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Boltzmann's Ergodic Hypothesis,
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Abstract. An overview of the history of Ludwig Boltzmann's more than one hundred year old ergodic hypothesis is given. The existing main results, the majority of which is connected with the theory of billiards, are surveyed, and some perspectives of the theory and interesting and realistic problems are also mentioned.

In 1964 Werner Heisenberg was elected a honorary doctor of Loránd Eötvös University, Budapest. In his inaugural lecture he made a point that sounded something like this: "A theoretical physicist feels best if there is no rigorously defined mathematical object behind his considerations". Certainly, Heisenberg was having the early years of quantum mechanics in his mind but what he said perfectly fitted the work of Ludwig Boltzmann as well. One could choose several areas of his interest to illustrate this statement, out of which the history of the ergodic hypothesis we are going to elaborate on is only one.#

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#Another striking example, perhaps not sufficiently widely known, is the case with the Boltzmann equation. He published it in 1872, B(1872), and the first mathematically satisfactory derivation of the equation was only obtained more than 100 years later in 1975 by Oscar Lanford, L(1975), though the picture is still not complete. Thus needless to say that Boltzmann's original argument was highly intuitive. At the same time, however, it was so much challenging for the great mathematician, David Hilbert that he included among his celebrated collection of 23 problems presented at the International Mathematical Congress held at Paris in 1900 the sixth one with the title "Mathematical Treatment of the Axioms of Physics" (see H(1900)). In its formulation, besides requiring an axiomatic approach to the theory of probabilities, Hilbert also says: "it is therefore very desirable that the discussion of the foundations of mechanics be taken up by mathematicians also. Thus Boltzmann's work on the principles of mechanics suggests the problem of developing mathematically

As it was so nicely explained in Professor Gallavotti's illuminating lecture at this conference, G[1994], though the rigorously defined mathematical object behind Boltzmann's considerations around the ergodic hypothesis was indeed missing, Boltzmann was ingenious in inventing mathematical paradigmas and in mastering mathematical calculations on them to find out the truth and to obtain convincing power, and even without having the mathematical object he understood many things better than we do now.

1. Boltzmann's ergodic hypothesis.

During the 1870s and 1880s, various forms of the ergodic hypothesis were used by Boltzmann in his works on the foundations of statistical mechanics (see e. g. B(1871) and B(1884); for a historic account also F(1989)). An advanced formulation of the hypothesis would sound as follows:

Boltzmann's Ergodic Hypothesis. *For large systems of interacting particles in equilibrium time averages are close to the ensemble, or equilibrium average.*

(Remark: In this paper — with the exception of section 10 — equilibrium averages always mean microcanonical ones, i. e. the Liouville measure on the submanifold of the phase space specified by the trivial invariants of the motion.)

More precisely, if f is a measurement (i.e. a function on the phase space of the system), then as N, the size of the system (for instance, the number of particles) tends to infinity, then

$$\frac{1}{T}\int_0^T f(S^t x)dt \to \int f(x)d\mu(x) \tag{1}$$

where μ is the equilibrium measure, and $S^t x$ is the time evolution of the phase point x.

We immediately note that if N varies, then f and μ also depend on N, and thus, for a mathematically strict statement one ought to specify the sense of the convergence in (1), too. Let us look at the main steps of the history of Boltzmann's hypothesis — without intending to provide a complete account though I think such a study should be done. One major incompleteness of our survey is that it does not go into the history of the quasi-ergodic hypothesis at all; as to some recent results about it see H(1991) and Y(1992).

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the limiting processes, there merely indicated, which lead from the atomistic view to laws of motion of continua". Boltzmann's law of motion of continua is, of course, his equation.

2. Finding a mathematical object, a notion and a problem (from Boltzmann to von Neumann, i.e. from 1870 until 1931).

It took quite a time until the mathematical object of the ergodic hypothesis was found. Indeed, only in 1929, Koopman, K(1931), began to investigate groups of measure-preserving transformations of a measure space or in other language, groups of unitary operators in a Hilbert space*. Koopman's idea was apparently in the air, and several mathematicians, including among other G. Birkhoff, M. S. Stone and A. Weil, contributed to the birth of ergodic theory; for a historic account see M(1990).

More precisely, let M be an abstract space, the phase space of the system and μ be a probability measure on (a σ-algebra of) M. The dynamics is a one-parameter group $S^{\mathbb{R}} = \{S^t : -\infty < t < \infty\}$ of measure preserving transformations, i.e. for every measurable subset $A \subset M$, and for every $t \in \mathbb{R}$ $\quad \mu(S^{-t}A) = \mu(A)$.

Here, of course, μ is the equilibrium measure of the system. Let finally, $f : M \to \mathbb{R}$ be a measurement such that $f \in L_2(\mu)$. Thus the object [i. e. $(M, S^{\mathbb{R}}, d\mu)$ with the functions f] is defined.

In 1931, von Neumann proved the first ergodic theorem, the so called

Mean Ergodic Theorem (N(1932)). *As $T \to \infty$,*

$$\frac{1}{T}\int_0^T f(S^t x)dt \to \bar{f}(x)$$

in the L_2-sense.

(The exact story of the first ergodic theorems is explained in the note of Birkhoff and Koopman, B-K(1932).)**

The proof of the mean ergodic theorem is not difficult but it is worth noting that — even more than 20 years later — Neumann very highly appreciated exactly this achievement among his various findings in the vast territory of his interest. In 1954, when answering a questionnaire of the American Mathematical Society, his works on the ergodic theorems were named by himself among his most important discoveries (the other two were the mathematicial foundations of quantum mechanics, and further operator-algebras, called today Neumann-algebras).

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*This progress was preceded by the success of Lebesgue's theory of measure which, on another path, also led, in 1933, Kolmogorov to the laying down the axiomatic foundations of probability theory.

**Though his name is not explicitly mentioned, Boltzmann's influence on von Neumann is also seen in the title of his earlier work on quantum ergodic theory, N[1929]: "Beweis des Ergodensatzes und des H-Theorems in der neuen Mechanik".

The limiting function $\bar{f}(x)$ satifies two further important properties:

(i)

$$\bar{f}(x) = \mathbb{E}(f/\mathcal{I})$$

where $\mathcal{I}$ is the σ-algebra of the invariant sets, or in other words $\bar{f}$ is the projection of f onto the subspace of functions invariant with respect to the dynamics $S^{\mathbb{R}}$;

(ii) $\int_M f d\mu = \int_M \bar{f}\mu$ whenever $f \in L_2$.

An extremely important consequence is the following: if the only invariant functions are the constants or, in other words, there are only trivial invariant sets (i.e. $\mu\big((S^{-t}A \setminus A) \cup (A \setminus S^{-t}A)\big) = 0$ implies $\mu(A) = 0$ or 1), then, first of all, $\bar{f}$ is a constant for every f and, moreover, by (ii), $\bar{f} = \int f d\mu$. Consequently, the ergodic theorem says that, then as $T \to \infty$,

$$\frac{1}{T}\int_0^T f(S^t x)dt \to \int_M f d\mu \tag{2}$$

in the L_2-sense.

This statement is much reminiscent to Boltzmann's hypothesis but here we still have just one fixed system and not various ones for different values of N. Anyway, define the system to be *ergodic*, if the only invariant functions are the constants. Then we know that, for ergodic systems, the relation (2), i.e. a version of the ergodic hypothesis holds.

Summarizing: we have a mathematical model (groups of measure preserving transformations), the notion of ergodicity and, finally, the problem of establishing the ergodicity of a system we are interested in from the mechanical point of view.

We note that, a bit later in 1931, Birkhoff, B(1931) (and also Khintchine) could, moreover, prove that the convergence in (2) holds almost everywhere as well.

This progress led to the birth of an independent branch of mathematics: ergodic theory. This theory then began his autonomous evolution within mathematics and several sub-branches were also born. Just to mention some, one of them studies various forms and generalizations of the ergodic theorems, another one stronger forms of stochasticity, a special branch — quite interesting for our present discussion — investigates the ergodicity of particular systems, among them those arising from mechanics, a further one the isomorphism problem of various dynamical systems,etc.

3. Proving the first relevant theorem (from Neumann to Sinai, from 1931 until 1970)

The methods for establishing the ergodicity of mechanical systems came from a different though related domain, from the theory of dynamical system. In 1938-39, Hedlund, He(1939) and Hopf, Ho(1939) found a method for demonstrating the ergodicity of geodesic flows on compact manifolds of negative curvature. Their main conceptual discovery was that the so called hyperbolic behaviour of dynamical systems could imply and, in fact, did imply ergodicity in the aforementioned models.

Hyperbolicity means, in other words, instability, i.e. the exponential divergence of trajectories starting arbitrarily close to each other, or else sensitivity to the initial conditions. The simplest example of a hyperbolic system is Arnold's famous cat, the linear automorphism of the torus (cat stands for a Continuous Automorphism of the Torus) . Indeed, if we consider the map T_A of the 2-torus $\mathbb{R}^2 \mid \mathbb{Z}^2$ onto itself defined by the (hyperbolic) matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, then we see that the image of the cat gets expanded in one direction and contracted in a transversal one, with the expansion (contraction) being the strongest in the eigendirection of the matrix corresponding to the eigenvalue $\lambda_u > 1 (\lambda_s < 1)$.

In 1942, very soon after Hedlund's and Hopf's fundamental results, the Russian physicist, N. S. Krylov discovered that systems of elastic hard balls show an instability similar to the one observed at geodesic flows on manifolds with negative curvature, cf. K[1942]. This finding and the progress of the ideas of Hedlund and Hopf in the theory of hyperbolic dynamical systems justified Sinai's stronger version of Boltzmann's ergodic hypothesis formulated in 1963 for the particular system of elastic hard balls.

The Boltzmann-Sinai Ergodic Hypothesis (S(1963)). *The system of N hard balls given on $\mathbb{T}^2$ or $\mathbb{T}^3$ is ergodic for any $N \geq 2$.*

Since mechanical systems also have conserved quantities, this conjecture is understood so that ergodicity is expected to hold on (connected components of) the submanifold of the phase space specified by the invariants of motion.

The conceptual surprise of this conjecture compared to Boltzmann's original formulation was that no large N was assumed. In fact, ergodicity (and further stronger mixing properties, like the K-property) was expected to hold for any fixed $N \geq 2$!

In 1970, Sinai, S(1970) was able to verify this conjecture in the case of $N = 2$ 2-dimensional discs moving on the 2-torus $\mathbb{T}^2$.

Before giving an insight into Sinai's approach, let us mention the limitations of this nice ergodic behaviour for systems with a fixed number of degrees of freedom.

4. Appearance of non-ergodic behaviour (cosequences of the KAM-theory, 1954-1974)

In nature, we have important examples of systems of interacting particles (or bodies) that are stable and not unstable like systems of hard balls.

The most striking example is — fortunately — the solar system. The fact that it consists of bodies of different masses is not of great importance, more significant is the fact that here the interaction is different.

The year 1954 brought two important discoveries. Kolmogorov's 1954 work, K(1954) and its later evolution — thanks first of all to the achievements of Arnold and Moser (in particular, A(1963) and M(1962)) in the 60's — indicated that we may well have a situation when invariant tori with dimension half of that of the phase space can fill a set of positive measure (we note that in completely integrable systems such invariant tori do foliate the whole phase space). Another, not so explicit, warning came from the numerial work of Fermi-Pasta-Ulam, F-P-U(1955) demonstrating that the asymptotic equipartition of the energy of modes may fail. As to a detailed exposition of this experiment and its effects we refer to the survey H(1983).

In the 1974 work of Markus-Meyer, M-M(1974) summarizing the previous progress there were two important statements out of which the first one is more remarkable for our discussion.

Theorem. *In the space of smooth Hamiltonians*

(1) *The nonergodic ones form a dense open subset;*
(2) *The nonintegrable ones form a dense open subset.*

Without going into technical details we note that the statements are formulated in the C^∞-topology and a Hamiltonian is called ergodic if, for almost every values of the energy, the system is ergodic on the corresponding submanifold of the phase space.

Thus, for generic Hamiltonians, we cannot expect ergodicity, and in the final sections of the paper we will return to the question of what kind of ergodic behaviour can then be expected for them. The forthcoming discussion will be focused on the comparatively simple case of hard ball systems.

5. Sinai's setup. Billiards (1970)

We start with a simple trick traditional both in mathematics and physics: instead of treating N particles we consider just one particle in a high dimensional phase space. More concretely: Let us assume, in general, that a system of $N(\geq 2)$ balls of unit mass and radii $r > 0$ are given on $\mathbb{T}^\nu$, the ν-dimensional unit torus ($\nu \geq 2$). Denote the phase point of the i'th ball by $(q_i, v_i) \in \mathbb{T}^\nu \times \mathbb{R}^\nu$. The configuration space $\tilde{\mathbf{Q}}$ of the N balls is a subset of $\mathbb{T}^{N\cdot\nu}$: from $\mathbb{T}^{N\cdot\nu}$ we cut out $\binom{N}{2}$ cylindric scatterers:

$$\tilde{C}_{i,j} = \left\{Q = (q_1, \ldots, q_N) \in \mathbb{T}^{N\cdot\nu} :\mid q_i - q_j \mid < 2r\right\},$$

$1 \leq i < j \leq N$. The energy $H = \frac{1}{2}\sum_1^N v_i^2$ and the total momentum $P = \sum_1^N v_i$ are first integrals of the motion. Thus, without loss of generality, we can assume that $H = \frac{1}{2}$ and $P = 0$ and, moreover, that the sum of spatial components $B = \sum_1^N q_i = 0$ (if $P \neq 0$, then the center of mass has an additional conditionally periodic or periodic motion). For these values of H, P and B, the phase space of the system reduces to $M := \mathbf{Q} \times \mathcal{S}_{N\cdot\nu-\nu-1}$ where

$$\mathbf{Q} := \left\{Q \in \tilde{\mathbf{Q}} \setminus \cup_{1\leq i<j\leq N}\tilde{C}_{i,j} : \sum_1^N q_i = 0\right\}$$

with $d := \dim \mathbf{Q} = N \cdot \nu - \nu$, and where $\mathcal{S}_k$ denotes, in general, the k-dimensional unit sphere. It is easy to see that the dynamics of the N balls, determined by their uniform motion with elastic collisions on one hand, and the billiard flow $\{S^t : t \in \mathbb{R}\}$ on $\mathbf{Q}$ with specular reflections on $\partial\mathbf{Q}$ on the other hand, are isomorphic and they conserve the Liouville measure $d\mu = const \cdot dq \cdot dv$.

We recall that a *billiard* is a dynamical system describing the motion of a point particle in a connected, compact domain $\mathbf{Q} \subset \mathbb{R}^d$ or $\mathbf{Q} \subset \mathbb{T}^d = \mathrm{Tor}^d$, $d \geq 2$ with a piecewise C^2-smooth boundary. Inside $\mathbf{Q}$ the motion is uniform while the reflection at the boundary $\partial\mathbf{Q}$ is elastic (the angle of reflection equals the angle of incidence, cf. Figure 1). Since the absolute value of the velocity is a first integral of motion, the phase space of our system can be identified with the unit tangent bundle over $\mathbf{Q}$. Namely, the configuration space is $\mathbf{Q}$ while the phase space is $M = \mathbf{Q} \times S_{d-1}$ where S_{d-1} is the surface of the unit d-ball. In other words, every phase point x is of the form (q, v) where $q \in \mathbf{Q}$ and $v \in S_{d-1}$. The natural projections $\pi : M \to \mathbf{Q}$ and $p : M \to S_{d-1}$ are defined by $\pi(q, v) = q$ and by $p(q, v) = v$, respectively.

Figure 1

Suppose that $\partial\mathbf{Q} = \cup_1^k \partial\mathbf{Q}_i$ where $\partial\mathbf{Q}_i$ are the smooth components of the boundary. Denote $\partial M = \partial\mathbf{Q} \times S_{d-1}$ and let $n(q)$ be the unit normal vector of the boundary component $\partial\mathbf{Q}_i$ at $q \in \partial\mathbf{Q}_i$ directed inwards $\mathbf{Q}$. In billiards, isomorphic to hard ball systems, the scatterers are convex cylinders if $N \geq 3$, and are (strictly convex) balls if $N = 2$. The observation of Krylov and Sinai was that a billiard with strictly convex scatterers behaves like a hyperbolic dynamical system, whereas in one with just convex scatterers there is some partial hyperbolicity. We will illustrate this observation after some definitions.

We say that a billiard is *dispersing* (a Sinai-billiard) if each $\partial\mathbf{Q}_i$ is strictly convex, and we say it is *semi-dispersing* if each $\partial\mathbf{Q}_i$ is convex. The billiards on Figures 2 and 3 are dispersing. Indeed, they correspond to the system of two discs on $\mathbb{T}^2$; the first one to the case $R < 1/4$ and the second one to the case $1/4 < R < 1/2$.

Figures 2, 3, 4

The third one is a semi-dispersing billiard given on $\mathbb{T}^3$ with two cylindric scatterers. This paradigm was the first semi-dispersing but not dispersing billiard whose ergodicity was established (cf. K-S-Sz(1989)).

Figure 5

The mechanism producing hyperbolicity in a dispersing billiard can be seen the best on Figure 5 borrowing the illustration from optics. Assume we have a strictly convex scatterer on $\mathbb{T}^d$ and imagine it is a mirror. Take, $x = (Q, V) \in M$, and the codimension one hyperplane ? through Q in the configuration space perpendicular to the velocity V. By attaching to points of ? velocities identical to V we obtain a wavefront $\tilde{?}$ in the phase space M. After one reflection from the mirror scatterer, our wavefront gets strictly convex while the linear distances measured on ? get uniformly expanded. This mechanism is exactly the one providing the (uniform) hyperbolicity of a dispersing billiard.

Sinai's 1970 work used the theory of uniformly hyperbolic smooth dynamical systems which had had an intensive progress in the 60s and culminated in the 1967 paper of Anosov and Sinai, A-S(1967). The serious difficulty Sinai had to cope with was that billiards were not smooth dynamical systems. Indeed, if a smooth wavefront gets reflected from a scatterer and it contains a tangency, then though the reflected wavefront will be continuous, its second derivative will have a jump at the tangency. This circumstance causes serious technical difficulties: in smooth uniformly hyperbolic dynamical systems the stable and unstable invariant manifolds, the fundamental tools of the theory are smooth and unbounded, whereas in billiards their smooth components can be arbitrarily small.

6. $N = 2$ balls (1970-1987). Local ergodicity of semi-dispersing billiards.

As mentioned earlier, Sinai, in 1970, in his celebrated paper obtained the first rigorous result in relation to the Boltzmann-Sinai ergodic hypothesis: he could show that $N = 2$ discs on the 2-torus $\mathbb{T}^2$ was a K-system.

In fact, his result was formulated for $2-D$ dispersing billiards (Sinai-billiards) with a finite horizon. A billiard has *finite horizon* if there is no collision-free trajectory in it. This condition is fulfilled by a two-billiard if $R > \frac{1}{4}$ (cf. Figure 3). In this case the configuration space consists of four connected components, and, of course, ergodicity is claimed on each of them. For the case of $R < \frac{1}{4}$ (cf. Figure 2), a $2-D$ billiard with infinite horizon, the corresponding result was proved by Bunimovich and Sinai in 1973, B-S(1973). On the basis of their work it was understood that a $2-D$ dispersing billiard was ergodic.

A multidimensional generalization of their theorem was only obtained in 1987. Indeed, Chernov and Sinai, S-Ch(1987) were, in general, investigating semi-dispersing billiards and introduced the basic notion of *sufficiency* of an orbit or equivalently of a phase point. The main consequence of sufficiency is that, in a suitably small neighbourhood of a sufficient point, the system is hyperbolic, though not uniformly. Next we present this notion in its minimal form as suggested in K-S-Sz (1990).

Figure 6

Our starting point is Figure 6, similar to Figure 5. It shows that, if a scatterer is not strictly convex but just convex, like e. g. a cylinder, then the image of the hyperplanar wavefront ? with parallel velocities will not be curved in the directions parallel with the constituent subspace of the cylinder, but in the transversal directions, only. However, the uncurved neutral directions can still die out after several reflections on differently oriented cylindric (or, in general, convex) scatterers.

Now for the definition of sufficiency. Assume that $S^{[a,b]}x$ is a finite trajectory segment, which is regular, i.e. it avoids singularities.

Let $S^a x = (Q, V) \in M$ and consider the hyperplanar wavefront $\tilde{?}(S^a x) := \{(Q + dQ, V) : dQ \text{ small} \in \mathbb{R}^d \text{ and } \langle dQ, V \rangle = 0\}$ (by denoting $\pi(x) = Q$ for $x = (Q, V)$ we see that, indeed, $\pi(\tilde{\gamma})$ is part of a hyperplane).

We say that the trajectory segment $S^{[a,b]}x$ is *sufficient* if $\pi(S^b\tilde{?})$ is strictly convex (see Figure 7). (To obtain a geometric or optical feeling of this notion, the reader is again suggested to imagine mirror-surfaced scatterers.) A phase point $x \in M$ is sufficient if its trajectory is sufficient (i.e. it contains a sufficient trajectory segment). In physical terms, sufficiency of a trajectory segment means that, during the time interval $[a, b]$, the trajectory of x encounters all degrees of freedom of the system.

Figure 7

If a trajectory segment is not sufficient, then the curvature of $\pi(S^b\tilde{?})$ at $\pi(S^bx)$ necessarily vanishes in certain directions forming the so-called *neutral subspace.* Simple geometric considerations (cf.K-S-Sz(1990)) show that a sufficient trajectory segment generates an expansion rate uniformly larger than 1 in some neighbourhood of the point S^ax. Then, by Poincaré recurrence and the ergodic theorem, it is not hard to see that, in some neighbourhood of S^ax, the relevant Lyapunov exponents of the system are not zero. In other words, in this neighbourhood, the system is hyperbolic. This observation should motivate the non-trivial

Fundametal theorem for semi-dispersing billiards (S-Ch(1987)). *Assume that a semi-dispersing billiard satisfies some geometric conditions and the Chernov — Sinai ansatz, a condition strongly connected with the singularities of the system.*

If $x \in M$ is a sufficient point, the it has an open neighbourhood U in the phase space belonging to one ergonent (i.e. ergodic component).

(A simplified and suitably generalized version of this theorem, the so-called 'transversal fundamental theorem' was given in K-S-Sz(1990). Moreover, a version of the fundamental theorem formulated for symplectic maps with singularities can be found in L-W(1994).) The property expressed in the statement is usually called *local ergodicity.* If almost every phase point of a semi-dispersing billiard is sufficient, then, of course, it may have at most a countable number of ergonents. In some cases it is not hard then to derive the *global ergodicity* of the system, i.e. to show that there is just one ergonent in the phase space. Note that it also follows from the general theory that, on each ergonent, the system is Kolmogorov mixing. A much important consequence is thus the following

Corollary (S-Ch(1987)). *Every dispersing billiard is ergodic, and, moreover, is a K-flow. In particular, the system of $N = 2$ balls on the ν-torus is a K-flow if $r < \frac{1}{2}$.*

(For details, see K-S-Sz(1990).)

7. $N \geq 3$ BALLS (1989-). GLOBAL ERGODICITY OF SEMI-DISPERSING BILLIARDS.

With the fundamental theorem for semi-dispersing billiards in mind, the proof of their global ergodicity boils down to

(1) first demonstrating the Chernov-Sinai ansatz, an important condition of the fundamental theorem, and

(2) to then showing that the subset of non-sufficient points is a topologically small set of measure zero; for instance, its topological codimension is not smaller than two.

In Sz(1993), we gave a sketch of the strategy worked out in our papers with A. Krámli and N. Simányi for the core part, and here we will just list the main results obtained so far.*

(1) in 1991, Krámli, Simányi and the present author, [KSSz-91] demonstrated the K-property of $N = 3$ balls on the $\nu-$torus whenever $\nu \geq 2$;

(2) in 1992, again the previous authors, [KSSz-92] improved their methods to get the ergodicity of $N = 4$ balls on the $\nu-$torus whenever $\nu \geq 3$;

(3) in 1992, Simányi, S(1992) was able to establish the so far strongest result for hard ball systems: the system of $N \geq 2$ balls is ergodic on the $\nu-$torus whenever $\nu \geq N$; his method is based on his Connecting Path Formula characterizing the neutral subspace of a trajectory segment.

The configuration of the cylindric scatterers of a billiard isomorphic to a hard ball system inherits the permutation symmetry of the balls. A natural generalization of hard ball systems is to investigate cylindric billiards in general, i. e. billiards with solely cylinders as scatterers. To this end consider compact affine subspaces $L^i : 1 \leq i \leq N,\ N \geq 1$ in the d-torus $\mathbf{T}^d$ (with dim $L^i \leq d - 2$), and denote $C^i := \{Q := (q_1, \dots, q_d) : \mathrm{dist}(Q, L^i) \leq r^i\},\ \ 1 \leq i \leq N$ where each r^i is positive. The billiard in $\mathbf{Q} := \mathbf{T}^d \setminus (\cup_{i=1}^N C^i)$ is *a billiard with cylindric scatterers.*

For cylindric billiards the following results have been obtained:

(1) in 1989, Krámli, Simányi and the present author, K-S-Sz(1989) considered a 3-dimensional orthogonal cylindric billiard (cf. Figure 4); they obtained its

*Most recently, in the Summer of 1994, Simányi and Szász, were able to prove Sinai's hypothesis in the general case $N, \nu \geq 2$.

K-property and thus this was the first semi-dispersing — but not dispersing — billiard whose ergodicity was shown.

(2) in 1993, motivated by a question of John Mather, the present author started a systematic study of cylindric billiards and found a sufficient and necessary condition for the ergodicity of a class of them: for *orthogonal cylindric billiards*, cf. Sz(1993), Sz(1994). These are characterized by the property that the constituent subspace of any cylindric scatterer is spanned by some of the coordinate vectors adapted to the orthogonal coordinate system where $\mathbf{T}^d$ is given;

(3) in 1994, Simányi and the present author, S-Sz(1994) found necessary and sufficient conditions for the K-property of a toric billiard with two arbitrary cylindric scatterers.

Since the class of cylindric billiards is relatively simple, one can hope for general necessary and sufficient conditions for the ergodicity (and the K-property) of these systems. Indeed, we next formulate a conjecture containing a general sufficinet condition.

Conjecture (Szász, 1992). *Assume that the configuration domain of a cylindric billiard is connected, and no pairs of the scatterers are tangent. If there is at least one sufficient point, then the billiard is K.*

8. The Boltzmann-Sinai ergodic hypothesis in pencase type models.

In order to resolve some difficulties on the way to establishing the Boltzmann-Sinai ergodic hypothesis, Chernov and Sinai, S-Ch(1985) suggested the study of a quasi-one-dimensional model of hard balls. It is given on an elongated torus of the type $(L\mathbb{T}^1) \times \mathbb{T}^{\nu-1}$ where L is a sufficiently large number compared to R (see Figure 8). The main assumption is

$$\frac{\sqrt{\nu-1}}{4} < R < \frac{1}{2}$$

ensuring that the order of balls (in the direction of $L\mathbb{T}^1$) is invariant under the dynamics. Thus the model, which was called by Chernov and Sinai *a pencase*, is realizable if $2 \le \nu \le 4$. If we number the balls in their order: $1, 2, \cdots, N$, then a particular feature of the model is that only the pairs of consecutive balls (i.e. $\{1,2\}, \{2,3\}, \cdots, \{N,1\}$) can interact.

Figure 8

The first result for a pencase type model was reached in 1992 by Bunimovich-Liverani-Pellegrinotti- Sukhov. Instead of a torus their model lives in a domain with dispersing boundaries (see Figure 9) and the sizes of the domain ensure that

Figure 9

(1) each ball is restricted to a fundamental domain of the "pencase" (the throats between them are smaller than $2R$);
(2) between consecutive collisions of a particular ball, it should always hit a dispersing boundary;
(3) the pairs of balls in neighbouring domains can, indeed, interact.

A billiard of this type is realizable in arbitrary dimension and the results of the aforementioned authors was that the system was K. This particular model was, in fact, the first one where the Boltzmann-Sinai ergodic hypothesis got settled for any N and $\nu \geq 2$.

Theorem (B-L-P-S(1992)). *The B-L-P-S pencase is a K-system for any* $N, \nu \geq 2$.

For some time it seemed so that the proof of ergodicity for the original Chernov-Sinai pencase was not easier than that for general hard ball systems. Nevertheless, — with Nándor Simányi — we could recently demonstrate the following

Theorem (S-Sz (1994-B)). *The Chernov-Sinai pencase is a K-system for any* $N \geq 2, \nu = 4$. *If* $\nu = 3$, *then the system has open ergodic components.*

The restriction $\nu \neq 2$ seems, at present, important whereas that of $\nu < 5$ only arises since the model, as invented by its authors, does not exist for $\nu \geq 5$. One could, however, introduce less realistic models that do exist for $\nu \geq 5$, too, and for them our proof would also work but we do not want to stay on them.

There is, however, another, more natural way to introduce models with a pencase-type interaction in any dimension. Consider, namely, N balls, numbered $1, 2, \ldots, N$ on the unit torus $\mathbb{T}^\nu$. The restriction is that only pairs of balls with neighbouring numbers, i.e. again only the pairs $\{1,2\}, \{2,3\}, \ldots, \{N-1, N\}, \{N, 1\}$ interact while other pairs can go through each other. This billiard is, of course, again a cylindric one.

Theorem (S-Sz (1994-B)). *The system with pencase-type interaction is a K-system whenever* $N \geq 2, \nu \geq$. *If* $\nu = 3$, *then the system has open ergodic components.*

(Froeschlé(1978) (cf. H(1983)) introduced the notion of *connectivity* as the ratio of the number of particles a given particle can interact with and of the number of all particles. His experiments suggested that this ratio can be related to the good ergodic properties of a system; in particular, below a critical value of the connectivity, a significant fraction of the phase space is occupied with invariant tori. Our theorem shows, however, that, for hard ball systems, the ergodic behaviour already appears at a connectivity arbitrarity close to zero.)

9. Ergodicity of systems with a fixed number of degrees of freedom.

¿From the work of Markus-Meyer mentioned in section 4 we know that ergodic Hamiltonians are in a sense exceptional. Nevertheless, it makes sense to look for possibly more of them since the mechanisms occurring in these can also help to understand the onset of chaotic behavior, for instance, the appearance of a large ergodic component in nonergodic systems.

In sections 5-7 we discussed billiard systems. Here we mention three classes of Hamiltonians, for which Donnay and Liverani, D-L(1991) could, in 1991, demonstrate ergodicity. These are systems of $N = 2$ particles on $\mathbb{T}^2$ interacting via a rotation-invariant pair potential $V(r)$. These system have the same conserved quantities as the system of two hard discs and we assume that $v_1^2+v_2^2 = 1,\ v_1+v_2 = 0,\ q_1+q_2 = 0$. We do not give here all the conditions since we are mainly interested in the qualitative description of these interactions.

Assume in all cases that for some $R > 0$

(1) $V(r) = 0$ if $r \geq R$;

(2) $V(r) \in C^2(O, R)$;

(3) $\lim_{r\to 0} r^2 V(r) = 0$;

(4) for $h(r) = r^2(1-2V(r))$, and for except one value of $r \in (O, R)$ $\quad h^{'}(r) > 0$.

Potentials in the first class are repelling ones (see Figure 10). The additional condition besides (1) — (4) is now

Figure 10

(5) $V(R-) = 0$ and $V^{'}(R-) < 0$.

Then, under some more conditions, the system is K. As it is evident from the conditions, V is, though continuous, not C^1 at $r = R$ (see Figure 11). Indeed, the jump of $V^{'}$ in R as required by (5) plays the same role as the effect of a reflection in a dispersing billiard. This phenomenon was first observed by Kubo in 1976 (K(1976)), and he, and later he and Murata, K-M(1981) could already establish the K- and the B-property of such systems under more restrictive conditions than those of Donnay and Liverani. It is a natural question whether the Kubo-type singularity can also lead to ergodicity in the case of several particles. In fact, we recall the following

Figure 11

Problem (Liverani-Szász, 1990). *Let $N = 3, \nu = 2$. Is it possible to find a Kubo-type interaction (i.e. one satisfying the conditions (1) — (5) formulated before) such that the system is ergodic?*

A simpler problem could be the generalization of the Kubo-Donnay-Liverani result for the case $N = 2, \nu \geq 3$ though, as observed by Wojtkowski, W(1990-C), in the multidimensional case new, unpleasant phenomena may arise.

The second class investigated by Donnay and Liverani contains attracting potentials. In 1987, Knauf, K(1987) showed that for attracting interactions with singularities at $r = 0$ of the type $-\frac{1}{r^{2(1-\frac{1}{n})}}$, $n = 2, 3, 4, \ldots$, the system was ergodic. Donnay and Liverani's main achievement was that they could get rid of the assumption that n was an integer (see Figure 10). Their main condition besides (1) — (4) is

(6) $V'(r) \geq 0$ if $\nu \in (0, R)$ and $V(R) = V'(R) = 0$.

¿From the conceptual point of view the most remarkable is their third class since here the potential is everywhere smooth. The basic feature of interactions in the third class is that, for some $r_c < R$, the circle of radius r_c is a closed orbit (see Figure 12). Interestingly enough this orbit plays the role of a singularity.

In all cases, the existence of potentials satisfying the aforelisted conditions is proved. For a given potential satisfying the appropriate requirements then ergodicity is fulfilled at sufficiently high energy. It is worth noting that having proved first that the Lyapunov exponent is non-zero, the proof of ergodicity can be obtained by a suitable adaptation of the fundamental theorem for semi-dispersing billiards (cf. section 6).

Figure 12

An interesting class of models was introduced and studied by Wojtkowski, W(1990-A) and W(1990-B). Here a one-dimensional system of N particles of different masses moves in an external field, and the interaction is elastic collision. The non-vanishing of Lyapunov exponents has been proved in several cases, but establishing global ergodicity still seems to be difficult.

10. Ergodicity of systems with an increasing number of degrees of freedom.

The situation when the number of particles increases exactly corresponds to Boltzmann's original question which — in modern terminology — could sound as follows: find, for a generic Hamiltonian, the asymptotic behaviour in the thermodynamic limit. This question is still not formulated precisely. From the various possible ways, the right one should, of course, be selected as dictated by the main applications. At present, as it seems to me, a very important application should be in the field of the derivation of hydrodynamic equation from microscopic, Hamiltonian principles. It is clear that, the so far strongest method worked out in the last decade by Varadhan and his coworkers, O-V-Y(1993) would require a form related to Boltzmann's hypothesis but we can still not select the right form (we note that the results obtained until now are valid for stochastic systems and not for purely Hamiltonian ones).

The conceptually simplest and widemost known form of a hypothesis is the following: denote as before the number of particles by N, and by $p(N)$ the relative measure of the phase space occupied by invariant tori. For simplicity, the interaction is fixed and $\frac{V}{N} = const$ (for definiteness, we assume that the system lives on the torus $V^{\frac{1}{\nu}}\mathbb{T}^\nu$). Then the first conjecture is that $p(N) \to 0$ as $N \to \infty$. A stronger conjecture would then require that the complement to the set of invariant tori contains a large ergodic component whose measure gets close to one.

Hénon (1983) and Galgani (1985) discussed in detail the situation and the connection of these conjectures to the one on the limiting equipartiton of energy between the modes of the system . The conclusion is that the picture is not clear at all. There are, on one hand, interactions when numerical work of Froeschlé and Scheidecker (1975) indicates that $p(N) \to 0$ as expected. They investigated a one-dimensional model with the Hamiltonian

$$H = \frac{1}{2\sigma}\sum_1^N p_i^2 + 2\pi G\sigma^2 \sum_{i<j} \mid q_i - q_j \mid$$

On the other hand, the famous Fermi — Pasta — Ulam (1955) experiment supported doubts about the conjectures by detecting the failure of the limiting equipartition of energy. This was also a one-dimensional model with the Hamiltonian

$$H = \frac{1}{2}\sum_1^N p_i^2 + \sum_1^{N-1} V(q_{i+1} - q_i)$$

where $V(q) = \frac{1}{2}q^2 + \alpha q^3$.

These works generated a vivid interest in the problem. For the contradictory views about it, the reader is suggested to consult the aforementioned papers of Hénon and Galgani and for a more recent review that of Galgani- Giorgili-Martinolli-Vanzini (1993).

As I learnt from Gregory Eyink, for establishing hydrodynamic equations in the sense of the approach of Varadhan's method, a weaker form of the conjecture would also be sufficient. It is not necessary to have one large ergodic component. It seems that a weakly increasing upper bound for the number of ergodic components and, of course, a good upper bound on $p(N)$ could be sufficient. The picture here, however, needs more elaboration and the problems seem very difficult.

11. Ergodicity of systems with an infinite number of degrees of freedom.

Since the situation with large but finite systems is so complicated, I expect that the solution of equilibrium statistical physics should be borrowed. Whereas even a rigorous definition of a phase transition in a finite system — not speaking about its demonstration — is not an easy task, the question gets much simpler for infinite systems. In my view, first the ergodicity of infinite systems should be understood.

The very first result for an infinite system was obtained in 1971 by Sinai and Volkovysky for the ideal gas: it was shown to be a K-system (V-S(1971)). (A weaker result was obtained by Dobrushin already in 1956, see D(1956).) For the first glance this sounds as a surprise since in the ideal gas there is no velocity mixing at all. Indeed, in the formulation of ergodicity one should be a bit cautious. By denoting the phase space by $M = \{\{(q_i, v_i) : i \in \mathbb{Z}\} : \{q_i\} \text{ is locally finite } \}$, the equilibrium measure is $P_\lambda(\{q_i : i \in \mathbb{Z}\}) \otimes \prod F(dv_i)$ where P_λ is a Poisson measure with density λ and $F(dv)$ is an arbitrary non-degenerate probability distribution in $\mathbb{R}^\nu$, and ergodicity holds with respect to this invariant measure. The proof reveals an apparently new mechanism of ergodicity : mixing — understood, of course, in time — is the result of the initial spatial mixing. In other words: the equilibrium measure is Poisson, i.e. a measure with independent increments. Now as time proceeds, in a fixed box of our observation, particles starting from more and more distant intervals appear and their numbers are, roughly speaking, independent. This phenomenon can be proved to provide mixing in time.

The same observation was used, in more delicate arguments, for showing the K-property for different variants of the Rayleigh-gas, among others, by Goldstein-Lebowitz-Ravishankar (1982), Boldrighini-Pellegrinotti-Presutti-Sinai- Soloveychik (1984) and L. Erdős-Tuyen (1991). In these models only one particle interacts with all the other ones and the equilibrium measure is still Poisson. A related model is the Lorentz gas where — similarly to the ideal one — there is no interaction between the particles, but the dynamics of each particle obeys a strong mixing in space. Based upon this mixing, Sinai demonstrated the K-property in S(1979).

Now a problem which I find very interesting and quite realistic is the following one:

Problem (Szász, 1990). *Consider an infinite pencase obtained as $N \to \infty$ of the finite ones was introduced in section 8. Prove that the natural Gibbs measure is ergodic. (Here, of course, the possible values of the dimension are $\nu = 2, 3, 4$.)*

(Infinite models also raise the question of existence of the dynamics, but for this model it was answered affirmatively by Alexander, A(1976).) In the proof of ergodicity two mechanisms can be exploited: the hyperbolic behaviour of the interaction as done for the finite pencase or the spatial mixing of the equilibrium distribution as in cases of the ideal gas or the Rayleigh one. At present, however, I do not see an easy way for any of these possibilities, in particular, for the second one. For the first one it is a natural idea to start building up the hyperbolic theory of infinite-dimensional dynamical systems and trying to define, for instance, the stable and unstable invariant manifolds first, and then to prove their existence.

In section 10 we already mentioned the problem of the derivation of hydrodynamic type equations. A sufficient condition in some cases for the method of O-V-Y(1993) to work is the following ergodicity type condition: *every "regular" state, invariant with respect to both translations in the space and the dynamics, is a mixture of canonical Gibbs measures.* This property is apparently stronger than ergodicity, but, as remarked in F-F-L(1994), to prove such an ergodicity for deterministic Hamiltonian systems is still a formiddable unsolved problem. (In fact, Fritz, Funaki and Lebowitz verify this property for a random Hamiltonian system, and their paper is, moreover, also recommended for further reference.) We add that regularity above means that the state has finite relative entropy (per unit volume) with respect to the Gibbs measure. This assumption implies, in particular, that the conditional distribution in any finite volume Λ, given the configuration outside Λ, is absolutely continuous with respect to the Lebesgue measure.

12. Concluding remarks

In this lecture I have been concentrating on the history of Boltzmann's ergodic hypothesis. I think that the second half of the title of the talk is already justified if we focus our interest to just the question of ergodicity. After more than one hundred years, ergodicity is still not established in the simplest mechanical model, in the system of elastic hard balls though I expect we are not far from a solution. But as to generic interactions, even the questions are not clearly posed and it might well be that there will not be a final understanding after the next hundred years either. And we have not touched upon more delicate, physically fundamental properties for whose proofs one should refine the methods used in studying ergodicity of the system involved. Without aiming at completeness we just mention the problems

(1) of the decay of correlations (cf. Ch(1994); here and in the forthcoming cases only the last reference, I am aware of, will be provided, where further ones can also be found),
(2) of the convergence to equilibrium, K-Sz(1983),
(3) of the calculation of and bounds on the entropy of mechanical systems, Ch(1991),
(4) and, finally, of the recurrence properties of such systems, K-Sz(1985).

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