surface* $\mathscr{R}(\gamma)$ is the union of all such lines, therefore it admits a parametrisation by the (single) surface patch $\sigma : I_s \times \mathbb{R} \to \mathbb{R}^3$, where

$$\sigma(s, v) = \gamma(s) + vV(s). \tag{4.1}$$

If $\gamma = (x, y, \theta)$ and $V = aX_1 + bX_2$, $a^2 + b^2 = 1$, we write equivalently

$$\begin{aligned}\sigma(s, v) &= \big(\tilde{x}(s, v), \tilde{y}(s, v), \tilde{\theta}(s, v)\big) \\ &= \big(x(s), y(s), \theta(s)\big) + v\big(a(s)(X_1)_{\gamma(s)} + b(s)(X_2)_{\gamma(s)}\big).\end{aligned}$$

However, as

$$\begin{aligned}(X_1)_{\gamma(s)} &= \cos\theta(s)\left(\frac{\partial}{\partial x}\right)_{\gamma(s)} + \sin\theta(s)\left(\frac{\partial}{\partial y}\right)_{\gamma(s)} \equiv (\cos\theta(s), \sin\theta(s), 0), \\ (X_2)_{\gamma(s)} &= \left(\frac{\partial}{\partial \theta}\right)_{\gamma(s)} \equiv (0, 0, 1),\end{aligned}$$

we have

$$\sigma(s, v) = (x(s) + va(s)\cos\theta(s), y(s) + va(s)\sin\theta(s), \theta(s) + vb(s)). \tag{4.2}$$

We calculate (denoting d/ds by dot)

$$\begin{aligned}&\tilde{x}_s = \dot{x} + v\dot{a}\cos\theta - va\sin\theta\dot{\theta}, \quad \tilde{y}_s = \dot{y} + v\dot{a}\sin\theta + va\cos\theta\dot{\theta}, \quad \tilde{\theta}_s = \dot{\theta} + v\dot{b}, \\ &\tilde{x}_v = a\cos\theta, \quad \tilde{y}_v = a\sin\theta, \quad \tilde{\theta}_v = b.\end{aligned}$$

Proposition 4.1. Provided that γ is not a horizontal straight line of the form

$$x = \text{const.}, \quad \theta = 2k\pi \pm \frac{\pi}{2},\ k \in \mathbb{Z}, \tag{4.3}$$

or,

$$y = \text{const.}, \quad \theta = 2k\pi \pm \pi,\ k \in \mathbb{Z}, \tag{4.4}$$

and V is not tangent to γ, the surface patch σ as in (4.2) is a regular surface patch for the straight ruled surface $\mathscr{R}(\gamma)$.

Proof. The surface patch σ is clearly smooth. Moreover, we have

$$\begin{aligned}
\partial(\tilde{y},\tilde{\theta}) &= b\dot{y} + v\sin\theta(\dot{a}b - a\dot{b}) + a\dot{\theta}(vb\cos\theta - \sin\theta),\\
\partial(\tilde{x},\tilde{\theta}) &= b\dot{x} + v\cos\theta(\dot{a}b - a\dot{b}) - a\dot{\theta}(vb\sin\theta + \cos\theta),\\
\partial(\tilde{x},\tilde{y}) &= a(\dot{x}\sin\theta - \dot{y}\cos\theta) - va\dot{\theta}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\sigma_s \times \sigma_v &= \Big(b\dot{y} + vb\dot{a}\sin\theta + vab\dot{\theta}\cos\theta - a\dot{\theta}\sin\theta - va\dot{b}\sin\theta\Big)\,\partial_x\\
&\quad - \Big(b\dot{x} + va\dot{b}\cos\theta - vab\dot{\theta}\sin\theta - a\dot{\theta}\cos\theta - va\dot{b}\cos\theta\Big)\,\partial_y\\
&\quad + \Big(a\dot{x}\sin\theta - a\dot{y}\cos\theta - v\dot{a}^2\dot{\theta}\Big)\,\partial_\theta.
\end{aligned}$$

We suppose now that the component of ∂_θ is equal to 0, i.e.,

$$a\dot{x}\sin\theta - a\dot{y}\cos\theta - va^2\dot{\theta} = 0$$

Then as a v-polynomial it is equal to 0 if and only if

$$a(\dot{x}\sin\theta - \dot{y}\cos\theta) = 0 \quad \text{and} \quad a^2\dot{\theta} = 0.$$

If $a = 0$ then since $V = aX_1 + bX_2$ is unit we obtain $b = \pm 1$; therefore the ∂_x and the ∂_y component, respectively, become $\pm\dot{y}$ and $\pm\dot{x}$ which cannot be both zero. If $a \neq 0$ then we must have $\dot{x}\sin\theta - \dot{y}\cos\theta = 0$ and $\dot{\theta} = 0$. Therefore γ is a horizontal straight line of the form (4.3) or (**??**).

In the first case, the ∂_y component vanishes and the ∂_x component becomes equal to

$$b\dot{y} \pm v(b\dot{a} - a\dot{b}).$$

Since $a \neq 0$ this vanishes only if $b = 0$; therefore $a = \pm 1$ and $V = \pm X_1$, is tangent to γ. The second case may be treated in a similar manner. □

From the proof of Proposition 4.1 we obtain the following.

Corollary 4.2. With the assumptions of Proposition 4.1, there exists an open $U \subset I_s \times \mathbb{R}$ so that $\mathscr{R}(\gamma)$ is the graph of some function $\theta(s,v)$, $(s,v) \in U$.

4.1.2 Horizontal normal and characteristic locus

The horizontal normal vector field as well as the charcteristic locus of a straight ruled surface are described in the following proposition.

Proposition 4.3. Let $\mathscr{R}(\gamma)$ be a straight ruled surface admitting a $\mathscr{C}^2$ parametrisation as in (4.1). Then at non characteristic points the unit horizontal normal vector field to $\mathscr{R}(\gamma)$ is

$$\nu^h_{\mathscr{R}(\gamma)} = \mathbb{J}V.$$

The characteristic locus $\mathfrak{C}(\mathscr{R}(\gamma))$ is the set

$$\left\{(s,v) \in I_s \times \mathbb{R} \ : \ \theta(s) = \tan^{-1}\left(\frac{\dot{y}(s)}{\dot{x}(s)}\right) \ \text{ and } \ b(s) = 0\right\}. \tag{4.5}$$

Proof. For the horizontal normal we calculate

$$\begin{aligned}\sigma_s \ = \ & \left(\dot{x}(s)\cos(\theta(s)+vb(s)) + \dot{y}(s)\sin(\theta(s)+vb(s)) + va(s)\right)X_1 \\ & +(\dot{\theta}(s)+v\dot{b}(s))X_2 \\ & + \left(\dot{x}(s)\sin(\theta(s)+vb(s)) - \dot{y}(s)\cos(\theta(s)+vb(s))\right)X_3,\end{aligned}$$

and

$$\sigma_v = a(s)X_1 + b(s)X_2 = V(s).$$

Therefore

$$N^h = (\sigma_s \wedge^{\mathscr{G}} \sigma_v)^h = \eta(s,v)\left(-b(s)X_1 + a(s)X_2\right) = \eta(s,v)\mathbb{J}V(s).$$

Here

$$\eta(s,v) = \dot{x}(s)\sin(\theta(s)+vb(s)) - \dot{y}(s)\cos(\theta(s)+vb(s)). \tag{4.6}$$

In our investigation to find all characteristic points of $\mathscr{R}(\gamma)$, we note first that the characteristic locus contains points $(s,0)$; at these points

$$\dot{x}(s)\sin\theta(s) - \dot{y}(s)\cos\theta(s) = 0.$$

Thus if γ is horizontal then all its points are characteristic. In the general case we may write the equation $\eta(s,v) = 0$ as

$$\theta(s) + vb(s) = \tan^{-1}\left(\frac{\dot{y}(s)}{\dot{x}(s)}\right).$$

This vanishes at points (s, v) where

$$\theta(s) = \tan^{-1}\left(\frac{\dot{y}(s)}{\dot{x}(s)}\right) \quad \text{and} \quad b(s) = 0.$$

Therefore the characteristic locus is thus given by (4.5) and the proof is complete. □

The following corollary gives a formula for the horizontal area of a regular straight ruled surface $\mathscr{R}(\gamma)$, away from characteristic points:

Corollary 4.4. Let $\mathscr{R}(\gamma)$ be a regular straight ruled surface in $\mathscr{G}$ with parametrisation given by Equation (4.1). Then its horizontal area is

$$\mathscr{A}^h(\mathscr{R}(\gamma)) = \iint_U |\eta(s,v)| ds dv,$$

where $U \subset I_s \times \mathbb{R}$ which does not comprise characteristic points and η is as in Equation (4.6).

4.1.3 Induced 1-form

The induced 1-form $\omega_{\mathscr{R}(\gamma)}$ of the straight ruled surface $\mathscr{R}(\gamma)$ is given by

$$\omega_{\mathscr{R}(\gamma)} = P(s,v)ds + Q(s,v)du, \tag{4.7}$$

where

$$\begin{aligned} P(s,v) &= \eta(s,v) + v\left(\dot{a}(s)\sin(vb(s)) - a(s)\dot{\theta}(s)\cos(vb(s))\right), \\ Q(s,v) &= a(s)\sin(vb(s)). \end{aligned}$$

4.2 Straight ruled surfaces are horizontally minimal

In this section we prove our main theorem.

Theorem 4.5. *A straight ruled surface $\mathscr{R}(\gamma)$ in $\mathscr{G}$ is horizontally minimal: Its horizontal mean curvature is identically zero.*

Proof. As we showed above, the unit horizontal normal vector field of γ is

$$\nu_\sigma^n = JV = \nu_1 X_1 + \nu_2 X_2 = -bX_1 + aX_2.$$

In case when

$$D = \cos\tilde{\theta}\partial(\tilde{x}, \tilde{\theta}) + \sin\tilde{\theta}\partial(\tilde{y}, \tilde{\theta}) \neq 0,$$

we may apply Proposition 3.17. For this we calculate

$$\partial(\nu_1, \tilde{\theta}) = \begin{vmatrix} -\dot{b} & 0 \\ \dot{\theta} + v\dot{b} & b \end{vmatrix} = -b\dot{b},$$

$$\partial(\tilde{x}, \nu_2) = \begin{vmatrix} \dot{x} + v\dot{a}\cos\theta - va\sin\theta & \dot{a} \\ a\cos\theta & 0 \end{vmatrix} = -a\dot{a}\cos\theta,$$

$$\partial(\tilde{y}, \nu_2) = \begin{vmatrix} \dot{y} + v\dot{a}\sin\theta + va\cos\theta & \dot{a} \\ a\sin\theta & 0 \end{vmatrix} = -a\dot{a}\sin\theta.$$

Thus we have

$$\begin{aligned} D \cdot H^h &= \partial(\nu_1, \tilde{\theta}) + \cos\theta\partial(\tilde{x}, \nu_2) + \sin\theta\partial(\tilde{y}, \nu_2) \\ &= -b\dot{b} - a\dot{a}\cos^2\theta - a\dot{a}\sin^2\theta \\ &= -a\dot{a} - b\dot{b} \\ &= 0. \end{aligned}$$

Here, the last equation is induced after differentiating the relation $a^2+b^2 = 1$. In case when $D = 0$ we have that the unit horizontal normal is X_2, i.e., $a = 0$, $b = 1$. But then we also have $H^h \equiv 0$. □

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