6.5

(1) α) $\int_D \frac{1}{\sqrt{xy}}\, dA$, $D = [0,1]\times[0,1]$

$D_{\eta,\delta} = [\eta, 1]\times[\delta, 1]$ $\quad 0 \le \eta \le 1$, $\quad 0 < \delta \le 1$

$$I_{\eta,\delta} = \left(\int_\eta^1 \frac{dx}{\sqrt{x}}\right)\left(\int_\delta^1 \frac{dy}{\sqrt{y}}\right) =$$

$$= 4[\sqrt{x}]_\eta^1 \,[\sqrt{y}]_\delta^1 =$$

$$= 4(1-\sqrt{\eta})(1-\sqrt{\delta}) \xrightarrow[(\eta,\delta)\to(0,0)]{} 4$$

d) $\int_0^1 \int_0^{e^y} \log x \, dx\, dy$

$x = e^y$, $y = \ln x$

($D_{\eta,\delta} = \{\eta \le y \le 1, \ \delta \le x \le e^y\}$

$$I_{\eta,\delta} = \int_\eta^1 \left(\int_\delta^{e^y} \log x \, dx\right) dy =$$

$$= \int_\eta^1 [x\log x - x]_\delta^{e^y}\, dy = \int_\eta^1 e^y y - e^y - \delta\log\delta + \delta \, dy$$

$= e - 2e + \delta(1-\log\delta) - ne^{n} + 2e^{n} - \delta(1-\log\delta)n =$

$$\ast \quad \lim_{\delta\to 0} \delta\log\delta = \lim_{\delta\to 0} \frac{\log\delta}{\frac{1}{\delta}} \overset{D'L}{=} =$$

$$= \lim \frac{\frac{1}{\delta}}{-\left(\frac{1}{\delta}\right)^{2}} = 0$$

Άρα $I = -e + 2$

② α) $D = \{ (x,y) \mid x \geq \alpha, \ \varphi_1(x) \leq y \leq \varphi_2(x) \}$

$D_n = \{ (x,y) \mid \alpha \leq x \leq n, \ \varphi_1 \leq y \leq \varphi_2 \}$

$$I = \iint_D f(x,y)\,dx\,dy = \lim_{n\to\infty} \iint_{D_n} f(x,y)\,dx\,dy$$

b) $\int_D xy\, e^{-(x^2+y^2)}\,dx\,dy, \quad x \geq 0, \quad 0 \leq y \leq 1$

$$I_n = \iint_{D_n} xy\, e^{-(x^2+y^2)}\,dx\,dy =$$

$$= \left(\int_0^n x e^{-x^2} dx\right)\left(\int_0^1 y e^{-y^2}\right) dy =$$

$$= \left[-\frac{e^{-x^2}}{2}\right]_0^n \cdot \left[-\frac{e^{-y^2}}{2}\right]_0^1 =$$

$$= \frac{1}{4}\left(-e^{-n^2}+1\right)\left(-e^{-1}+1\right) =$$

$$\xrightarrow[n\to\infty]{} \frac{1}{4}\left(1-\frac{1}{e}\right)$$

⑤ $\displaystyle\int_D \frac{x+y}{x^2+2xy+y^2}\,dx\,dy \quad , D=[0,1]^2$

$$D_{n,\delta} = [n,1]\times[\delta,1]$$

$$I_{n,\delta} = \int_n^1 \int_\delta^1 \frac{1}{x+y}\,dy\,dx =$$

$$= \int_n^1 \left[\ln(x+y)\right]_{y=\delta}^{y=1} dx =$$

$$= \int_n^1 \ln(1+x) - \ln(\delta+x)\,dx =$$

$$= \int_n^1 \ln(1+x)\,dx - \int_n^1 \ln(\delta+x)\,dx =$$

$$= \left[(x+1)\ln(x+1) - (x+1) - (\delta+x)\ln(\delta+x) + (x+\delta)\right]_n^1$$

$$= 2\ln 2 - 2 - (\delta+1)\ln(\delta+1) + (\delta+1) -$$

$$\to 2\ln 2.$$

(6) $f \geq 0$ μη φραγμένη μη συνεχής στο ∂D

$$f(x,y) \leq g(x,y)$$

Αν $\iint_D g \, dxdy$ υπάρχει τότε $\iint_D f$ υπάρχει

$$\iint_D g \, dxdy = \lim_{(\eta,\delta)\to(0,0)} \iint_{D_{\eta,\delta}} g \, dxdy$$

$$f \leq g$$

Άρα $\iint_{D_{\eta,\delta}} f \, dx dy \leq \int_{D_{\eta,\delta}} g \, dxdy$

Άρα $\lim_{(\eta,\delta)\to(0,0)} \iint_{D_{\eta,\delta}} f(x,y) \, dx dy \leq \iint_D g \, dx dy$

(7) $I = \iint_D \frac{\sin^2(x-y)}{\sqrt{1-x^2-y^2}} \, dxdy$, $D: \{x^2+y^2 \leq 1\}$

Έστω $I_\delta = D(0,\delta)$ $\forall (x,y) \in D(0,\delta)$

$$\frac{\sin^2(x-y)}{\sqrt{1-x^2-y^2}} \leq \frac{1}{\sqrt{1-x^2-y^2}}$$

$$\iint_{D(0,\delta)} \frac{\sin^2(x-y)}{\sqrt{1-x^2-y^2}}\,dx\,dy \leq \iint_{D(0,\delta)} \frac{dx\,dy}{\sqrt{1-x^2y^2}}\,dx\,dy$$

$$= \int_0^{\delta}\int_0^{2\pi} \frac{r}{\sqrt{1-r^2}}\,d\theta\,dr = 2\pi\left[\frac{\sqrt{1-r^2}}{-2}\right]_0^{\delta} =$$

$$= -\pi\left(\sqrt{1-\delta^2} - 1\right) =$$

$$= \pi\left(1 - \sqrt{1-\delta^2}\right) \xrightarrow[\delta \to 1]{} \pi$$

(11) $$I = \iiint_{\omega} \frac{(x^2+y^2+z^2)^{1/4}}{\sqrt{z+(x^2+y^2+z^2)^2}}\,dx\,dy\,dz$$

$x^2+y^2+z^2 = \alpha^2$, $x \geq 0$, $y \geq 0$, $z \geq 0$

(προβλημα στο $(0,0,0)$))

παιρνω $W_\varepsilon = \{(x,y,z) \mid \varepsilon^2 \leq x^2+y^2+z^2 \leq \alpha^2\}$

Σφαιρικές

$$I_\varepsilon = \int_\varepsilon^\alpha \int_0^{\pi/2}\int_0^{\pi/2} \frac{\rho^{5/2}\sin\varphi}{\sqrt{\rho\cos\varphi + \rho^4}}\,d\theta\,d\rho\,d\varphi =$$

$$= \frac{\pi}{2}\int_\varepsilon^\alpha \int_0^{\pi/2} \frac{\rho^{5/2}\sin\varphi}{\sqrt{\rho\cos\varphi + \rho^4}}\,d\varphi\,d\rho =$$

$$= \frac{\pi}{2} \int_{\varepsilon}^{\alpha} \left[-2\rho^{3/2} \sqrt{\rho\cos\varphi + \rho^4} \right]_{\varphi=0}^{\varphi=\pi/2} =$$

$$= -\pi \int_{\varepsilon}^{\alpha} -\rho^{3/2}\sqrt{\rho^4} + \rho^{3/2}\sqrt{\rho+\rho^4}\, d\rho =$$

$$= \pi \int_{\varepsilon}^{\alpha} \rho^{7/2} - \rho^{3/2}\sqrt{\rho+\rho^4}\, d\rho =$$

$$= \pi \left[\frac{\rho^{\frac{9}{2}}}{\frac{9}{2}} \right]_{\varepsilon}^{\alpha} - \pi \int_{\varepsilon}^{\alpha} \rho^2 \sqrt{1+\rho^3}\, d\rho =$$

$$= \pi \frac{2}{9} \alpha^{9/2} - \frac{4\pi}{9} \varepsilon^{9/2} - \pi \left[\frac{2}{9} (1+\rho^3)^{3/2} \right]_{\rho=\varepsilon}^{\rho=\alpha}$$

$$= \frac{2\pi}{9} \left(\alpha^{9/2} - \varepsilon^{9/2} \right) - \frac{\pi 2}{9} \left[(1+\alpha^3)^{3/2} - (1+\varepsilon^3)^{3/2} \right]$$

$$\xrightarrow{\varepsilon \to 0} \frac{2\pi}{9} \alpha^{9/2} - \frac{2\pi}{9} (1+\alpha^3)^{3/2} + \frac{2\pi}{9}$$

Scanned with CamScanner

(x,y,z) , $x^2+y^2+z^2 \geq 1$

$$\int_D \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^2}$$

$$I_M = \int_{D_M} \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^2}$$

όπου $D_M = \{(x,y,z) \mid 1 \leq x^2+y^2+z^2 \leq M^2\}$

$$= \int_0^{\pi}\int_0^{2\pi}\int_1^{M} \frac{\rho^2 \sin\varphi}{(\rho^2)^2}\, d\rho\, d\theta\, d\varphi =$$

$$= \int_0^{\pi}\int_1^{M} \frac{2\pi \sin\varphi}{\rho^2}\, d\rho\, d\varphi =$$

$$= 2\pi \left(\int_1^{M} \frac{d\rho}{\rho^2}\right)\left(\int_0^{\pi} \sin\varphi\, d\varphi\right) =$$

$$= 2\pi \left[-\frac{1}{\rho}\right]_{\rho=1}^{\rho=M} \left[-\cos\varphi\right]_{\varphi=0}^{\varphi=\pi} =$$

$$= 2\pi \left(-\frac{1}{M} + 1\right)(1+1) =$$

$$= 4\pi\left(1 - \frac{1}{M}\right) \xrightarrow{M \to \infty} 4\pi$$

Επαναληπτικές

(4) $\iiint_\omega z\, dx\, dy\, dz$

$\omega : \{ x=0,\ y=0,\ z=0,\ z=1,\ x^2+y^2=1,\ x\geq 0,\ y\geq 0 \}$

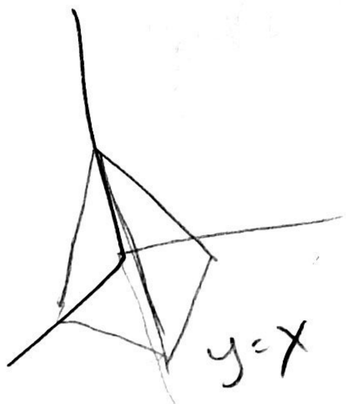
$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^1 z\, dz\, dy\, dx =$$

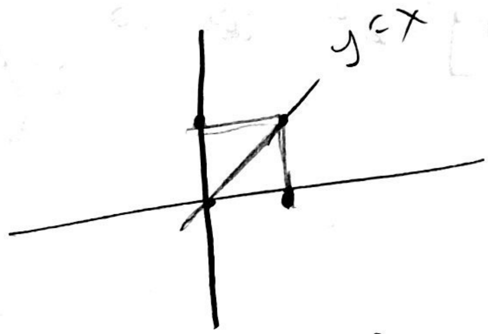
$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\frac{z^2}{2} \right]_{z=0}^{z=1} dy\, dx =$$

$$= \frac{1}{2} \int_0^1 \int_0^{\sqrt{1-x^2}} dy\, dx = \frac{1}{2} \int_0^1 \sqrt{1-x^2}\, dx =$$

$$= \frac{1}{2} \frac{\pi}{4} = \frac{\pi}{8}$$

(7) $\iiint_\omega (1-z^2)\, dx\, dy\, dz$: ω πυραμίδα με κορυφή το $(0,0,1)$ και κορυφές της βάσης $(0,0)$, $(1,0)$, $(0,1)$ και $(1,1)$





Χωρίζουμε σε 2 χωρία:

Το επίπεδο που περνά από $(0,0,1),(1,0,1)(1,1,0)$

$$\boxed{x+z=1}$$

και το επίπεδο που περνά από τα $(0,0,1),(1,1,0)$ $(0,1,0)$

$$\boxed{y+z=1}$$

$$I=\iiint_{\omega_1}+\iiint_{\omega_2}=\int_0^1\int_0^x\int_0^{1-x}(1-z^2)\,dz\,dy\,dx+\int_0^1\int_0^y\int_0^{1-y}(1-z^2)\,dz\,dx\,dy=$$

$$=\int_0^1\int_0^x\left[z-\frac{z^3}{3}\right]_{z=0}^{z=1-x}dy\,dx+\int_0^1\int_0^y\left[z-\frac{z^3}{3}\right]_{z=0}^{z=1-y}dx\,dy=$$

$$=\int_0^1\int_0^x(1-x)-\frac{(1-x)^3}{3}\,dy\,dx+\int_0^1\int_0^y(1-y)-\frac{(1-y)^3}{3}\,dx\,dy=$$

$$=\int_0^1\left[(1-x)y-\frac{(1-x)^3}{3}y\right]_{y=0}^{y=x}dx+\int_0^1\left[(1-y)x-\frac{(1-y)^3}{3}x\right]_{x=0}^{x=y}dy=$$

$$=\int_0^1(1-x)x-\frac{(1-x)^3}{3}x\,dx+\int_0^1(1-y)y-\frac{(1-y)^3y}{3}\,dy$$

$$=\ldots=\frac{3}{10}$$

(10) όγκος ανάμεσα στον κώνο $x^2+y^2=z^2$ και το επίπεδο $2z-y-2=0$.

$$\begin{cases} x^2+y^2=z^2 \\ 2z-y-2=0 \end{cases} \Rightarrow \begin{cases} x^2+y^2=z^2 \\ z=\frac{2+y}{2} \end{cases}$$

$$x^2+y^2=\frac{(2+y)^2}{4}$$

$$\Rightarrow x^2+y^2=\frac{4+4y+y^2}{4} \Rightarrow x^2+\frac{3y^2}{4}-y=1$$

$$\frac{x^2}{\frac{2}{3}}+\frac{(y-2/3)^2}{\frac{8}{9}}=1$$

Έλλειψη

$$V=\iint_{pr_{xy}\omega}\left(\int_0^{\frac{y+2}{2}} dz\right)dx\,dy=\frac{1}{2}\iint_{pr_{xy}\omega} y+2\,dx\,dy$$

Ελλειπτικές συντεταγμένες

$$x=\frac{\sqrt{2}}{3}r\cos\theta,\quad y=\frac{2}{3}+\frac{\sqrt{2}}{3}r\sin\theta$$

$r\in[0,1]$

$\theta\in[0,2\pi]$

$\int = \frac{4}{9} r$

Άρα $V = \int_0^1 \int_0^{2\pi} \left(\frac{\sqrt{8}}{3} r\sin\theta + \frac{8}{3}\right)\frac{4}{9} r \, d\theta \, dr =$

$= \int_0^1 \left[-\frac{4}{9}\cdot\frac{\sqrt{8}}{3} r^2 \cos\theta + \frac{32}{27} r\cdot\theta \right]_{\theta=0}^{\theta=2\pi} dr =$

$= \int_0^1 \left(\frac{32}{27}\cdot 2\pi \cdot r \right) dr =$

$= \frac{64\pi}{27} \left[\frac{r^2}{2} \right]_0^1 = \frac{32\pi}{27}$

(12) C_1, C_2 κύλινδροι με άπειρο μήκος, διάμετρο 2 με ~~κοινούς~~ άξονες τους x, y αντίστοιχα.

όγκος $C_1 \cap C_2$.

$C_1: y^2 + z^2 = 1 \qquad C_2: x^2 + z^2 = 1$ $\boxed{4 \text{ τρίγωνα}}$

$C_1 \cap C_2: \quad y^2 + 1 - x^2 = 1 \Rightarrow y^2 = x^2 \Rightarrow x = \pm y$

$V = 4 \cdot \int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dz \, dx dy =$

$= 8 \int_0^1 \int_{-y}^{y} \sqrt{1-y^2} \, dx dy = 8 \int_0^1 \left[\sqrt{1-y^2}\cdot x \right]_{x=-y}^{x=y} dy$

$$= 16\int_0^1 y\sqrt{1-y^2}\,dy = 16\left[-2\,\frac{(1-y^2)^{3/2}}{2\cdot 3}\right]_{y=0}^{y=1} =$$

$$= \frac{16}{3}$$

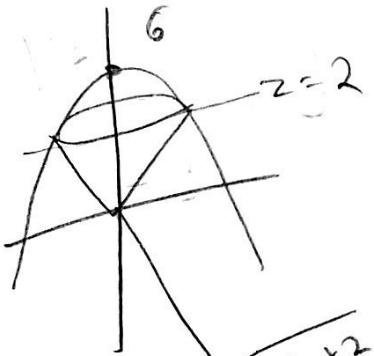
$z \le 6 - x^2 - y^2, \quad z \ge \sqrt{x^2+y^2}$ (14)

$z = \sqrt{x^2+y^2} = 6 - x^2 - y^2$

$\Rightarrow \quad z = 6 - z^2$

$z^2 + z - 6 = 0, \quad \Delta = 1 - 4(-6) = 25$

$z_{1,2} = \frac{-1 \pm 5}{2} = \begin{cases} 2 \\ -3 \end{cases}$



για $z = 2$.

$x^2 + y^2 = 4$

κυκλος κεντρου 0
ακτινας 2.

$$V = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{6-x^2-y^2} 1 \, dz \, dy \, dx =$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 6 - x^2 - y^2 - \sqrt{x^2+y^2} \, dy \, dx$$

πολικες $V = \int_0^{2\pi} \int_0^2 (6 - r^2 - r) r \, dr \, d\theta =$

$$= \int_0^{2\pi} \left[\frac{6r^2}{2} - \frac{r^4}{4} - \frac{r^3}{3} \right]_0^2 d\theta = \dots$$

$$\iiint_S \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{3/2}}$$

$S: \left\{ x^2+y^2+z^2=a \;\text{ και }\; x^2+y^2+z^2=b^2, \quad 0<b<a \right\}$

σφαιρικές

$$\int_b^a \int_0^{2\pi} \int_0^{\pi} \frac{\rho^2 \sin\varphi}{(\rho^2)^{3/2}}\, d\varphi\, d\theta\, d\rho =$$

$$= \int_b^a \int_0^{2\pi} \left[\frac{-\rho^2 \cos\varphi}{(\rho^2)^{3/2}} \right]_{\varphi=0}^{\varphi=\pi} d\theta\, d\rho =$$

$$= \int_b^a \int_0^{2\pi} \frac{2\rho^2}{(\rho^2)^{3/2}}\, d\theta\, d\rho =$$

$$= 4\pi \int_b^a \frac{\rho^2}{\rho^{6/2}}\, d\rho = 4\pi \left[\ln\rho \right]_{\rho=a}^{\rho=b}$$

$$= 4\pi \ln \frac{a}{b}$$

21 α) $\displaystyle\int_0^\infty \int_0^y x e^{-y^3}\, dx\, dy =$

$$= \int_0^\infty \left[\frac{x^2}{2} e^{-y^3} \right]_{x=0}^{x=y} dy = \int_0^\infty \frac{y^2}{2} e^{-y^3}\, dy$$

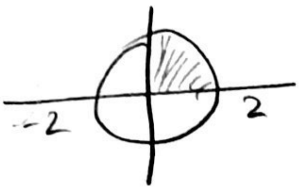
$$I_M = \int_0^M \frac{y^2}{2} e^{-y^3}\, dy = \left[-\frac{e^{-y^3}}{6} \right]_0^M = \frac{1-e^{-M^3}}{6}$$

$$I = \lim_{M\to\infty} I_M = \frac{1}{6}.$$

b) $\iint_B (x^4+2x^2y^2+y^4)\,dx\,dy = \iint_B (x^2+y^2)$

B: κυκλικός δίσκος με $\rho=2$, κέντρο $(0,0)$

1ο τεταρτημόριο



$x = r\cos\theta$

$y = r\sin\theta$

$$\int_0^2\int_0^{\pi/2} r^4\cdot r \; d\theta\, dr = \int_0^2 r^5 \frac{\pi}{2}\, dr =$$

$$= \frac{\pi}{2}\left[\frac{r^6}{6}\right]_0^2 = \frac{2^6\cdot\pi}{12} = \frac{2^4\pi}{3} = \frac{16\pi}{3}$$

(23) $\int_{R^3} \exp\left[-(x^2+y^2+z^2)^{3/2}\right] dx\,dy\,dz$

$$I_R = \iiint_{B(0,R)} e^{-(x^2+y^2+z^2)^{3/2}}\, dx\,dy\,dz$$

Σφαιρικές

$$\int_0^R \int_0^{2\pi}\int_0^{\pi} e^{-\rho^3}\rho^2 \sin\varphi \; d\varphi\, d\theta\, d\rho =$$

$$= 2\pi \int_0^R \int_0^{\pi} e^{-\rho^3}\rho^2\sin\varphi\; d\varphi\, d\rho = 2\pi\int_0^R \left[-e^{-\rho^3}\rho^2\cos\varphi\right]_{\varphi=0}^{\varphi=\pi} d\rho =$$

$$= 2\pi \int_0^R 2e^{-\rho^3}\rho^2 \, d\rho = 4\pi \int_0^R e^{-\rho^3}\rho^2 \, d\rho =$$

$$= 4\pi \left[-\frac{e^{-\rho^3}}{3} \right]_0^R = 4\pi \left(-\frac{e^{-R^3}}{3} + \frac{1}{3} \right)$$

$$\xrightarrow{R \to \infty} \frac{4\pi}{3}$$