Abridged notation

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> They that are ignorant of Algebra cannot imagine the wonders in this kind are to be done by it: and what further improvements and helps advantageous to other parts of knowledge the sagacious mind of man may yet find out, it is not easy to determine. This at least I believe, that the ideas of quantity are not those alone that are capable of demonstration and knowledge; and that other, and perhaps more useful, parts of contemplation, would afford us certainty, if vices, passions, and domineering interest did not oppose and menace such endeavors.
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> *J. Locke, An Essay Concerning Human Understanding, B 4, Ch. 3, sec. 18*

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1 Introduction

Abridged notation is the use of the symbol of an equation in a certain coordinate system and its manipulation respecting the rules of the geometric object it represents. For example, a quadratic equation, representing a conic, written in a cartesian coordinate system

$$f(x,y) \;=\; Ax^2+2Bxy+Cy^2+2Dx+2Ey+F \;=\; 0\,, \tag{1}$$

is represented by $f=0$ and we speak of the conic f. A linear equation representing a line and written in the same coordinate system

$$\alpha(x,y) \;=\; Ax+By+C \;=\; 0\,, \tag{2}$$

is represented by $\alpha=0$ and we speak of the line α. For a second line represented by β, we can form the expression

$$g_{\mu,\nu} \;=\; \mu\cdot f+\nu\cdot\alpha\cdot\beta \quad \text{with} \quad \mu,\nu\in\mathbb{R}\,, \tag{3}$$

representing, through $g_{\mu,\nu}=0$ also a conic depending on $\{f,\alpha,\beta\}$ and the constants $\{\mu,\nu\}$, the conic being the same if we divide by one of the constants and consider instead the conic $f+\lambda\cdot\alpha\cdot\beta$ with $\lambda=\nu/\mu$ or the conic $\lambda\cdot f+\alpha\cdot\beta$ with $\lambda=\mu/\nu$. In the sequel we'll prefer the notation $f+\lambda\cdot\alpha\cdot\beta$ having in mind that λ can be written as $\lambda=\nu/\mu$ which leads to an equivalent representation of the corresponding conic in the form $f_{\mu,\nu}$ of equation (3).

The aim is to deduce properties of $g_\lambda=f+\lambda\cdot\alpha\cdot\beta$ from those of $\{f,\alpha,\beta\}$ possibly without entering into calculations with coordinates. A trivial example is the fact, that if the line α intersects the conic f at points $\{A,B\}$ and line β intersects f at $\{C,D\}$, then all conics $\{g_\lambda,\lambda\in\mathbb{R}\}$, forming a so-called *"pencil of conics"*, pass through these four points (see figure 1). A consequence of this is, that if D tends to coincide with A along a fixed

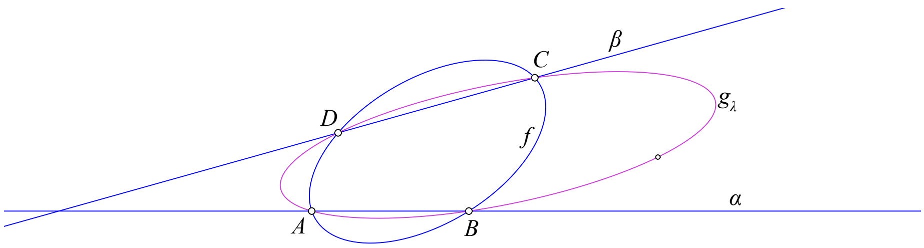

Figure 1: Conics $g_\lambda=f+\lambda\cdot\alpha\cdot\beta$

line γ through A, then the common chord AD of all these conics, which is part of the line γ, tends to the tangent of each one, hence all these conics have the same tangent equal to γ at the point of coincidence $A=D$ (see figure 2).

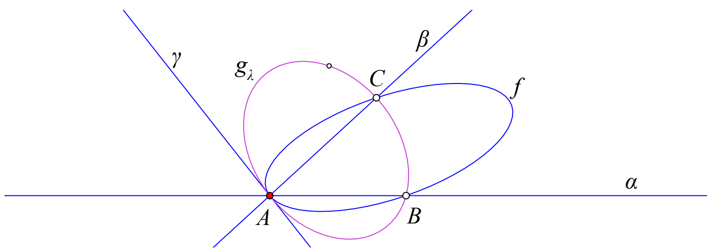

Figure 2: Conics passing through $\{A,B,C\}$ with common tangent at A

Applying the same reasoning for points $\{A, B\}$ of figure 1 and letting them tend towards $\{A \to B\ ,\ D \to C\}$ we see that for lines α tangent to f at B and β tangent to f at C the conics of the pencil $\{g_\lambda = f + \lambda \cdot \alpha \cdot \beta\}$ are all tangent to lines $\{\alpha, \beta\}$ respectively at points $\{B, C\}$ (see figure 3).

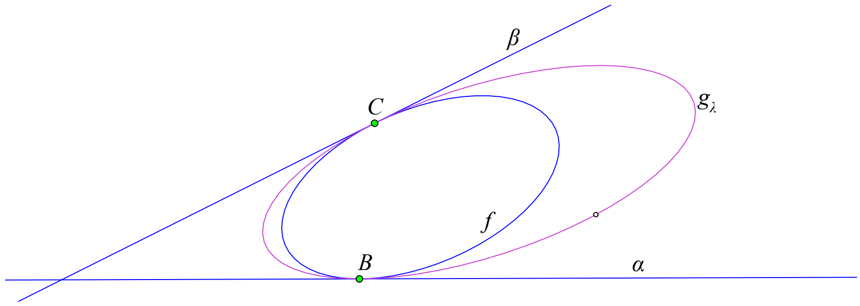

Figure 3: Conics tangent to $\{\alpha, \beta\}$ respectively at $\{B, C\}$

The preceding examples are typical for the use of *abridged notation*, which often involves relations between several conics. Below we'll see more such examples.

2 Abridged notation, cartesian coordinates, homogenization

As we said, to symbols representing lines and conics underlie coordinates and some times, in order to draw certain conclusions, we pass to the analytic expression of the symbol by the corresponding coordinates. An example offer the existence proofs of lines passing through two given points and conics passing through five points. In fact, the coefficients of a line α satisfying equation (2) are solutions of an homogeneous system with a 3×3 matrix M determined by the given points $\{(x_i, y_i), i = 1, 2\}$. From elementary linear algebra we know that in order to have solutions of the homogeneous equation, the determinant of the matrix must vanish:

$$M = \begin{pmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{pmatrix} \quad \text{and} \quad \alpha(x, y) = \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0\,. \tag{4}$$

Analogous is also the case of the conics. The coefficients of a conic f given by equation (1) and passing through five given points $\{(x_i, y_i), i = 1, ..., 5\}$ must satisfy an homogeneous system with coefficients determined by these points, leading again to its equation in terms of a vanishing determinant of a 6×6 matrix M.

$$M = \begin{pmatrix} x^2 & 2xy & y^2 & 2x & 2y & 1 \\ x_1^2 & 2x_1y_1 & y_1^2 & 2x_1 & 2y_1 & 1 \\ x_2^2 & 2x_2y_2 & y_2^2 & 2x_2 & 2y_2 & 1 \\ x_3^2 & 2x_3y_3 & y_3^2 & 2x_3 & 2y_3 & 1 \\ x_4^2 & 2x_4y_4 & y_4^2 & 2x_4 & 2y_4 & 1 \\ x_5^2 & 2x_5y_5 & y_5^2 & 2x_5 & 2y_5 & 1 \end{pmatrix}, f(x, y) = \begin{vmatrix} x^2 & 2xy & y^2 & 2x & 2y & 1 \\ x_1^2 & 2x_1y_1 & y_1^2 & 2x_1 & 2y_1 & 1 \\ x_2^2 & 2x_2y_2 & y_2^2 & 2x_2 & 2y_2 & 1 \\ x_3^2 & 2x_3y_3 & y_3^2 & 2x_3 & 2y_3 & 1 \\ x_4^2 & 2x_4y_4 & y_4^2 & 2x_4 & 2y_4 & 1 \\ x_5^2 & 2x_5y_5 & y_5^2 & 2x_5 & 2y_5 & 1 \end{vmatrix} = 0. \tag{5}$$

The proof of uniqueness of the conic through five given points is more involved. It can be done using the Chasles-Steiner definition of conics ([Eve63, p.303], [Bak71, p.73]) or using *Bezout's theorem* ([Ber87, II, p.182], [Bix06, p.195]), both guaranteeing, that two conics having in common more that four real intersection points are identical.

In our discussion so far we used cartesian coordinate systems w.r.t. orthogonal axes with units of the same length. We'll consider also their extension to the corresponding *"homogeneous coordinates"*, by which the point

$$(x, y) \quad \text{corresponds to} \quad (x', y', z') \quad \text{with} \quad x = x'/z' \;,\; y = y'/z' \;.$$

Various triples $\{(x', y', z'), (x'', y'', z''), ...\}$ such that $x'/x'' = y'/y'' = z'/z'' = k \neq 0$ define the same point $(x = x'/z', y = y'/z')$ and *"homogenize"* the equations of lines and conics:

$$Ax + By + C = 0 \quad \Leftrightarrow \quad Ax' + By' + Cz' = 0 \tag{6}$$

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0 \quad \Leftrightarrow$$
$$Ax'^2 + 2Bx'y' + Cy'^2 + 2Dx'z' + 2Ey'z' + Fz'^2 = 0 \tag{7}$$

The points $\{(x, y, 0)\}$ (we drop now the primes) belong to the *"line at infinity"*, whose equation is $z = 0$. With this convention, two non-identical lines have always an intersection point, which is an ordinary point, if the lines are not parallel, otherwise it is their common point at infinity. In fact, two parallel lines whose equations are respectively $Ax + By + Cz = 0$ and $Ax + By + C'z = 0$ with $C' \neq C$, are satisfied by the point at infinity $(-B, A, 0)$ representing their common *"direction"*. The interpretation of a point at infinity $(U, V, 0)$ as a *direction* of parallel lines $\{Vx - Uy + C = 0, C \in \mathbb{R}\}$ is used throughout this study. This is compatible with the interpretation of $(U, V, 0)$ as common point of the line $Vx - Uy = 0$ and the line at infinity $z = 0$. Two parallel lines

$$\alpha \;:\; Ax + By + Cz = 0 \quad \text{and} \quad \alpha' \;:\; Ax + By + C'z = 0$$

result each one from the other by adding a multiple of the line at infinity:

$$\alpha' = \alpha + (C - C')z \;.$$

3 Pencils of conics

We encountered them in section 1. A pencil of conics is a family of conics $\{f_{\lambda,\mu}\}$ depending on two distinguished conics $\{f_1, f_2\}$ and two arbitrary real numbers $\{\lambda, \mu\}$. We say *"the pencil is generated by $\{f_1, f_2\}$"*:

$$f_{\lambda,\mu} \;=\; \lambda \cdot f_1 \;+\; \mu \cdot f_2 \quad \text{with} \quad \lambda, \mu \in \mathbb{R} \;. \tag{8}$$

Figure 4 shows two conics $\{f_1, f_2\}$ and also some conics of the pencil $\{f_{\lambda,\mu}\}$ they generate. By the aforementioned *theorem of Bezout*, two conics intersect, counting the multiplicities at their intersections, at four points. In the example we have four distinguished intersection points $\{A, B, C, D\}$. All conics of the pencil pass through these four points and their points cover the entire plane. In fact, a point P of the plane, different from the points $\{A, B, C, D\}$, defines a unique conic of the pencil $\{f_{\lambda,\mu}\}$ passing through it. This follows from equation

$$f_{\lambda,\mu}(P) = 0 \quad \Leftrightarrow \quad \frac{f_1(P)}{f_2(P)} = -\frac{\mu}{\lambda},$$

determining the necessary parameters $\{\lambda, \mu\}$ up to a non-zero multiplicative factor k, since (λ, μ) and $(k\lambda, k\mu)$ define the same conic, since the same conic is represented by an equation $f = 0$ and a non-zero multiple $kf = 0$. This implies, that we can express almost all the members of the pencil, setting $k = \mu/\lambda$ or $k = \lambda/\mu$, by combinations of the form

$$f_1 + k \cdot f_2 \quad \text{and also as combinations} \quad kf_1 + f_2 \quad \text{with} \quad k \in \mathbb{R} \;.$$

An obvious property of the pencil is, that every pair of its members generates by linear combinations the whole pencil. Also, as is noticed in figure 4, every pencil contains as members three pairs of lines passing through the intersection points $\{A, B, C, D\}$ called *"singular members"* of the pencil, since they represent degenerate conics (products of lines). Since the pencil is generated by any two of its members we can select this *"singular"* members, for example the lines $\{\alpha = AB, \beta = CD, \gamma = AD, \delta = BC\}$ and write all the members as combinations

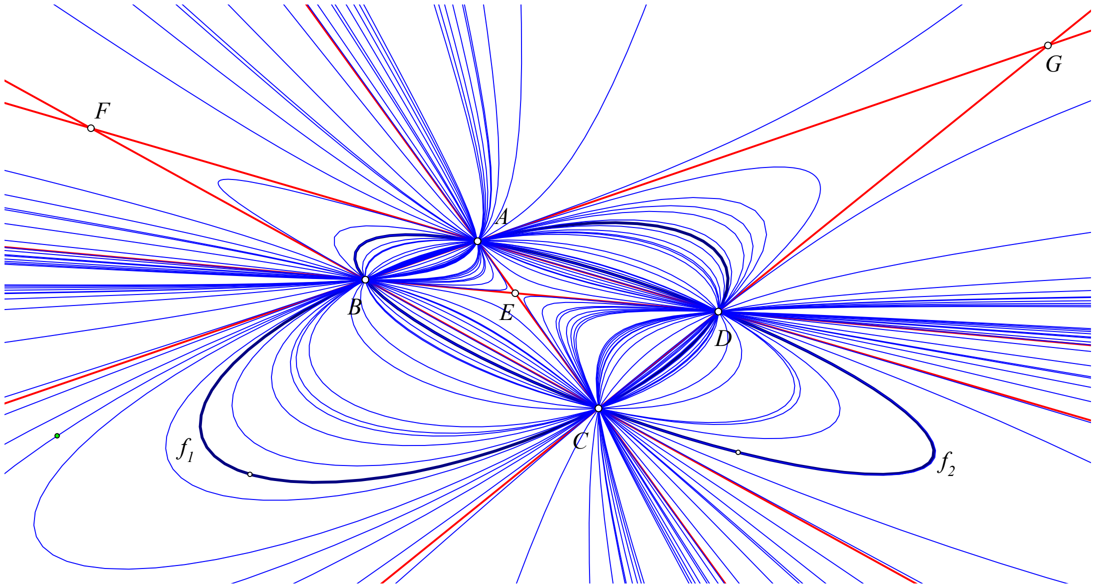

Figure 4: A pencil of conics generated by the two conics $\{f_1, f_2\}$

$$f \;=\; \lambda \cdot \alpha \cdot \beta \;+\; \mu \cdot \gamma \cdot \delta\,. \tag{9}$$

Taking into account these facts, we see that the pencil of conics $\{g_\lambda = f + \lambda \cdot \alpha\beta\}$ of figure 1 comprises *all* conics passing through the four points $\{A, B, C, D\}$. Analogously the pencil of conics $\{g_\lambda\}$ of figure 2 comprises *all* the conics passing through $\{B, C\}$ and tangent to γ at A. Analogously also the pencil $\{g_\lambda\}$ of figure 3 comprises *all* conics tangent to lines $\{\alpha, \beta\}$ respectively at their points $\{B, C\}$.

By the way, we can ask for each one of these pencils, whether they contain a circle for a particular value $\lambda = \lambda_0$. For the pencil of figure 1 the answer is obviously: *when the four points* $\{A, B, C, D\}$ *are on a circle*. For the pencil of figure 2 the answer is: *when the angle* $\widehat{ACB}$ *equals the angle between line* γ *and* α, and in the case of figure 3 the answer is: *when points* $\{B, C\}$ *are at the same distance from the intersection of lines* $\{\alpha, \beta\}$.

In general, two arbitrary conics corresponding to the equations $\{f_1 = 0, f_2 = 0\}$ have four intersection points in common real or imaginary. Consequently, they have at most six chords in common, real or imaginary, forming by pairs degenerate conics belonging to the pencil. Since the condition of degeneration for a conic expressed through equation (1) is the vanishing of the determinant

$$\begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix} = 0\,,$$

if the coefficients of the conics $\{f_1, f_2\}$ are real, then the vanishing of the corresponding determinant for the member $f = f_1 + \lambda f_2$ of the pencil will create a polynomial of third degree in λ. Consequently, there will be one or three real roots. In any case, for any pencil we'll have at least one real λ and at least one corresponding degenerate conic. Figure 5 shows some members and the unique real degenerate conic, consisting of two intersecting lines, of a pencil generated by two ellipses without a common point. The two points

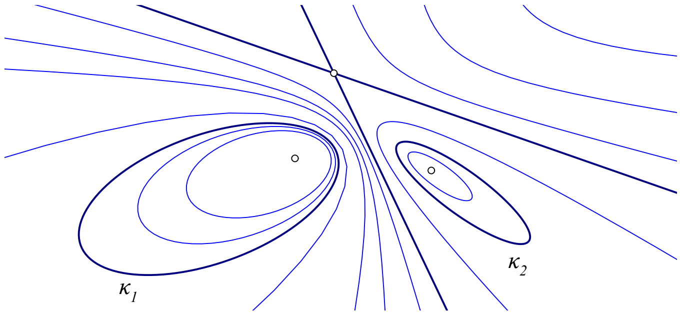

Figure 5: A pencil containing one only real degenerate conic

around which concentrate the ellipses contained in the pencil are the real intersections of two pairs of complex conjugate lines representing the imaginary degenerate conics of the pencil.

We should notice that the abridged notation and the relations between the symbols are valid in both cases of underlying coordinate systems: the cartesian as well as its homogenization. Depending on the particular configuration considered, the interpretation of the relations with one of these systems may be clearer than that with the other. By times the problematic ingredient is the *"line at infinity"*, which in homogeneous coordinates has the form $Cz = 0$ with a constant $C \neq 0$. This line is represented in abridged notation with the constant C. In a cartesian system though, which describes only ordinary points of the plane, we do not have a representation of the line at infinity. In this case we admit that the line at infinity is represented by the apparently absurd (because of $C \neq 0$) equality

$$C = 0 : \quad \text{line at infinity represented in a cartesian system,}$$

having in mind that the symbol represents the line $0x + 0y + Cz = 0$. The role of the constant, meant to represent the line at infinity, can be observed also in the standard representation of a line and a conic. The line, $\alpha = Ax + By + C$ is the sum of two lines, better understood when we use the homogenization: $\alpha = (Ax + By) + (Cz)$: a sum of an ordinary line, and the line at infinity. Thus, any line can be viewed as a particular member of a pencil: the pencil of its parallel lines $\{Ax + By + t = 0\,,\ t \in \mathbb{R}\}$.

The same observation can be made for the general conic

$$f = Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F\ .$$

It is a member of a pencil of the kind described by equation (9): $f = \lambda \cdot \alpha \cdot \beta \ + \ \mu \cdot \gamma \cdot \delta$. This is again better understood by interpreting f with homogeneous coordinates:

$$f = (Ax^2 + 2Bxy + Cy^2) + z(2Dx + 2Ey + Fz) \ = \ \alpha \cdot \beta + \gamma \cdot \delta\,.$$

With $(Ax^2 + 2Bxy + Cy^2) = \alpha \cdot \beta$, since the equation $Ax^2 + 2Bxy + Cy^2 = 0$ represents a product of two real or complex lines $\{\alpha, \beta\}$ (see file **Quadratic equation**), $\gamma = 1 \cdot z$ represents the line at infinity and $\delta = 2Dx + 2Ey + Fz$.

4 Interpretation of the members of a pencil

The members of a general pencil of conics, which always admits the representation of equation (9), can be described geometrically in the *"generic"* case, in which the four points

$\{A, B, C, D\}$ are real and distinct, as in figure 4. The description results from the property of the line equation $\alpha = Ax + By + C = 0$, when it is *"normalized"*, i.e. when its normal $\overline{n} = (A, B)$ is a unit vector. Then, for a point P not lying on the line, the quantity $\alpha(P)$ is the signed distance of P from the line, the sign being positive for P lying on the half plane pointed by the normal $\overline{n}$ and negative for P lying on the other half plane defined by the line α. In case $\overline{n}$ is not a unit vector $\alpha(P)$ is a constant multiple of this distance, since setting $k = \sqrt{A^2 + B^2}$ we can write

$$a = Ax + By + C = k\left(\frac{A}{k}x + \frac{B}{k}y + \frac{C}{k}\right) = k(A'x + B'y + C'),$$

and $\overline{n'} = (A', B')$ is a unit vector.

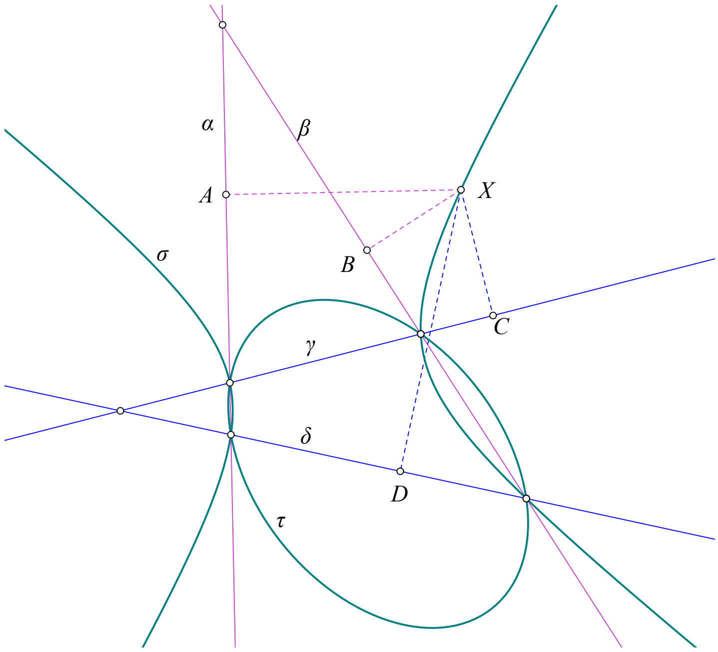

Figure 6: Conics satisfying $\frac{\alpha \cdot \beta}{\gamma \cdot \delta} = \pm k$

Theorem 1. *Given four lines $\{\alpha, \beta, \gamma, \delta\}$ in general position, the conic of the pencil*

$$f = \lambda \cdot \alpha \cdot \beta + \mu \cdot \gamma \cdot \delta$$

is the locus of points X whose signed distances from these lines, denoted by the same letters, satisfy the condition

$$\frac{\alpha \cdot \beta}{\gamma \cdot \delta} = k \text{ (constant)}. \tag{10}$$

Proof. This is trivial, since a point X on the conic f satisfies

$$0 = f(X) \quad \Leftrightarrow \quad \frac{\alpha \cdot \beta}{\gamma \cdot \delta} = -\frac{\mu}{\lambda},$$

and vice versa, thereby proving the statement. Figure 6 shows an example of two conics defined by this method. The conic σ corresponds to a constant $k = -(\mu/\lambda) > 0$ and the conic τ is obtained from the same formula by changing the constant to $-k$. $\square$

Corollary 1. *For any quadrangle inscribed in a conic and having the pairs of opposite side-lines* $\{(\alpha,\beta),(\gamma,\delta)\}$ *the corresponding distances* $\{\alpha',\beta',\gamma',\delta'\}$ *of a point* P *of the conic from these lines define a constant ratio independent of the position of* P *(see figure 7):*

$$\frac{\alpha'\cdot\beta'}{\gamma'\cdot\delta'}=k \quad \text{(constant).}$$

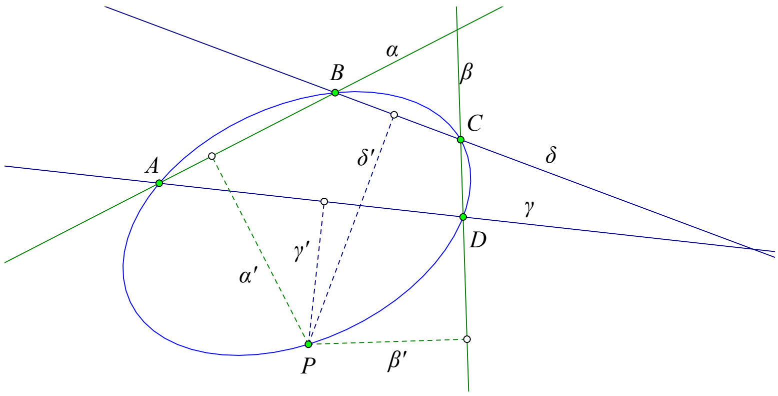

Figure 7: Property of quadrilateral inscribed: $\frac{\alpha'\cdot\beta'}{\gamma'\cdot\delta'}=k$

Corollary 2. *Given four points* $\{A,B,C,D\}$ *on the conic, the cross ratio* $k=P(AB;CD)$ *of four lines* $\{PA,PB,PC,PD\}$ *is constant for any point* P *of the conic.*

Proof. This is based on the formula for the altitude $h_a=\frac{bc\sin(A)}{a}$ of the triangle with sides $\{a,b,c\}$ and the formula for the cross ratio $P(AB;CD)$ of four lines through a point, expressed through the angles between them. Regarding the altitudes, we apply the formula and obtain (see figure 7):

$$\alpha'=\frac{|PA||PB|\sin(\widehat{APB})}{|AB|}, \quad \beta'=\frac{|PC||PD|\sin(\widehat{DPC})}{|DC|},$$
$$\gamma'=\frac{|PA||PD|\sin(\widehat{APD})}{|AD|}, \quad \delta'=\frac{|PB||PC|\sin(\widehat{BPC})}{|BC|}.$$

This, after the cancellation of $\{|PA|,|PB|,\ldots\}$ gives

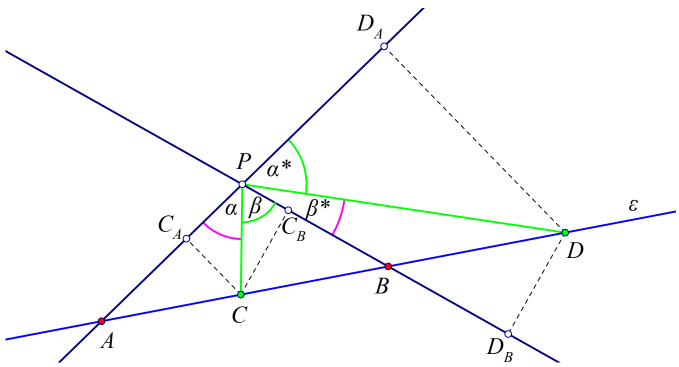

Figure 8: Cross ratio $(AB;CD)$ in terms of angles

$$\frac{\alpha'\cdot\beta'}{\gamma'\cdot\delta'}=\frac{\sin(\widehat{APB})}{\sin(\widehat{APD})}\cdot\frac{\sin(\widehat{DPC})}{\sin(\widehat{BPC})}\cdot\frac{|AD|\cdot|BC|}{|AB|\cdot|DC|}$$

It follows that the expression containing the four sines is constant, for points P lying on the conic, and this is exactly the cross ratio $P(AB;CD)$ of the four lines, which, per definition, is measured by the cross ratio $(AB;CD)$ defined on any line ε intersecting the four lines (see figure 8). Latter expressed through the angles between these lines is given by the following formula (see file **Cross ratio**) completing the proof:

$$(AB;CD) = \frac{CA}{CB} : \frac{DA}{DB} = \frac{\sin(\alpha)}{\sin(\alpha^*)} : \frac{\sin(\beta)}{\sin(\beta^*)}. \qquad \square$$

5 Bitangent conics and pencils

"Bitangent conics" are called two conics which have two common points and their tangents at these points coinciding. We encountered *"Bitangent pencils"* already in the introduction (see figure 3). They are families of conics which pass through the same two points $\{A, B\}$

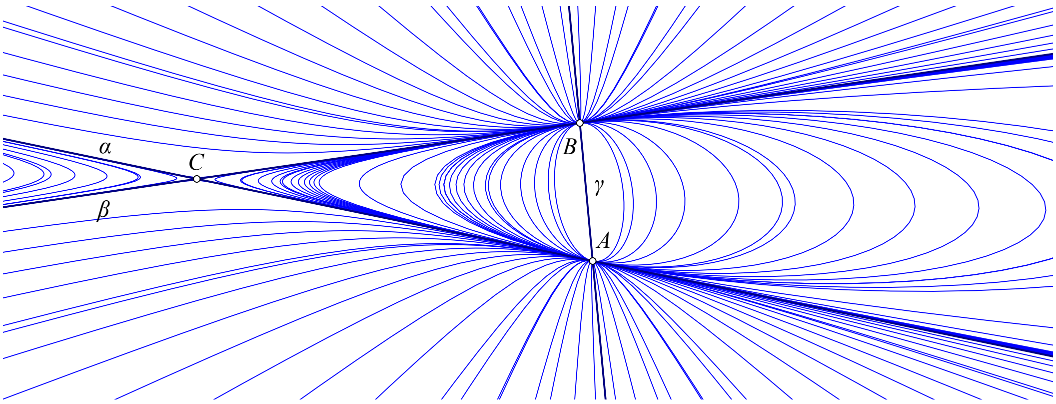

Figure 9: A bitangent pencil of conics $\{f_{\lambda,\mu} = \lambda \cdot \alpha \cdot \beta + \mu\gamma^2\}$

having there the same tangent (see figure 9). These are special cases of the general pencil $\lambda \cdot \alpha \cdot \beta + \mu \cdot \gamma \cdot \delta$ for which $\gamma = \delta$. Thus, they comprise all conics of the form

$$f_{\lambda,\mu} \ = \ \lambda \cdot \alpha \cdot \beta \ + \ \mu \cdot \gamma^2$$

where $\{\alpha, \beta, \gamma\}$ are given lines.

They contain two particular *"singular"* members, which are *"degenerate"* conics represented by (i) the pair of lines $f_{1,0} = \alpha \cdot \beta$ and (ii) the *"double"* line $f_{0,1} = \gamma^2$. Every conic tangent to the lines $\{\alpha, \beta\}$ correspondingly at $\{A, B\}$ is a member of this pencil. Also, every conic κ can be considered as a member of such a pencil. It suffices to take two points $\{A, B \in \kappa\}$ and consider their tangents $\{\alpha, \beta\}$ there and the line joining the contact points $\gamma = AB$. The conic can be written then as a combination

$$\kappa = \lambda \cdot \alpha \cdot \beta + \mu \cdot \gamma^2 \,.$$

From corollary 1 we draw also the following conclusion:

Corollary 3. *For every pair of tangents $\{\alpha, \beta\}$ of a conic and the line γ of their contact points, the ratio of the distances $\{\alpha', \beta', \gamma'\}$ of a point P of the conic from these lines (see figure 10):*

$$k \ = \ \frac{\alpha' \cdot \beta'}{\gamma'^2}$$

is constant. Conversely, if for a point P this expression of distances from three lines is constant, then P lies on a member-conic of the bitangent pencil $\{\lambda\alpha \cdot \beta + \mu\gamma^2\}$.

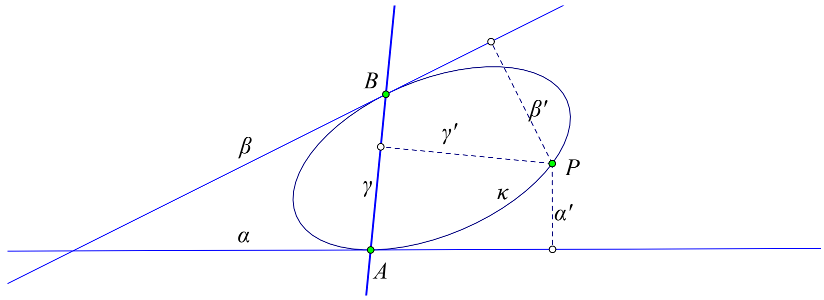

Figure 10: The constant ratio property $\frac{\alpha' \cdot \beta'}{\gamma'^2} = k$

From this corollary we can deduce a fundamental property of hyperbolas. First we should notice that every hyperbola can be represented in homogeneous coordinates as a member of a special bitangent pencil.

$$h = \lambda \alpha \cdot \beta + \mu z^2 , \tag{11}$$

with $\gamma = 1 \cdot z$ the line at infinity and $\{\alpha, \beta\}$ arbitrary lines. The intersection points $\{A, B\}$ of the lines $\{A = \alpha \cap \gamma, B = \beta \cap \gamma\}$, through which pass all the conics of the pencil, are at infinity. Since, e.g. the line $\beta = Dx + Ey + Fz$ intersects the line at infinity $z = 0$ at the point at infinity $B = (-E, D, 0)$. This is a point of the conic h and line β is the tangent of the conic there, since it does not contain a second point of the conic. This shows that β is an asymptote of the conic. Analogously α is also an asymptote and the conic is a hyperbola. Thus, dividing with λ and setting $k = \mu/\lambda$ we can say that

$$h = \alpha \cdot \beta + k \cdot z^2$$

describes the most general hyperbola with asymptotes $\{\alpha, \beta\}$. Disregarding the place of the hyperbola in the plane, we may assume that one of the asymptotes, α say, is given by the $y-$axis: $\alpha : x = 0$, so that the general form simplifies in this case, after dividing by k, to

$$h = x(Dx + Ey + Fz) + z^2 ,$$

or in non-homogeneous coordinates

$$h = x(Dx + Ey + F) + 1 .$$

Figure 11 shows an example and suggests a property resulting from corollary 3.

Corollary 4. *The product of the distances $|PP_\alpha||PP_\beta|$ of a point P of the hyperbola h from the asymptotes is constant and the area of the triangle $PP_\alpha P_\beta$ is also constant. Also, if $\{Q_\alpha, Q_\beta\}$ denote the parallel along an asymptote projections of P on the other asymptote and O is the center of the hyperbola, then the parallelogram $PQ_\alpha OQ_\beta$ has the same area for all positions of of P on h.*

Proof. This is obvious, since the constant ratio in this case is

$$\frac{\alpha \cdot \beta}{\gamma^2} = \frac{x}{z} \cdot \left(D\frac{x}{z} + E\frac{y}{z} + F \right) \quad \text{a multiple of the distances from} \quad \alpha, \beta.$$

The second claim is obviously an immediate consequence of the first one. The third results from the relations involving the angle ϕ of the asymptotes:

$$|PQ_\alpha| = \frac{|PP_\alpha|}{\sin(\phi)} \ , \ |PQ_\beta| = \frac{|PP_\beta|}{\sin(\phi)} . \qquad \square$$

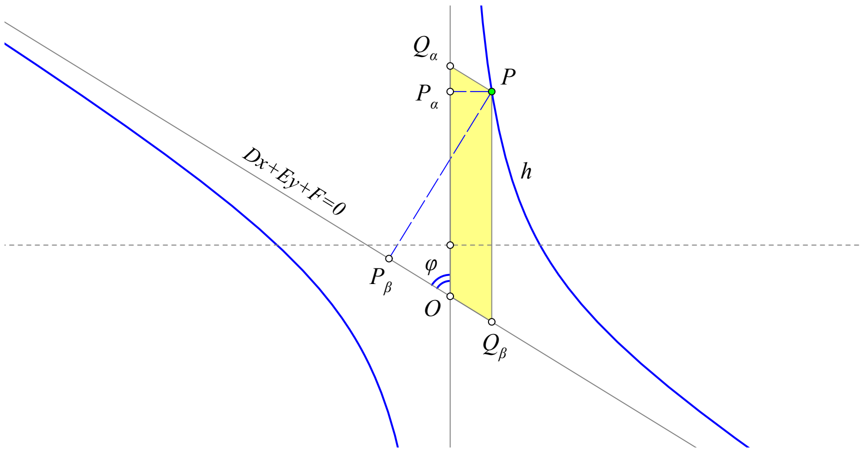

Figure 11: Hyperbola with asymptotes $\{x = 0\ ,\ Dx + Ey + F = 0\}$

We notice here, how easily and almost without any computation, we come to detailed quantitative relations by simply interpreting the general relations between the symbols established by the methods of the *abridged notation*. With some additional effort, we can obtain more properties similar to those of the last corollary. The following exercises are examples of a further elaboration of the obtained results, reflecting the spirit of the method: First we obtain general relations between the symbols, and then we interpret the obtained relations and work with geometric and analytic arguments.

Exercise 1. *Let h be a hyperbola with asymptotes $\{\alpha, \beta\}$ intersecting at its center O. Show, that the tangent ε at a point $P \in h$ intersects the asymptotes at the points $\{S_\alpha, S_\beta\}$ and the triangle $S_\alpha P S_\beta$ has twice the area of the parallelogram $Q_\alpha O Q_\beta P$, where $\{Q_\alpha, Q_\beta\}$ denote the parallel projections along an asymptote of P on the other asymptote (see figure 12), which has a constant area for $P \in h$. Also point P is the middle of the segment $S_\alpha S_\beta$.*

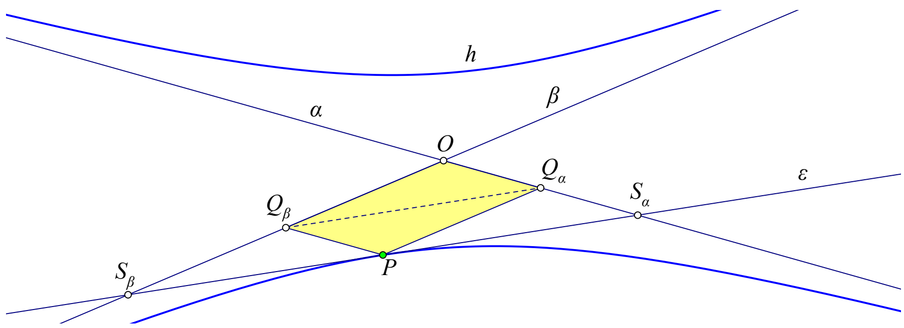

Figure 12: Hyperbola property $\triangle S_\alpha P S_\beta$ has constant area

Hint: Start by *defining* $\{S_\alpha, S_\beta\}$ to be the symmetrics respectively of O w.r.t. to $\{Q_\alpha, Q_\beta\}$ and their line ε containing P. To see that ε is the tangent at P it suffices to show that it has no other point in common with the hyperbola h, i.e. to show that for all other points $P \neq P' \in \varepsilon$ the corresponding parallelogram $Q_\alpha O Q_\beta P'$ has area different from that of $Q_\alpha O Q_\beta P$. But this is an elementary property of signed areas, formulated in the next exercise.

Exercise 2. *Let P' be a point on the base-line $\varepsilon = BC$ of the triangle ABC and $\{P, E, F\}$ be the middles of $\{BC, AC, AB\}$. Project point P' parallel to the sides $\{AB, AC\}$ to points on $\{I, J\}$ on these sides (see figure 13). Then, the parallelogram $P'IAJ$ has a signed area which is different from that of $PEAF$ for all $P' \in \varepsilon$.*

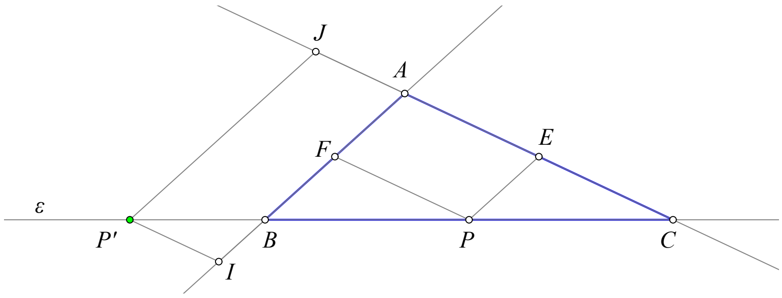

Figure 13: An area property

Hint: Considering the variable $x = AI$, the ratio of the signed areas is:

$$\frac{(IAJP')}{(FPEA)} = \frac{P'I \cdot IA}{PF \cdot FA} = \frac{IB}{FB} \cdot \frac{AI}{AF} = \frac{2d - x}{d} \cdot \frac{x}{d} \quad \text{with} \quad d = AF = \frac{AB}{2}.$$

This ratio is 1 if and only if $(x-d)^2 = 0 \Leftrightarrow x = d$.

Exercise 3. *Let $\gamma = AB$ be a diameter of a central conic κ and $\{\alpha, \beta\}$ the tangents at its endpoints. Project a point $P \in \kappa$ on the diameter and the tangents and consider the circle $\nu = (A'B'C')$ passing through the projections. Let C'' be the other than $C' \in \gamma$ point of intersection of the circle ν with line PC'. Then the ratio PC'/PC'' is constant.*

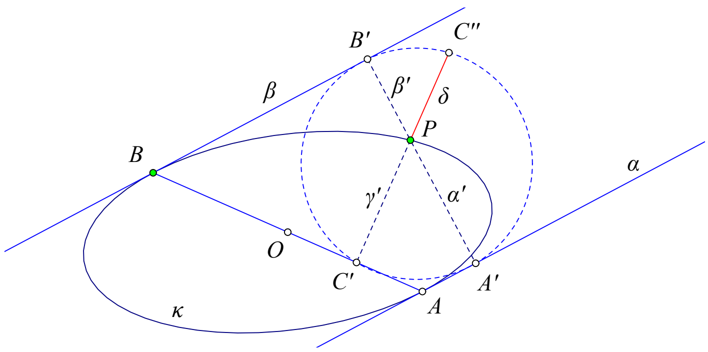

Figure 14: A ratio property for $P \in \kappa$: $\frac{\gamma'}{\delta} = k$ (constant)

Hint: Apply corollary 3 the lines $\{\alpha, \beta\}$ being the tangents from a point at infinity and implying $(\alpha' \cdot \beta')/\gamma'^2$ is a constant for $P \in \kappa$.

6 Abridged notation, line formularium

Every conic can be considered as a member of a bitangent pencil. It suffices to consider two tangents $\{\alpha, \beta\}$ and the line γ joining their contact points$\{A, B\}$. Besides the possibility to describe all conics κ passing through these two points and having there the tangents $\{\alpha, \beta\}$ through a simple expression involving the three lines

$$\kappa = \gamma^2 + \alpha \cdot \beta,$$

the system can be used to parameterize the conic κ. Before to discuss the details of this subject we need to make some remarks on a system of coordinates based on the three lines $\{\alpha, \beta, \gamma\}$. The intersection points $\{A = \beta \cap \gamma, B = \gamma \cap \alpha, C = \alpha \cap \beta\}$ together with an additional arbitrary point D define a system of *"homogeneous coordinates"* or *"projective coordinates"* (see file **Projective plane**), in which every point P is represented by a formal sum

$$P \;=\; u \cdot A + v \cdot B + w \cdot C, \quad \text{we write} \quad P(u,v,w)\;.$$

D is the *"unit"* point with coordinates $D(1,1,1)$. The quadruple $\{A,B,C,D\}$ is a *"projective base"* of the plane and the lines

$$\alpha = 0, \quad \beta = 0, \quad \gamma = 0$$

are the *"coordinate axes"* coinciding respectively with $\{u = 0, v = 0, w = 0\}$ (see figure 15).

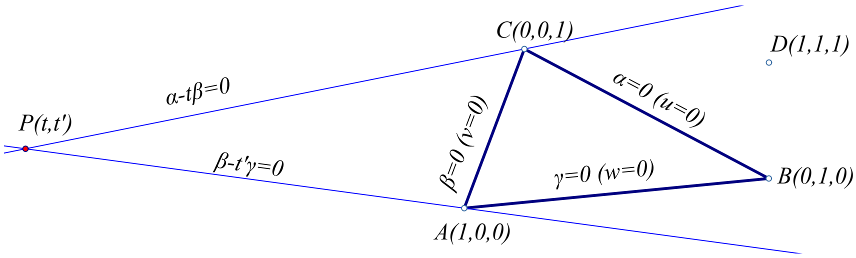

Figure 15: Projective base $\{A,B,C,D\}$

Just like in the homogenized cartesian coordinate system, which is a particular case of this general one, where every line is represented as a linear combination of the lines $\{x = 0, y = 0, z = 0\}$:

$$px + qy + rz = 0,$$

so in the above general homogeneous system we can represent every line of the plane by a linear combination of the three lines

$$p\alpha + q\beta + r\gamma = 0,$$

the coefficients $\{p,q,r\}$ defined up to a non zero multiplicative constant. Also every point P is defined through its homogeneous coordinates $\{(u,v,w)\}$ defined up to a non zero multiplicative constant.

In the context of the *abridged notation* another convenient way to determine the location of a point relative to the lines $\{\alpha,\beta,\gamma\}$ is the one defined in *nr-1* of the list below, the subsequent *nrs* giving corresponding consequences of this method.

1. A point P is preferably represented as intersection of two lines belonging respectively to the pencils of lines at A and C :

$$P(t,t') : \quad \text{intersection of lines:} \quad \alpha - t\beta = 0 \quad \text{and} \quad \beta - t'\gamma = 0. \tag{12}$$

   We speak of the point (t,t'). Equation $\alpha - t\beta = 0$ represents a line through C and equation $\beta - t'\gamma = 0$ represents a line through point A. Eliminating β from equations (12) we find $\alpha - tt'\gamma = 0$ representing the line passing through $B(\alpha = \gamma = 0)$ and the point $P(t,t')$.

2. The condition that this point belongs to line $a\alpha + b\beta + c\gamma = 0$ is

$$att' + bt' + c = 0. \tag{13}$$

3. The coefficients $\{a,b,c\}$ of the line $a\alpha + b\beta + c\gamma = 0$ passing through the two points $\{(t,t'),(s,s')\}$ satisfy the system of equations:

$$att' + bt' + c = 0, \qquad ass' + bs' + c = 0. \tag{14}$$

4. Thus, solving for $\{a, b, c\}$, we find, up to multiplicative constant:

$$a = t' - s', \qquad b = ss' - tt', \qquad c = t's'(t - s). \tag{15}$$

5. Replacing in *nr-2*, we get the nice condition of *"collinearity"* of three points (t, t'), (s', s'), (u, u') :

$$\begin{aligned} (t' - s')uu' + (ss' - tt')u' + t's'(t - s) &= 0 \quad \Leftrightarrow \\ uu'(t' - s') + ss'(u' - t') + tt'(s' - u') &= 0. \end{aligned} \tag{16}$$

6. For two lines represented by equations $\{a\alpha + b\beta + c\gamma = 0\ ,\ a'\alpha + b'\beta + c\gamma' = 0\}$ their intersection point $P(t, t')$ satisfies $att' + bt' + c = 0$ and $a'tt' + b't' + c' = 0 \quad \Rightarrow$

$$t = \frac{bc' - b'c}{a'c - ac'}, \qquad t' = \frac{a'c - ac'}{ab' - a'b}. \tag{17}$$

7. Thus, the condition of *"concurrency"* of three lines results by replacing this into *nr-2* for a third line, which after simplification leads to:

$$a''(bc' - b'c) + b''(a'c - ac') + c''(ab' - a'b) = 0. \tag{18}$$

   This is the determinant of the coefficients of the three lines.

8. The connection with the usual notation, which prefers to write a point in the coordinate base as $P = uA + vB + wC$, results by replacing this into the system of equations 12, with corresponding $\{\alpha = 0 \leftrightarrow u = 0,\ \ \beta = 0 \leftrightarrow v = 0,\ \ \gamma = 0 \leftrightarrow w = 0\}$ leading to the relations:

$$t = \frac{u}{v}, \quad t' = \frac{v}{w} \quad \text{and} \quad (u, v, w) = k(tt', t', 1) \quad \text{with arbitrary} \quad k \neq 0. \tag{19}$$

9. The *"cross ratio"* of four lines $\{\sigma_i = \alpha - t_i\beta = 0, i = 1, 2, 3, 4\}$ passing through the point C can be defined by the expression

$$(\sigma_1\sigma_2; \sigma_3\sigma_4) = \frac{t_1 - t_3}{t_2 - t_3} : \frac{t_1 - t_4}{t_2 - t_4}. \tag{20}$$

   This is compatible with the definition of the cross ratio of four points on a line. This means that on every line η intersecting these four lines, the cross ratio of the four intersection points on η coincides with this one.

10. The equation of two *"harmonic conjugate"* lines w.r. to the lines $\alpha = 0$ and $\beta = 0$ passing through C is

$$\alpha - t\beta = 0, \qquad \alpha + t\beta = 0. \tag{21}$$

11. Eliminating β from two line equations $\{a\alpha + b\beta + c\gamma = 0\ ,\ a'\alpha + b'\beta + c\gamma' = 0\}$ we obtain the equation

$$(ab' - a'b)\alpha - (bc' - cb')\gamma \ = \ 0\,, \tag{22}$$

    representing the line joining the base point $B(\alpha = \gamma = 0)$ of the coordinate system to the intersection point (see *nr-6*) of the two lines.

7 Abridged notation, conics formularium

In this section the main object is a conic κ. We investigate its properties considering it as a member of a bitangent pencil and refer it to a coordinate system consisting of its tangents $\{\alpha, \gamma\}$ at its points $\{C, A\}$ and the chord of contacts β ([Car76, p.336]) (see figure 16). Within this framework the following are valid properties:

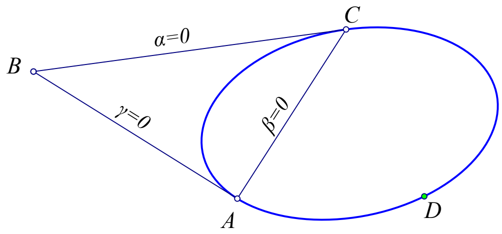

Figure 16: Bitangent conics with common chord $\beta = AC$

1. The equation of the conic obtains the *"bitangent conic"* form
$$\alpha \cdot \gamma - \beta^2 = 0. \tag{23}$$

2. More general
$$\alpha \cdot \gamma - k \cdot \beta^2 = 0, \quad \text{with variable k} \tag{24}$$
represents all conics passing through $\{A, C\}$ and being tangent there to lines $\{\gamma, \alpha\}$ correspondingly. The particular one of *nr-1* is obtained by requiring from the conic to pass through the unit point D, which since $\alpha(D) = \beta(D) = \gamma(D) = 1$, gives $k = 1$.

3. In this coordinate system we can describe the conic $\alpha \cdot \gamma - \beta^2 = 0$ by a very simple parametrization. In fact, point P on the conic can be determined as intersection of two lines passing through B and A (see figure 17).

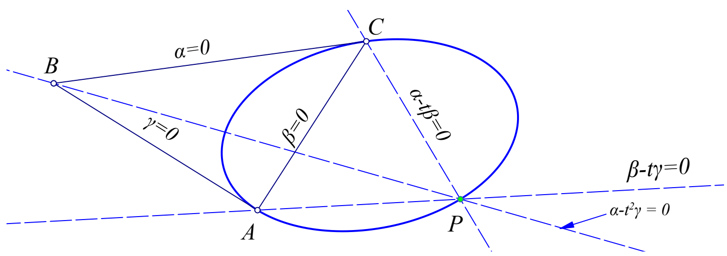

Figure 17: Determination of a point on the conic

$$\{\alpha - t \cdot \beta = 0 \ , \ \beta - t' \cdot \gamma = 0\} \overset{\alpha \cdot \gamma = \beta^2}{\Longrightarrow} \quad \beta = t \cdot \gamma \quad \Rightarrow \quad \alpha = t^2 \cdot \gamma\,.$$

Any pair out of the three lines
$$\alpha = t \cdot \beta \ , \ \beta = t \cdot \gamma \ , \ \alpha = t^2 \cdot \gamma \tag{25}$$
can be used to determine through their intersection the position of P in terms of the position of the lines. This defines a parametrization, identifying points P on the conic with the corresponding parameter t, and we can speak of *"the point t on the conic"*.

4. For an arbitrary line $p\alpha + q\beta + r\gamma = 0$, the intersection points with the conic are the *"points t"* found through the solution of the system:

$$p\alpha + q\beta + r\gamma = 0, \qquad \alpha - t^2\gamma = 0, \qquad \beta - t\gamma = 0,$$

which leads to the quadratic equation:

$$pt^2 + qt + r = 0. \tag{26}$$

5. The equation of the line (chord) through points $P(t)$ and $P'(t')$ is

$$(tt')\gamma - (t+t')\beta + \alpha = 0. \tag{27}$$

This, because it is satisfied by the points

$$P(t) : \{\alpha - t^2\gamma = 0,\ \beta - t\gamma = 0\} \quad \text{and} \quad P'(t') : \{\alpha - t'^2\gamma = 0,\ \beta - t'\gamma = 0\}.$$

6. If all these chords are to pass through a fixed point $Q(p,q,r)$, then, replacing into equation (27) the coordinates of the point, we see that an equation of the form

$$(tt')r - (t+t')q + p = 0 \tag{28}$$

must be satisfied with constants $\{p,q,r\}$. This represents the necessary and sufficient condition for *"chords tt' "* to pass through a fixed point.

7. Allowing complex numbers for $\{t,t'\}$, the preceding equation can represent any line of the plane, since every line in this case has two real or complex intersection points with the conic. For real conics $\{t,t'\}$ are complex conjugate, hence tt' and $t+t'$ are real.

8. The equation of the *"tangent at $P(t)$ "* is

$$t^2\gamma - 2t\beta + \alpha = 0, \tag{29}$$

since it represents the limit position of the chord-line of *nr-5*, when t' tends to coincide with t. The converse is also true, since one can reverse the arguments and show the following theorem.

Theorem 2. *Any one-parameter line-equation, which can be put into the form of equation (29), with parameter t, represents a tangent to the conic $\alpha \cdot \gamma - \beta^2 = 0$.*

The theorem has many applications in finding a conic enveloping a one parameter family of lines.

9. Requiring that the tangent of *nr-8* passes through a point

$$P(s,s') : \{\alpha = s\beta,\ \beta = s'\gamma\} \Rightarrow \alpha = ss'\gamma \quad \text{implies} \quad t^2 - 2ts' + ss' = 0.$$

Point t being on the conic implies $\{t = \beta/\gamma,\ t^2 = \alpha/\gamma\}$ hence the equation

$$\alpha - 2s'\beta + ss'\gamma = 0,$$

which represents the *"polar"* of $P(s,s')$.

10. A line through $B(\alpha = \gamma = 0)$ is of the form $\alpha - k\gamma = 0$ and intersects the conic at the points $\pm\sqrt{k}$. It is trivially seen that $\alpha - t^2\gamma = 0$ represents a line passing through the points $\{t, -t\}$ of the conic $\alpha\gamma - \beta^2 = 0$.

Theorem 3. *If the coefficients of a variable line $L_t : A_t x + B_t y + C_t z = 0$ are given by the quadratic equations w.r.t. to the parameter t :*

$$\begin{aligned} A_t &= a_{11}t^2 + a_{12}t + a_{13}, \\ B_t &= a_{21}t^2 + a_{22}t + a_{23}, \\ C_t &= a_{31}t^2 + a_{32}t + a_{33}, \end{aligned}$$

then the line L_t envelopes a conic with equation $\alpha \cdot \gamma - \beta^2 = 0$. In this the lines are given by coefficients corresponding to the columns of the preceding matrix:

$$\begin{aligned} &\alpha : a_{13}x + a_{23}y + a_{33}z = 0, \\ &\beta : -\frac{1}{2}(a_{12}x + a_{22}y + a_{32}z) = 0, \\ &\gamma : a_{11}x + a_{21}y + a_{31}z = 0. \end{aligned}$$

Proof. Apply theorem 2, writing the equation of the line L_t in the equivalent form

$$L_t \ : \ t^2\gamma - 2t\beta + \alpha \ = \ 0 \, . \qquad \square$$

Exercise 4. *With the notation of this section, show that for the conic $f_k = \alpha\gamma - k\beta^2 = 0$:*

1. *The point $P : \alpha - t\beta = \beta - t'\gamma = 0$ is on f_k if and only if $t = kt'$.*
2. *The chord of f_k through the points $\{(kt, t), (kt', t')\}$ is $\alpha - k(t + t')\beta + k(tt')\gamma = 0$.*
3. *The tangent of f_k through the point (kt, t) is $\alpha - 2kt\beta + kt^2\gamma = 0$.*

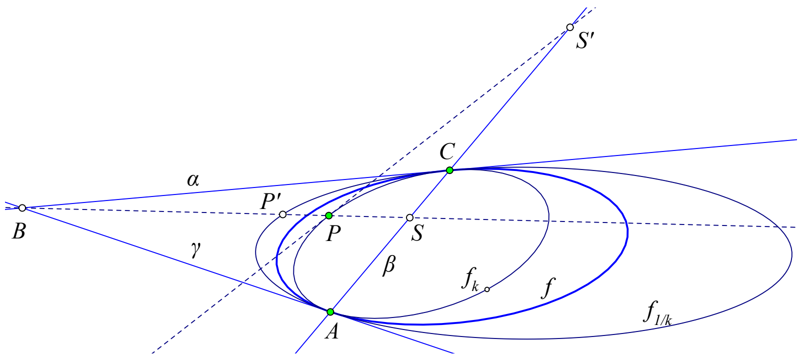

Figure 18: The bitangent pencil members $\{f_k, f_{1/k}\}$

Theorem 4. *With the notation of this section, consider a member-conic $f_k = \alpha\gamma - k\beta^2$ of the bitangent pencil. For each point $P \in f_k$ let P' be the pol of the tangent to f_k at P w.r.t. the member-conic $f = \alpha\gamma - \beta^2$. Then, the following are valid properties (see figure 18):*

1. *The point $P(kt, t)$ has corresponding $P'(t, kt)$. Thus, the correspondence $P \mapsto P'$ sends the member-conic f_k to the member-conic $f_{1/k}$.*
2. *The line through P, P' passes also through $B : (\alpha = \gamma = 0)$.*
3. *The pol S' of PP' w.r.t. to every member-conic of the pencil is the harmonic conjugate of the intersection $S = PP' \cap AC$ w.r.t. $\{A, C\}$.*

Proof. *nr-1*. We need exercise 4, which shows that the points of f_k are of the form $P(kt, t)$ and the tangent at such a point is $\alpha - 2kt\beta + kt^2\gamma = 0$. If this is the polar of a point $P(s, s')$ w.r.t. the conic f, then it should be equal to $\alpha - 2s'\beta + ss'\gamma = 0 \quad \Rightarrow \quad s' = tk$, from which follows that $(s, s') = (t, kt)$, which is a point of $f_{1/k}$.

nr-2. The line $a\alpha + b\beta + c\gamma = 0$ through $\{P(kt, t), P'(t, kt)\}$ has according to *nr-4* of section 6 $b = 0$ hence is of the form $a\alpha + c\gamma = 0$ which is a line through B.

nr-3 is a consequence of the general pol-polar reciprocity, according to which if : *the pol A of a line α is on a line β then the pol B of β is also on line α.* □

Remark 1. This is a particular case of the *"polarity transformation"* w.r.t. to a given conic f. This transformation corresponds to every point P of the plane the polar line α_P w.r.t. f, and to every line α of the plane its pol P_α w.r.t. f. If in a cartesian coordinate system the conic f is represented by a symmetric matrix $\{M : f(X) = X^t M X = 0, X \in \mathbb{R}^3\}$, then the polarity (points $\mapsto$ lines) is described by the linear transformation $Y^t = X^t M$, the line vector Y^t representing the coefficients of the polar line of the column vector X representing a point of the plane. The inverse transformation (lines $\mapsto$ points) is represented by the inverse matrix $X^t = Y^t M^{-1}$.

It can be proved in general that the polarity w.r.t. some conic f, maps in the sense used above, every conic g to another conic g'. As we used it above, g' consists of the pols of tangents of g w.r.t. f. In general applying such a polarity to the members of a pencil we do not obtain again members of the same pencil. In the particular case though of a bitangent pencil, as we saw, this happens indeed and the polarity w.r.t. a member f applied to another member f_t of the pencil gives again a member $f_{t'}$ of the same pencil.

Theorem 5. *Every genuine conic can be represented in a homogeneous coordinate system in parametric form with quadratic polynomials, whose coefficients build an invertible matrix A in the form:*

$$\left.\begin{aligned} u &= a_{11}t^2 + a_{12}t + a_{13}, \\ v &= a_{21}t^2 + a_{22}t + a_{23}, \\ w &= a_{31}t^2 + a_{32}t + a_{33}. \end{aligned}\right\} \quad \text{with determinant} \quad |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0\,. \tag{30}$$

Conversely, every such parametrization in an arbitrary coordinate system defines a genuine conic.

Proof. In fact, from equations (25) and (19) we know that every conic can be represented in a system of two tangents and the chord of contact of the conic by a parametrization of the form:

$$u = t^2\,,\ v = t\,,\ w = 1,$$

which corresponds to the simplest case of A coinciding with the unit matrix $A = I_3$. For the converse, given the above relations in the coordinate system (u, v, w), it suffices to change the coordinate system by the inverse matrix $B = A^{-1}$ and pass to the system

$$\begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} = B \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} t^2 \\ t \\ 1 \end{pmatrix} \quad \Rightarrow \quad u' \cdot w' - v'^2 = 0\,,$$

showing that the curve is a conic, having $\{u' = 0, w' = 0\}$ as tangents and $v' = 0$ is the line of contacts. □

Remark 2. Using vector notation equation (30) could be written $U = t^2 K + tL + M$, where $\{K, L, M\}$ denote the columns of the matrix A. Referring to barycentric coordinates, the vectors $\{K, L, M\}$ could represent *"absolute barycentrics"*, i.e. the sum of the coordinates

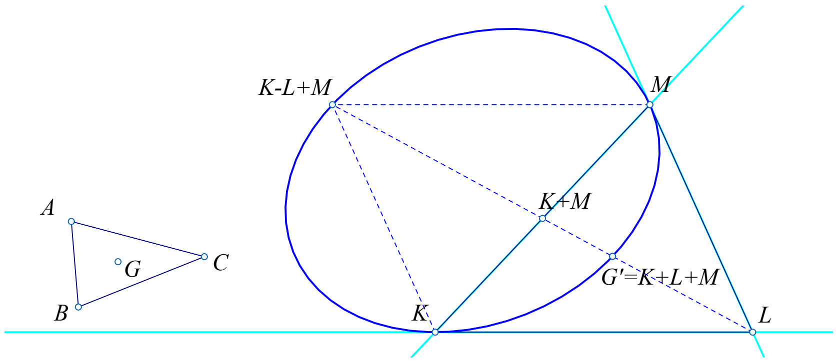

Figure 19: Conic $U = t^2K + tL + M$ for $\{K, L, M\}$ in absolute barycentrics

of each vector be 1. Then, the conic generated by U is the one seen in figure 19. This is a special conic. Namely, an ellipse tangent, as expected to the lines $\{LK, LM\}$ at $\{K, M\}$ and passing through the centroid $G' = K + L + M$ of the triangle KLM, obtained for the value of $t = 1$. The conic passes also through the symmetric $K - L + M$ of L w.r.t the middle $K + M$ of the side KM, obtained for the value of $t = -1$ and which is the harmonic conjugate of G' w.r.t the couple of points L and $K + M$. The triangle ABC and its centroid G are used to define the barycentric coordinates.

Notice that, since the shapes involved in this figure are geometrically defined through the location of the points $\{K, L, M\}$, their appearance and geometric properties do not depend on the form and the location of the triangle of reference ABC.

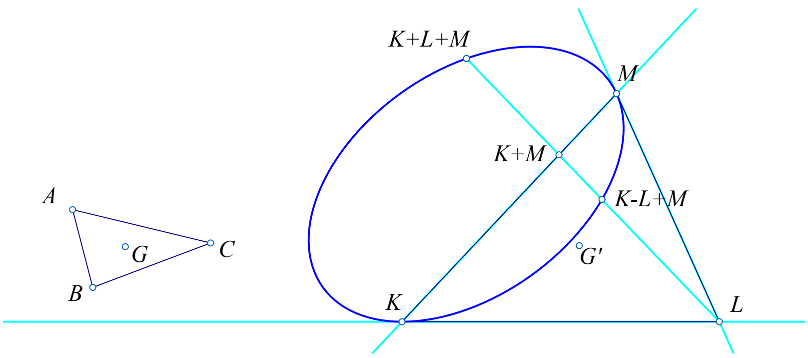

Figure 20: Conic $U = t^2K + tL + M$ for $\{K, L, M\}$ in general barycentrics

Figure 20 uses the same points $\{K, L, M\}$ with the preceding one, but now the coordinates of the vectors are not *"absolute"*, i.e. their sums are not 1 but some distinct constants $\{s_1, s_2, s_3\}$. The conic has the same formal characteristics, passing through the points $\{K, M, K + L + M, K - L + M\}$ but now $K + M$ is essentially an arbitrary point on line KM and $K + L + M$ does not coincide with the centroid G' of the triangle KLM. Points $\{K + L + M, K - L + M\}$ however are again harmonic conjugate w.r.t. the couple of points $\{L, K + M\}$.

Maintaining the same location of the points $\{K, L, M\}$ but multiplying their absolute barycentric coordinates with varying factors $\{s_1K, s_2L, s_3M\}$ we obtain all possible members of the bitangent pencil of conics tangent to lines $\{LK, LM\}$ at $\{K, M\}$. Again the shapes and properties of these conics do not depend on the location and shape of the triangle of reference ABC.

Remark 3. Continuing the preceding remark, we could consider a slight generalization and instead of the conic described by the parameterization $U = t^2K + tL + M$, equivalent to $U = t(tK + L) + M$, replace in one place t with a homographicaly related variable $s(t) = (at + b)/(ct + d)$, i.e. consider the parameterization $U = t(s(t)K + L) + M$. Using the fact that vectors of coordinates representing the points of the plane are defined up to a non-zero multiplicative constant, we see that this reduces to a parameterization like the one of the preceding remark, involving only some other coordinate vectors depending on $\{K, L, M\}$ and the homographic relation:

$$
\begin{aligned}
U = t(s(t)K + L) + M \; &= \; t\left(\frac{at+b}{ct+d}K + L\right) + M \\
&= t\left((at+b)K + (ct+d)L\right) + (ct+d)M \\
&= t^2[aK + cL] + t[bK + dL + cM] + dM \,.
\end{aligned}
$$

The same remark holds also if we use a second homographic relation $s'(t)$ and consider the parameterization

$$U \;=\; s'(t)[s(t)K + L] + M \,.$$

8 Cross ratio on the conic

We re-examine here the *"cross ratio"* of four points $\{P(t_i),\;\; i = 1,2,3,4\}$ *on the conic* defined by the cross ratio of the four lines joining these points with an arbitrary point P of the conic, seen in corollary 2 to be independent of the particular choice of P.

Theorem 6. *The cross ratio of four points* $\{P_i = P(t_i),\;\; i = 1,2,3,4\}$ *on the conic can be expressed using the values of* $\{t_i\}$ *through*

$$(P_1P_2; P_3P_4) \;=\; (t_1t_2; t_3t_4) \;=\; \frac{t_1 - t_3}{t_2 - t_3} : \frac{t_1 - t_4}{t_2 - t_4} \,.$$

Proof. Since this cross ratio is computed using any line ε and its intersections $\{Q_i\}$ with lines $\{P_0P_i, i = 1,2,3,4\}$ (see figure 21), for any point P_0 of the conic, we can use the (coordinate) line $\varepsilon = \alpha = 0$. The chords through the points (t_i, t_0) of the conic are given

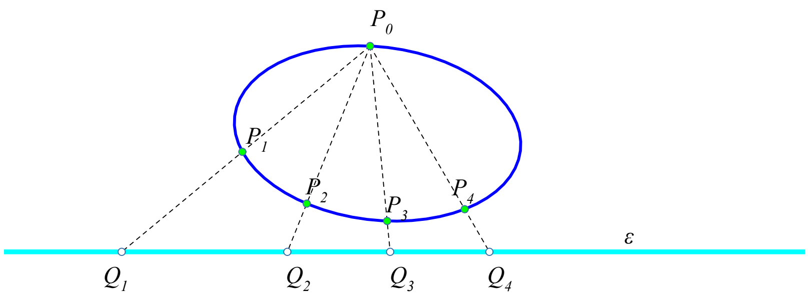

Figure 21: Cross ratio of 4 points on the conic $(P_1P_2; P_3P_4) = (Q_1Q_2; Q_3Q_4)$

by equation (27):

$$t_it_0\gamma - (t_i + t_0)\beta + \alpha = 0 \,.$$

Their intersection points $\{(s_i, s_i')\}$ with $\alpha = 0$ satisfy the preceding equation and, by definition also the equations

$$
\begin{aligned}
&\alpha - s_i\beta = 0 \quad \text{and} \quad \beta - s_i'\gamma = 0 \quad \Rightarrow \quad s_i = 0 \quad \Rightarrow \\
&t_it_0\gamma - (t_i + t_0)\beta = 0 \quad \text{and} \quad \gamma = \frac{\beta}{s_i'} \quad \Rightarrow \quad s_i' = \frac{t_it_0}{t_i + t_0} \,.
\end{aligned}
$$

Thus, the cross ratio expressed through the four lines $\{\beta - s'_i\gamma = 0, i = 1,2,3,4\}$ is (see file **Cross ratio**) $(s'_1 s'_2; s'_3 s'_4)$ and since, by the last equation, the pairs $\{(s_i, t_i)\}$ are connected by a *"homographic relation"* the corresponding cross ratios are equal. □

Theorem 7. *Given four points $\{P_i, i = 1,2,3,4\}$ in general position and a number c there is a unique conic κ whose points $\{P\}$ joined with the four given points define quadruples of lines with constant cross ratio $P(P_1P_2; P_3P_4) = c$.*

Proof. This follows from the existence and uniqueness of a conic passing through four points $\{P_i, i = 1,2,3,4\}$ and having at P_1 a given tangent ε ([Pam14, 8.1]). In fact, if there is such a conic, then the cross ratio of the four lines is independent from the position of P on the conic and for P tending to P_1 the chord PP_1 tends to the tangent ε at P_1 (see figure 22). Thus, the ratio is equal to that of the quadruple consisting of ε and the other three lines $\{P_1P_2, P_1P_3, P_1P_4\}$. Since the three last lines are fixed, the line ε is uniquely determined by the given value c of the cross ratio and by the aforementioned theorem there is a unique conic passing through the four points and tangent to P_1 at ε.

Also the existence of such a conic for a given value c of the cross ratio can be proved using a similar figure. In fact, given the four points, consider an arbitrary fifth P_5 and the conic κ' through the five points. All points P of this conic joined to the $P'_i s$ define four lines having constant cross ratio, c' say. As in the preceding part of the proof, this cross

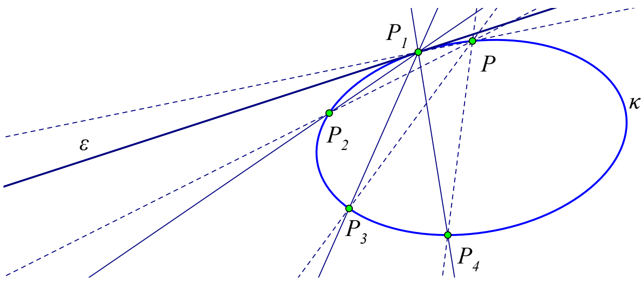

Figure 22: The cross ratio $P(P_1P_2; P_3P_4)$ computed for $P = P_1$

ratio is the same with that of the quadruple of lines consisting of the tangent ε' to κ' at P_1 and the the three other lines $\{P_1P_2, P_1P_3, P_1P_4\}$. Thus, in order to show the existence of the desired conic it suffices to show, that as ε rotates about P_1 the corresponding cross ratio of the four lines $\{\varepsilon, P_1P_2, P_1P_3, P_1P_4\}$ takes all possible values, thus also the given value c. This though is easily seen by measuring the cross ratio using the intersections of the four lines with the x axis. These points, identified with their coordinates: $\{x, p, q, r\}$, x being variable and the other numbers being constant, leading to the expression of the cross ratio:

$$c' = \frac{x-q}{p-q} : \frac{x-r}{p-r} = \frac{p-r}{p-q} \cdot \frac{x-q}{x-r},$$

so that c' is a function of x obtaining all possible values, as claimed. □

By the way, the preceding argument suggests another aspect of the pencil of conics passing through four points in general position $\{A, B, C, D\}$. The members of the pencil are all conics passing through these points and having at one of the points, A say, tangents in all possible directions (see figure 23).

Theorem 8. *The cross ratio of four points $\{P_i = P(t_i), i = 1,2,3,4\}$ on the conic is equal to the cross ratio of the four intersection points $\{Q_i, i = 1,2,3,4\}$ of their tangents with the tangent at any fifth point P_0 of the conic (see figure 24).*

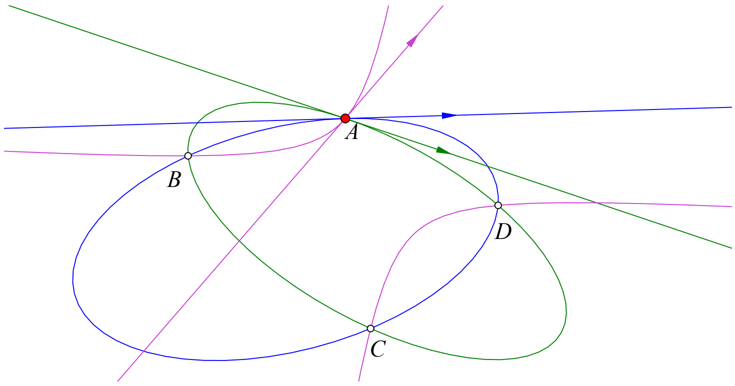

Figure 23: Members of the pencil through $\{A, B, C, D\}$

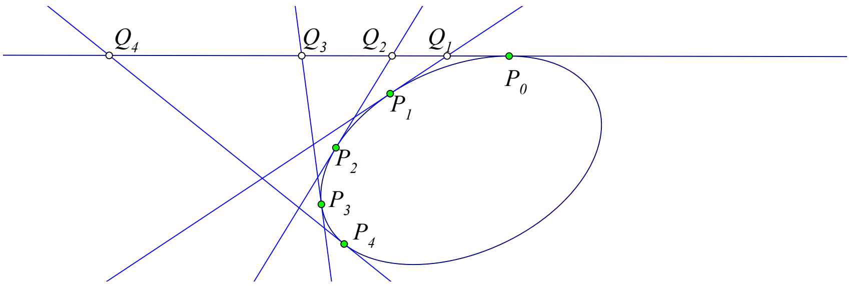

Figure 24: The equality of cross ratios: $(P_1P_2; P_3P_4) = (Q_1Q_2; Q_3Q_4)$

Proof. By equation (29), the tangents at $\{P(t_i), P(t_0)\}$ are given by:

$$t_i^2\gamma - 2t_i\beta + \alpha = 0\,,\ t_0^2\gamma - 2t_0\beta + \alpha = 0\,.$$

Solving this homogeneous system for the values of $\{\alpha, \beta, \gamma\}$, we find that

$$(\alpha, \beta, \gamma) = k(-2t_0t_i, t_0 + t_i, 2) \quad \text{with a constant} \quad k \neq 0 \quad \Rightarrow \quad \frac{\alpha}{\gamma} = -t_0t_i\,.$$

Hence the points $\{Q_i\}$ are on the lines $\{\alpha + t_0t_i\gamma\}$ whose cross ratio is

$$(Q_1Q_2; Q_3Q_4) = ((t_0t_1)(t_0t_2); (t_0t_3)(t_0t_4)) = (t_1t_2; t_3t_4)\,. \qquad \square$$

Corollary 5. *Four fixed tangents of a conic intersected by a variable fifth tangent define on it quadruples of points having constant cross ratio.*

Remark 4. This property of the conics is so important, that Chasles in his monumental work [Cha65] deduces from this the whole theory of conics discussed in the book.

Corollary 6. *Consider three arbitrary but fixed points $\{A, B, C\}$ on a parabola and a variable point P on it. Then the tangents at these three points intercept on the tangent at P segments having constant ratio as P varies on the parabola (see figure 25).*

Proof. This is a consequence of theorem 8 and the fact that the parabola is tangent to the line at infinity. This implies that the cross ratio formed on arbitrary tangents by the four fixed tangents at $\{A, B, C\}$ and the line at infinity is constant. But the cross ratio of four points one of which is at infinity becomes the usual ratio of the segments defined by the three other points. In figure 25 the tangent at the variable P defines segments with constant ratio: $\frac{A'B'}{B'C'} = k$. $\square$

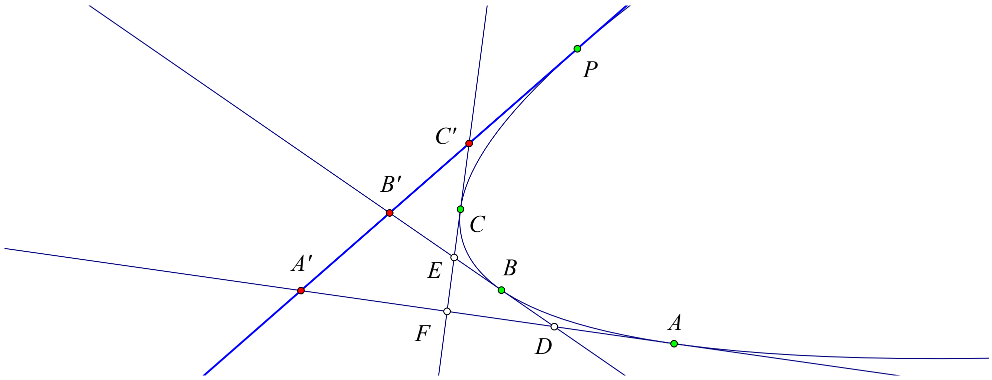

Figure 25: Segments of constant ratio $\frac{A'B'}{B'C'}$ on the tangent of a variable P

In the same figure, holding the points $\{P, B, A\}$ fixed and varying the tangent at P, we see that the ratio $\frac{FE}{EC'}$ is constant. When point C tends to coincide with P the segments $\{FE, EC'\}$ having all the time the same constant ratio tend to coincide with $\{A'B', B'P\}$. When point C tends to coincide with B the segments tend to coincide with $\{DB, BB'\}$ and when C tends to coincide with A the segments $\{FE, EC'\}$ tend to coincide with $\{AD, DA'\}$. We conclude that $\frac{A'B'}{B'P} = \frac{DB}{BB'} = \frac{AD}{DA'}$ and the proof of the following corollary:

Corollary 7. *Two tangents $\{SP, SQ\}$ of a parabola intersected at the points $\{P', Q'\}$ by the tangent at X, define segments having the same ratio $\frac{SP'}{P'P} = \frac{QX}{XP'} = \frac{QQ'}{Q'S}$ (see figure 26-(I)).*

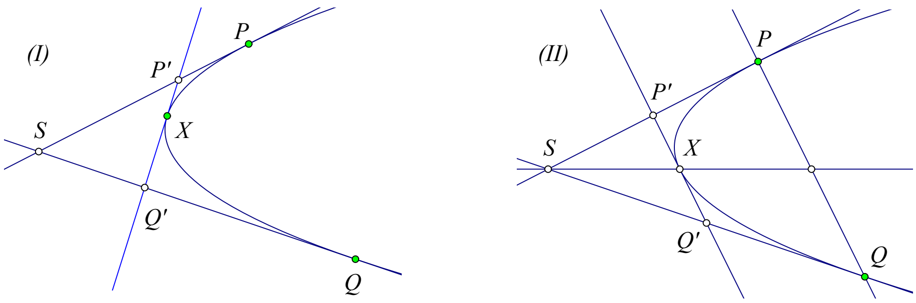

Figure 26: Same ratio $\frac{SP'}{P'P} = \frac{Q'X}{XP'} = \frac{QQ'}{Q'S}$ Characteristic of parabola

This implies also a characteristic property of the parabola:

Corollary 8. *The line $P'Q'$ joining the middles of the tangents $\{SP, SQ\}$ of a parabola is tangent to it at its middle X and line SX is parallel to the axis of the parabola (see figure 26-(II)).*

Corollary 7 is behind the popular visualization of parabolas by dividing the sides of an angle $\widehat{ABC}$ in equal parts: $\{AA_1 = A_1A_2 = ... = a;\ BB_1 = B_1B_2 = ... = b\}$ and joining the points with lines $\{\alpha_i = A_iB_i\}$, which are tangents to a parabola and suggest its existence as the *"envelope"* of these lines (see figure 27).

9 The Chasles-Steiner definition of a conic

This is a definition using the concept of *"homographic relation"* or *"homography"* between points of two lines or/and relations between lines of two pencils (see file **Homographic**

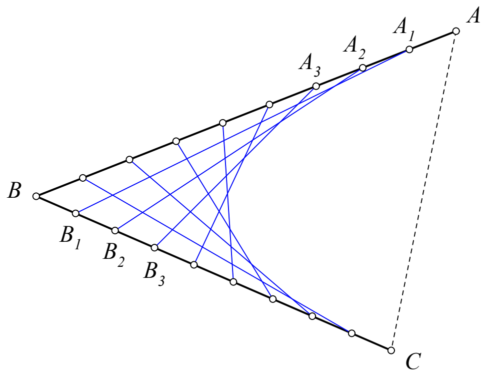

Figure 27: Tangents $\{\alpha_i = A_iB_i,\ i = 1, 2, ...\}$ to a parabola

relation). Using coordinates on two lines $\{\alpha, \beta\}$ the homographic relation is described by an invertible transformation of the form

$$y = \frac{ax+b}{cx+d} \quad \text{with} \quad a, b, c, d \in \mathbb{R} \quad \text{and} \quad ad - bc \neq 0\,. \tag{31}$$

The basic properties of this transformation are:

1. They maintain the same form when changing to other coordinate systems of the two lines in the form $\{x' = px + q\ ,\ y' = rx + s\}$.
2. They preserve the cross ratio of related points: $(y_1y_2; y_3y_4) = (x_1x_2; x_3x_4)$.
3. They maintain the same form when composing with a relation to the points of a third line. More general, the concatenation of any number of homographies is also a homography.

A homographic relation between two pencils $\{A^*, B^*\}$ of lines through the points, respectively, $\{A, B\}$ can be defined by reducing it to a homographic relation between the points of a line ε (see figure 28). To the line $\lambda \in A^*$ we correspond the line $\mu \in B^*$ via the coordinates $\{x, y\}$ of their intersections $\{A', B'\}$ with line ε using a relation of the form (31). A detailed discussion of the concept and several examples can be found in the afore-

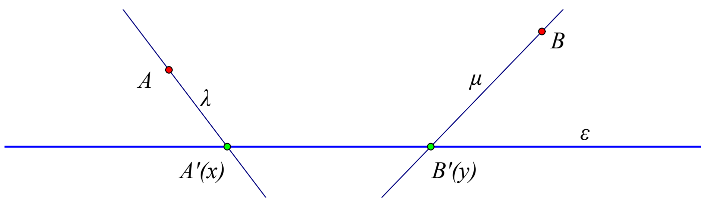

Figure 28: Homography between lines $\{\lambda, \mu\}$ of the pencils $\{A^*, B^*\}$

mentioned reference **Homographic relation**. The Chasles-Steiner definition relays on the following theorem.

Theorem 9. *The intersection points $X = \lambda \cap \mu$ of two lines through $\{A, B\}$ corresponding under a homographic relation $\mu = f(\lambda)$ between the pencils of lines $\{f : A^* \to B^*\}$ is a conic passing through $\{A, B\}$. Conversely, selecting two arbitrary points $\{A, B\}$ on a conic and considering a variable point X on it, the correspondence $f : AX \to BX$ defines a homographic relation between the pencils of lines $\{A^*, B^*\}$.*

Proof. The converse is a direct consequence of the property *nr-3* of section 7 visualized in figure 17. Selecting two points $\{A, B\}$ on the conic defines the coordinate system consisting of the tangents $\{\alpha, \beta\}$ at these points and the line of contacts γ, as in the aforementioned reference. In this system the homographic relation between the pencils $\{A^*, B^*\}$ is described by the correspondence $A^* \ni (\alpha - t\gamma) \mapsto (\gamma - t\beta) \in B^*$ represented by the identity transformation $t \mapsto t$.

For the direct part of the theorem, consider two points $\{A, B\}$ and a homographic relation between the pencils of lines $\{A^*, B^*\}$ represented on an auxiliary line ε by a transformation $x' = f(x)$ (see figure 29). Consider also four particular points $\{C, D, E, F\}$ created by such intersections and the conic κ passing through these points and B. Because the corresponding line coordinates $\{x, ..., x', ...\}$ of the four points are pairwise related homographically, the cross ratios of the pencils $\{A(CD; EF), B(CD; EF)\}$ are equal. By theorem 7 point A will be also on the conic. Consider now one of the points, E say, varying on the conic κ. By the proved converse part of the theorem, the corresponding lines $\{AE, BE\}$, as E varies on the conic, define on ε a homographic relation f'. But this relation coincides with f at three points, corresponding to $\{C, D, F\}$. By the fundamental property of homographic relations, *to be identical if they coincide at three points*, we get $f' = f$, thereby proving the claim. □

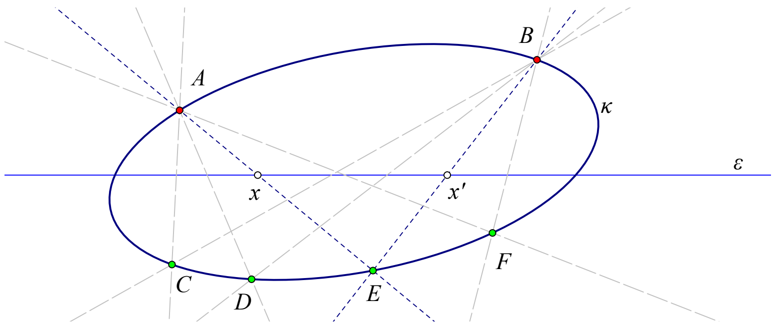

Figure 29: Chasles-Steiner definition of the conic κ

Remark 5. The conic κ is genuine if E does not obtain the position of the intersection of lines $E_0 = \varepsilon \cap AB$. Expressed independently from the auxiliary line ε and in terms of the homography $f : A^* \to B^*$ this happens when $f(AB) = BA$. In this case the conic is degenerate consisting of the line AB and a second line ζ.

The following corollaries present some example applications of the theorem.

Corollary 9. *Consider three fixed points $\{A, B, C, D \in BC\}$ and a fixed line AD. On line AD choose points $\{B', C'\}$ such that $B'A/B'C' = k$ is a fixed constant. The intersection points P of the variable lines $\{BB', CC'\}$ generate in general a hyperbola and in one case a parabola.*

Proof. That this is a conic follows immediately by taking the origin of coordinates on AD to be at A and setting $\{AB' = x, AC' = y\}$ (see figure 30). This implies the relation

$$\frac{x}{x - y} = k \quad \Rightarrow \quad y = \frac{k - 1}{k} x,$$

which is a very simple homography between the pencils of lines $\{B^*, C^*\}$ as required by the Chasles-Steiner method and defines a conic passing through $\{B, C\}$ and also through A for $x = y = 0$.

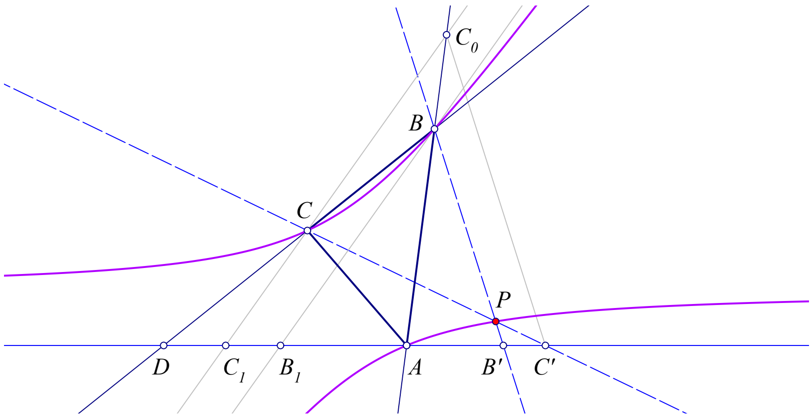

Figure 30: Hyperbola defined by a triangle a point and a number

To see that this is a hyperbola it suffices to notice that it intersects the line at infinity at two distinct points. The first is determined by the direction of the line AD, since B' going to infinity implies that C' does the same and lines $\{BB', CC'\}$ become parallel to AD. The other point at infinity lying on the conic is found as follows. Note first that since $B'A/B'C'$ is constant the parallel to BB' from C' intersects AB at a fixed point C_0. Line C_0C intersects AD at a point C_1 and the parallel to C_0C from B defines on AD a point B_1 and obviously $B_1A/B_1C_1 = k$. Thus, the other point at infinity on the conic is the point determined by the parallels $\{C_0C_1, BB_1\}$.

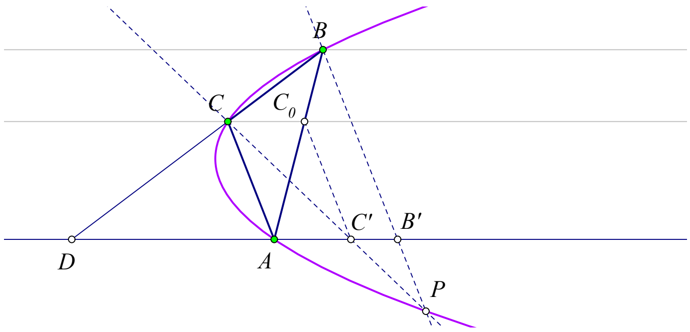

Figure 31: Special case: parabola when k is such that $CC_0 \parallel AD$

The special case of a parabola (see figure 31) arises when the value of the ratio k is such that $BA/BC_0 = k$ defines a line CC_0 parallel to CD.

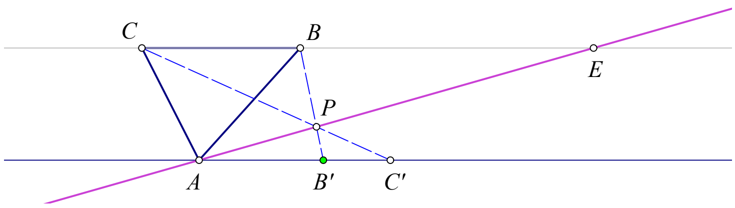

Figure 32: Special case: the geometric locus is line AE

A special case occurs also when line AD is parallel to BC. Then line AP intersects BC at a point E such that $BE/BC = k$, hence E is fixed and P moves on line AE (see figure 32). □

Corollary 10. *Let $\{A, B, C\}$ be three points on a hyperbola (resp. parabola) κ and consider the* *parallel to an asymptote (resp. axis of the parabola) through A line AD (see figure 30). Then, for any point P on the hyperbola (resp. parabola) lines $\{PB, PC\}$ intersect line AD at points $\{B', C'\}$ such that $B'A/B'C'$ is constant.*

Proof. The proof for the hyperbola (and analogously for the parabola) can be based on corollary 9. After it, forgetting for the moment the given hyperbola κ, we determine another hyperbola κ' describing the locus of a point P as in this corollary. For this take an arbitrary point P_0 on κ and determine the ratio $k = B'A/B'C$. Construct then κ' by the data $\{A, B, C, D, k\}$ as in corollary 9. The two hyperbolas $\{\kappa, \kappa'\}$ coincide then on the four points $\{A, B, C, P_0\}$ and the point at infinity determined by line AD. Hence they are identical. □

Corollary 11. *The line ζ revolves about the point A intersecting the conic κ at the points $\{B, C\}$. The middle D of BC describes then a conic κ' homothetic to κ passing through A and through its center K, if the conic κ is central (see figure 33).*

Proof. I sketch the proof for an ellipse and a point A in its interior. The case of a point outside, of a hyperbola and a parabola, can be handled similarly with minor modifications and are left as exercises. Under this hypothesis, consider the polar ε of A w.r.t. κ and the intersections $\{X = \varepsilon \cap \zeta\ ,\ Y = \varepsilon \cap DK\}$. The line ζ is the polar of Y and $\{X, Y\}$ are

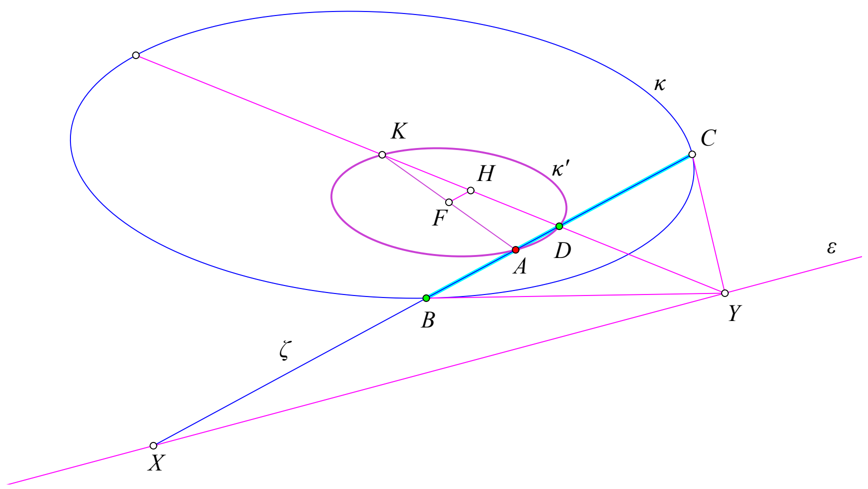

Figure 33: The geometric locus of middles D of chords BC

homographically related. This is a basic example of homographic relation, associating to the point X on the polar of A the pol Y of $\zeta = XA$. A simple proof of that can be given for the circle (see file **Homographic relation**) and the general case can be reduced to that using a projectivity mapping the circle onto the conic κ.

From this it follows that the map between the pencils of lines $f : A^* \to K^*$ sending line $\zeta = AX$ to KY is a homography, hence the intersection points $\{D = AX \cap KY\}$ of lines corresponding via f is a conic.

That the two conics are homothetic follows from the fact that they have all their conjugate diameters parallel. This follows by first observing that AK is a common diameter for both conics. This, because sending X on ε to infinity, point Y obtains the position $Y = AK \cap \varepsilon$ and ζ becomes tangent to κ' at A parallel to ε. Similarly sending Y on ε to infinity, X obtains the position $X = AK \cap \varepsilon$ and ζ becomes tangent to κ' at K. This shows that the two conics have the directions of ε and AK as common conjugate directions, hence the center F of κ' is the middle of AK. For an arbitrary point $D \in \kappa'$ the direction of the line AD is conjugate to that of KD w.r.t. κ but also w.r.t. κ', since the line FH joining the middles of $\{AK, DK\}$ is parallel to AD. □

10 Some applications, Pascal's theorem

If $\{f, \alpha, \beta\}$ represent respectively a circle $(x - x_0)^2 + (y - y_0)^2 - r^2 = 0$ and two lines, the points P of the conic $\{f + \lambda\alpha\beta = 0, \lambda \in \mathbb{R}\}$ have the property $\frac{f(P)}{\alpha(P)\beta(P)} = -\lambda$ a constant. Since $f(P)$ represents the *"power"* of P w.r.t. the circle f, we have:

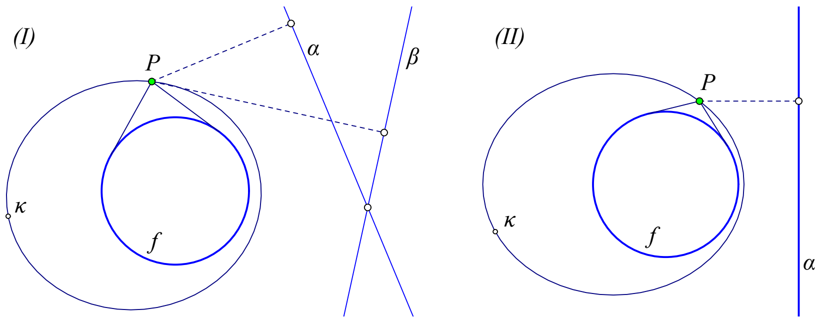

Figure 34: For $P \in \kappa$ the ratio $\frac{f(P)}{\alpha(P)\beta(P)}$, resp. $\frac{f(P)}{\alpha^2}$

Corollary 12. *If f is a circle and $\{\alpha, \beta\}$ two lines, the points P of the conic $f + \lambda\alpha\beta$ have the ratio of the power of P w.r.t. f to the product of distances from $\{\alpha, \beta\}$ constant (see figure 34-(I)) and vice versa. If for a given circle and two lines this ratio is constant, then the point is on a member conic of the pencil $\{f + \lambda\alpha\beta\}$. If the two lines coincide $\alpha = \beta$, then the members of the pencil $\{f + \lambda\alpha^2\}$ are characterized by the analogous property to have the ratio $\frac{f}{\alpha^2}$ of the power to the square of distance constant (see figure 34-(II)).*

In the limit case, in which the circle becomes smaller and smaller ending to a point Q the preceding property becomes one of the traditional definitions of conics:

Corollary 13. *A conic is the geometric locus of points P for which the ratio of distances $\frac{f}{\alpha}$ from a point Q and a line α is constant (see figure 35).*

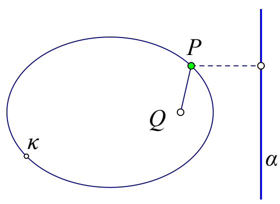

Figure 35: A traditional definition of a conic: $\frac{f}{\alpha} = k$ constant

Corollary 14. *Two conics $\{f_1, f_2\}$ bitangent to a third f_3 with lines joining the contact points respectively $\{\alpha, \beta\}$ intersecting at the point I (see figure 36), have a pair of common chords (EG, FH) intersecting at I and being harmonic conjugate to the pair (α, β).*

Proof. This seemingly complicated property admits, using the abridged notation, a remarkable simple formal proof. In fact, f_1 being bitangent to f_3 can be expressed in the

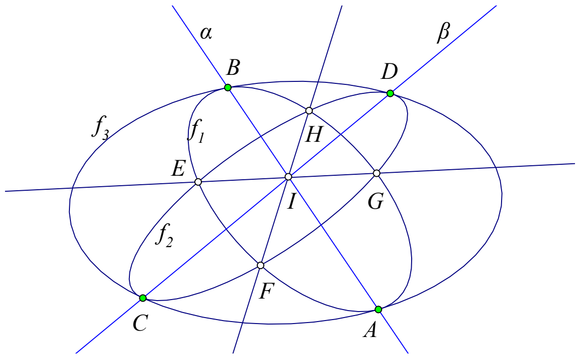

Figure 36: Conics $\{f_1, f_2\}$ bitangent to f_3

form $f_1 = f_3 + \lambda_1 \alpha^2$. Analogously, the bitangent property with f_2 leads to the expression $f_2 = f_3 + \lambda_2 \beta^2$. Thus the conic $f_1 - f_2 = \lambda_1 \alpha^2 - \lambda_2 \beta^2 = 0$ is a product of lines

$$\alpha = \pm \beta \sqrt{\frac{\lambda_2}{\lambda_1}}\,.$$

This is a line equation representing two lines harmonic conjugate to the pair (α, β). □

Corollary 15. *The two diagonals of an inscribed in a conic f quadrangle $ABCD$ and the two diagonals of the corresponding tangential at the vertices of the former quadrangle $EFGH$ intersect at a point I and form harmonic conjugate pairs of lines $I(AB; EF) = -1$ (see figure 37).*

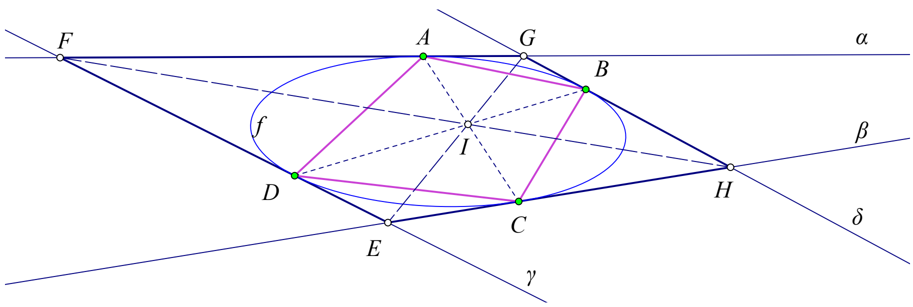

Figure 37: Conics with a common chord δ

Proof. This is a special case of the corollary 14: $\{ f_1 = \alpha\beta\,,\, f_2 = \gamma\delta\,,\, f_3 = f\ \}$. □

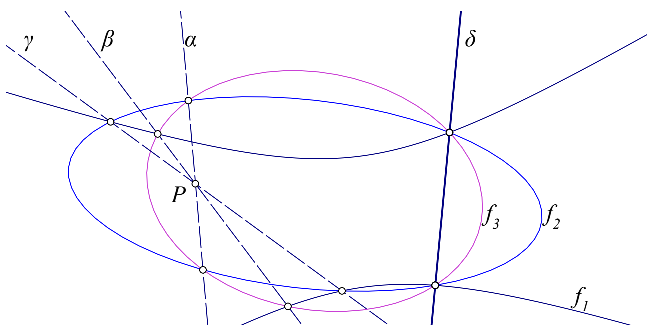

Figure 38: Conics with a common chord δ

Corollary 16. *If the conics $\{f_1,f_2,f_3\}$ have a common chord δ, then the other chords of intersections $\{\alpha,\beta,\gamma\}$ of the pairs of conics pass through the same point (see figure 38).*

Proof. It suffices to write the conics $\{f_2,f_3\}$ as members of pencils:

$$f_2 \;=\; f_1+\lambda_1\gamma\cdot\delta \quad \text{and} \quad f_3 \;=\; f_1+\lambda_2\beta\cdot\delta\,.$$

Then, the other than δ common chord of $\{f_2,f_3\}$ satisfies the equation

$$0 \;=\; f_2-f_3 \;=\; (\lambda_1\gamma-\lambda_2\beta)\cdot\delta \quad \Rightarrow \quad \lambda_1\gamma-\lambda_2\beta \;=\; 0\,.$$

This being identical with α, shows that later is a combination of $\{\beta,\gamma\}$ hence passes through their intersection point. □

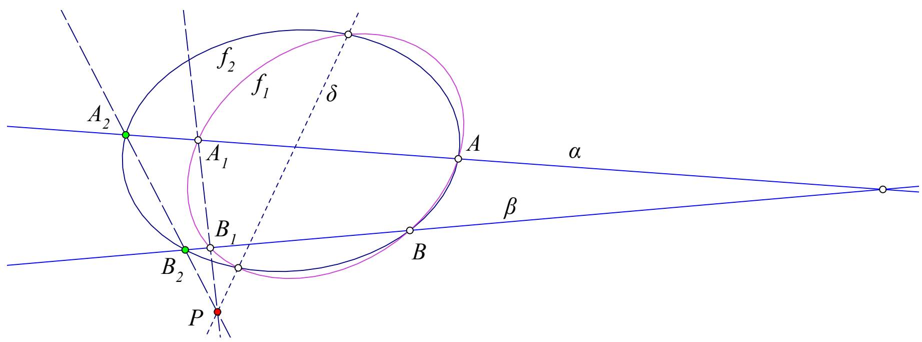

Figure 39: Conics with a common chord δ

Corollary 17. *If $\{A,B\}$ are common points of the conics $\{f_1,f_2\}$ and the lines $\{\alpha,\beta\}$ through them intersect the conics in points $\{A_1,B_1\in f_1\}$ and $\{A_2,B_2\in f_2\}$, then the intersection point $P=A_1B_1\cap A_2B_2$ is on a common chord δ of the two conics (see figure 39).*

Proof. This is a special case of the preceding corollary for which the third conic is the (degenerate) product of the two lines $f_3=\alpha\cdot\beta$. □

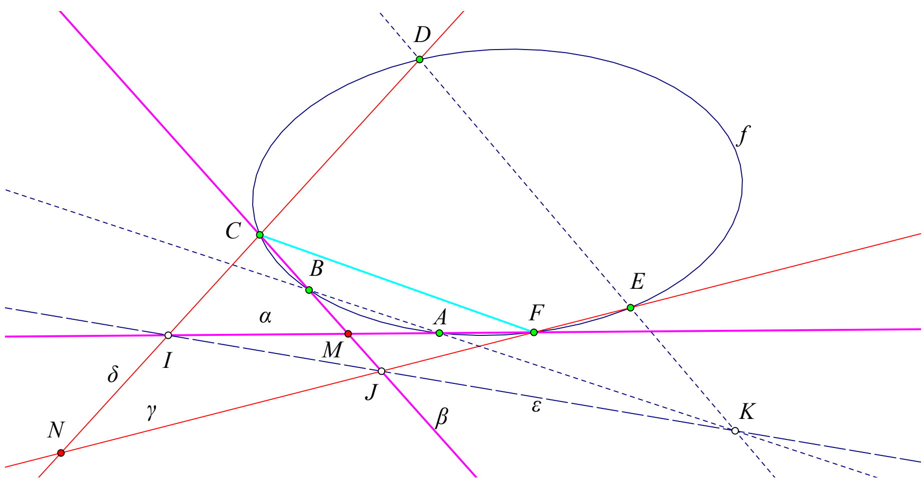

Figure 40: Pascal's mystic hexagram theorem

Theorem 10 (Pascal's theorem)**.** *Consider an hexagon $ABCDEF$ inscribed in the conic f. Then, the pairs of opposite sides intersect on three collinear points $\{I, J, K\}$ (see figure 40). Conversely, if the opposite sides of a hexagon intersect in three collinear points, then the hexagon can be inscribed in a conic.*

Proof. Start with a chord CF of the conic f joining opposite vertices of the hexagon and define the side-lines adjacent to $\{C, F\}$:

$$\alpha = AF \;\; , \;\; \beta = BC \quad \text{and} \quad \gamma = FE \;\; , \;\; \delta = CD \, .$$

The three conics $\{ f \; , \; \alpha \cdot \beta \; , \; \gamma \cdot \delta \; \}$ have the chord CF in common and by corollary 16 their other common chords $\{IJ, AB, DE\}$ pass through a common point K.

The converse follows from the direct part of the theorem. In fact, if the condition of collinearity is valid, then consider the conic κ' defined by the first five points $\{A, B, C, D, E\}$ and its intersection E' with the line FE. By the direct part of Pascal's theorem the intersections of the line pairs $\{I = AC \cap CD \; , \; J = AF \cap BC \; , \; K' = AB \cap DE'\}$ will be collinear. But $AB \cap IJ = K$ hence $K' = K$ and consequently $E' = E$ as claimed. □

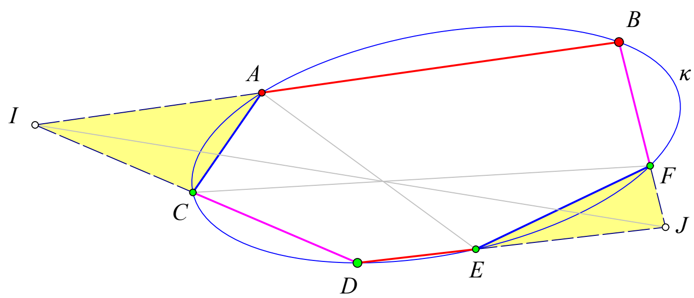

Figure 41: A condition of perspectivity for two triangles

Corollary 18 (Perspective triangles)**.** *The triangles $\{ACI, EFJ\}$ formed on two opposite sides of a hexagon by the extensions of the other sides are perspective (see figure 41), if and only if the hexagon can be inscribed in a conic.*

Proof. Follows from Pascal's theorem 10. If there is a conic κ as claimed, then the sides of the two triangles are pairwise opposite of an inscribed hexagon, hence the pairs of opposite sides intersect at three collinear points and the triangles are perspective. Conversely, if they are perspective, then the three corresponding sides define a hexagon satisfying the condition of Pascal's theorem, hence there is a conic κ through the six vertices as shown in figure 41. □

Remark 6. Corollary 18 can be used to test if six points lie on a conic. If the triangles of one pair of this kind are perspective, then the triangles of the other two pairs constructed similarly on opposite sides will be also perspective.

Remark 7. Notice that Pascal's theorem is valid also in the case of degenerate hexagons, i.e. when some consecutive vertices coincide, a case where the line joining two points, must be replaced with the tangent at the (double) point. This is the case with figure 42, where at A we take the tangent to the conic.

The figure suggests also a recipe for the geometric construction of the tangent at a point A of the conic, using points on the conic, their lines and their intersections.

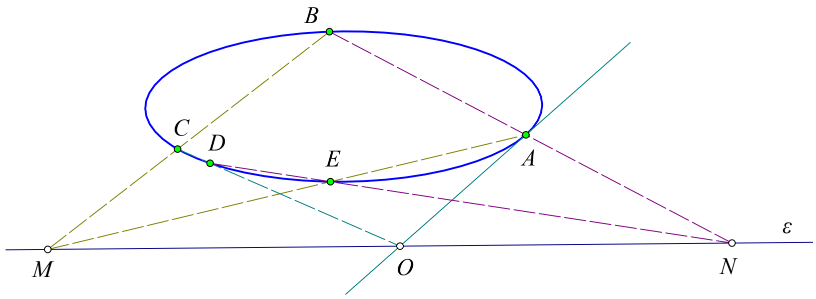

Figure 42: Pascal's theorem for pentagons, constructing the tangent at A

To draw the tangent at A we take four other points on the conic $\{B, C, D, E\}$ and join them as shown. All dotted lines are known, and points $\{M, N\}$ are immediately constructed. Then O is determined as intersection point of the lines MN and CD, the tangent being line AO.

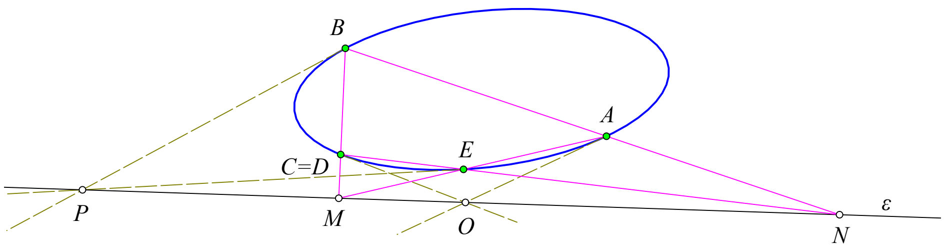

Figure 43: Pascal's theorem for quadrangles

Letting C coincide with D we obtain a quadrangle $ABDE$ inscribed in the conic (see figure 43) and Pascal's theorem for quadrangles coincides with corollary 21, according to which the intersections $\{M, N\}$ of opposite sides and the intersections of tangents at opposite vertices $\{O, P\}$ are collinear and build a harmonic quadruple $(PO; MN) = -1$.

Corollary 19. *All the conics of the pencil through the four points $\{A, B, D, E\}$ have the intersections $\{O, P\}$ of their tangents at opposite vertices on the same line $\varepsilon = MN$, where $\{M, N\}$ the intersections of pairs of opposite sides of the quadrangle (see figure 43). Further, points $\{O, P\}$ are harmonic conjugate w.r.t. $\{M, N\}$.*

Remark 8. On the occasion of figure 43 we should notice the *"autopolar triangle"* w.r.t. to the conic defined by the quadrangle $ABDE$. It consists of the *"diagonal triangle"* of the quadrangle, which has vertices the intersection points $\{M, N, F\}$ of pairs of opposite sides (see figure 44). By its definition, the polar of each vertex of this triangle w.r.t. any conic $\{\kappa, \kappa', \ldots\}$ of the pencil through the points $\{A, B, D, E\}$ is the opposite side-line of the triangle.

Remark 9. In the preceding figure, letting D converge to E we obtain a triangle inscribed in the conic κ and Pascal's theorem relates the tangents at the vertices with the sides of the triangle (see figure 45). According to this, the tangents at the vertices and the opposite to them side-lines intersect at three collinear points $\{A'', B'', C''\}$. The figure shows also the pol D w.r.t. κ of the line ε carrying these points, which is simultaneously the *"trilinear polar"* of ε. The figure is a basic one in the theory of *"triangle conics"* i.e. the conics circumscribing a triangle. In this context point D is called the *"perspector "* of the conic κ. It is then proved, that κ is generated by the *"tripols"* of the lines through D ([Yiu13, p.114]).

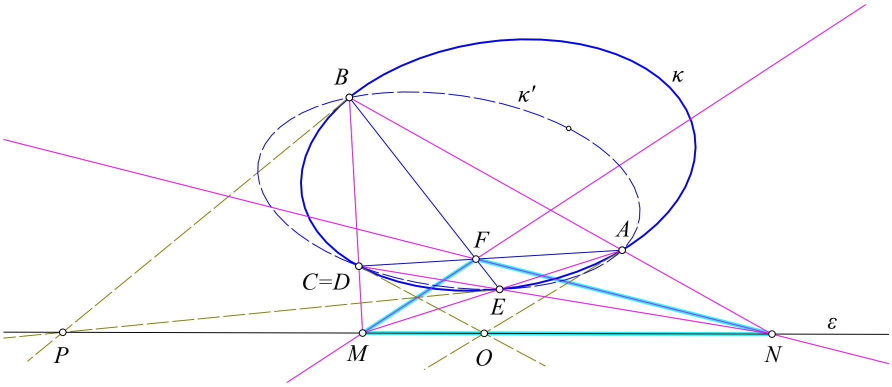

Figure 44: The autopolar $\triangle FMN$ w.r.t to all members $\{\kappa, \kappa', ...\}$ of the pencil

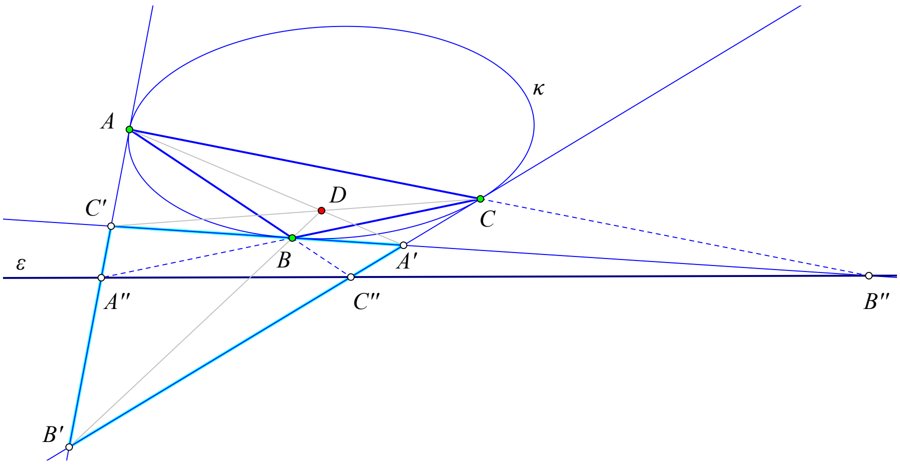

Figure 45: Pascal's theorem for triangles

11 Maclaurin's conic generation method

This recipe producing conics, despite its historical precedence over the Steiner-Chasles method, can be given a simple explanation using this method and the basic example of homography between lines. The basic example is the *"perspectivity"* between two lines from a point not lying on any of them. By this we have two fixed lines $\{\alpha, \beta\}$ and a fixed

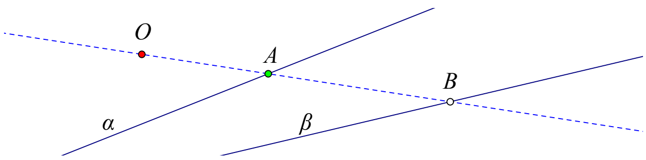

Figure 46: Perspectivity from O between lines α and β

point O *"the perspector"*. Every point $A \in \alpha$, joined to O, defines a line intersecting β at a point B. The perspectivity f_O from O maps $A \mapsto B$. It is easily proved that this defines a homography between the two lines $\{\alpha, \beta\}$ (see file **Homographic relation**). The proof given below uses also the basic property of homographies to produce by their concatenation a homography too.

Theorem 11 (Maclaurin)**.** *The variable triangle ABC has all its sides passing all the time through three fixed points $\{AB \ni C_0\ ,\ BC \ni A_0\ ,\ CA \ni B_0\}$. Also two of its vertices $\{B, C\}$ glide on two*

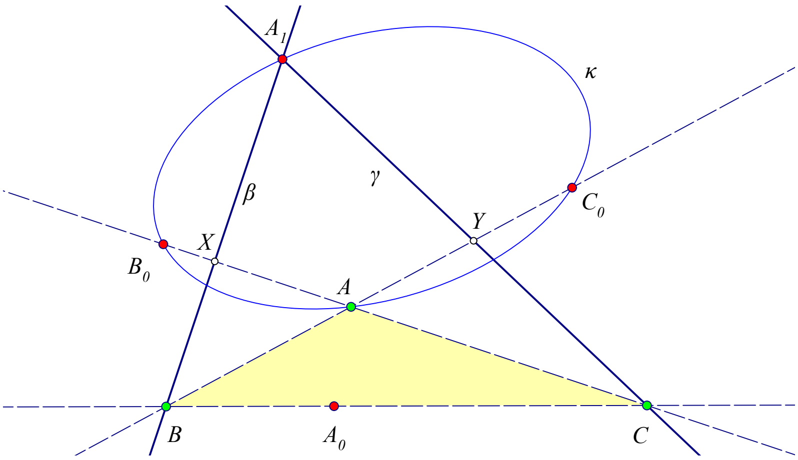

Figure 47: Maclaurin's theorem for the variable triangle ABC

fixed lines $\{\beta,\gamma\}$. *Then the third vertex* A *describes a conic* κ *passing through* $\{B_0,C_0\}$ *and the intersection of lines* $A_1=\beta\cap\gamma$ *(see figure 47).*

Proof. Point $A=B_0C\cap C_0B$ satisfies the hypothesis of the Chasles-Steiner theorem 9. In fact, the perspectivity from $B_0: f:\beta\rightarrow\gamma$ maps a point $X\in\beta$ to $C\in\gamma$. The perspectivity from $A_0: g:\gamma\rightarrow\beta$ maps point C to $B\in\beta$. Finally the perspectivity from $C_0: h:\beta\rightarrow\gamma$ maps point B to Y. Thus, the correspondence $f':X\mapsto Y$, being the composition of three perspectivities $f'=h\circ g\circ f$, defines a homographic relation between the pencils $\{B_0^*,C_0^*\}$ and theorem 9 applies, proving that A describes a conic passing through $\{B_0,C_0\}$. That the conic passes through A_1 is seen by letting B converge towards A_1. Then all the points $\{A,B,C,X,Y\}$ tend to coincide with A_1, showing that κ passes through A_1. □

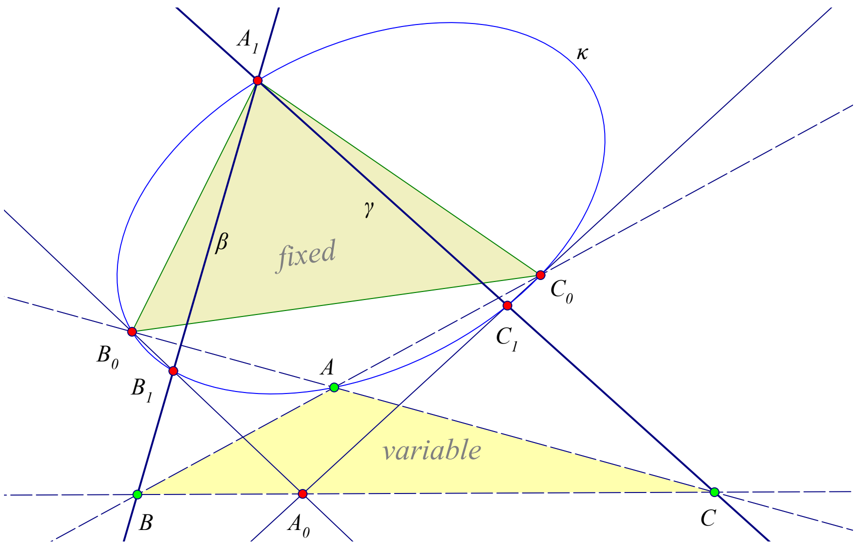

Figure 48: The conic passes through $\{B_1,C_1\}$

Remark 10. The conic κ of the preceding theorem passes also through the intersection points of the lines $\{B_1=\beta\cap A_0B_0\ ,\ C_1=\gamma\cap A_0C_0\}$ (see figure 48). This is seen by letting B converge towards B_1 and seeing that A converges also to this point. Analogously is seen that C_1 belongs to κ. This implies that κ can be constructed as a conic passing through five known points: $\{A_1, B_1, C_1, A_0, B_0\}$.

Remark 11. Given five points like $\{A_1, B_0, B_1, C_1, C_0\}$, Maclaurin's construction gives a means to construct arbitrary many additional points of the conic passing through those five points. The additional points result as vertices A of triangles ABC pivoting around A_0, which is determined by the five given points.

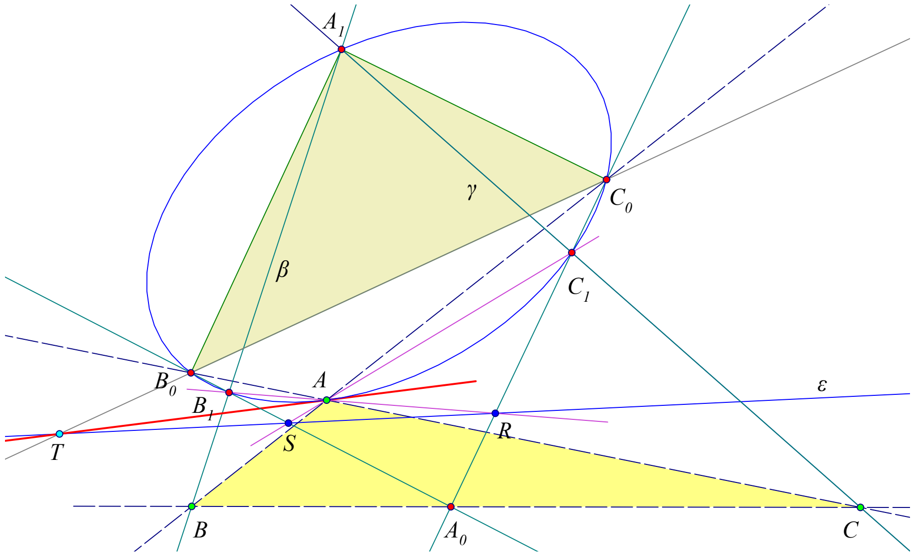

Figure 49: Construction of the tangent at A

Also the tangent at A can be easily located by means of the theorem of Pascal applied to pentagon $B_0B_1AC_1C_0$. By this theorem the intersection point T of the tangent and line B_0C_0 and the intersections of the line-pairs $\{R = B_1A \cap C_1C_0\ ,\ S = C_1A \cap B_1B_0\}$ will lie on a line ε. Thus, the tangent AT can be constructed by first finding $\{R, S\}$, then constructing $T = SR \cap B_0C_0$ and joining it to A.

Maclaurin's theorem admits an inverse, whose proof is trivial:

Theorem 12 (Maclaurin's inverse)**.** *Given a conic κ and three points $\{A_1, B_0, C_0\}$ on it, consider two fixed lines $\{\beta, \gamma\}$ through A_1. For every point A of the conic define the intersection points $\{B, C\}$ of line-pairs $\{B = AC_0 \cap \beta, C = AB_0 \cap \gamma\}$. Then line BC, as A varies on the conic, pivots around a fixed point A_0, which is the intersection point of the lines $A_0 = B_0B_1 \cap C_0C_1$, where $\{B_1, C_1\}$ are the second intersection points of the conic with lines $\{\beta, \gamma\}$.*

Proof. Use Maclaurin's construction to define a conic κ' with the given data. This conic then will have with κ the common points $\{A_1, B_1, C_1, B_0, C_0\}$, thus the two conics will coincide. □

In the case in which the fixed points $\{B_0, C_0\}$ are respectively on the lines $\{\beta, \gamma\}$, the conic κ becomes tangent to lines $\{A_0B_0, A_0C_0\}$ at corresponding points $\{B_0, C_0\}$ and A_0 is the pole of line B_0C_0 (see figure 50). The inverse of Maclaurin's construction becomes Pascal's theorem for quadrilaterals: For quadrilateral $\{A_1B_0AC_0\}$, the intersection points $\{B, C\}$ of pairs of opposite sides and the tangents at pairs of opposite vertices meet on a line BC.

Corollary 20. *With the notation and conventions of this section, the tangents at the other two vertices $\{A, A_1\}$ intersect at a point A_2 lying also on line BC. In addition, A_2 is harmonic conjugate to A_0 w.r.t. $\{B, C\}$.*

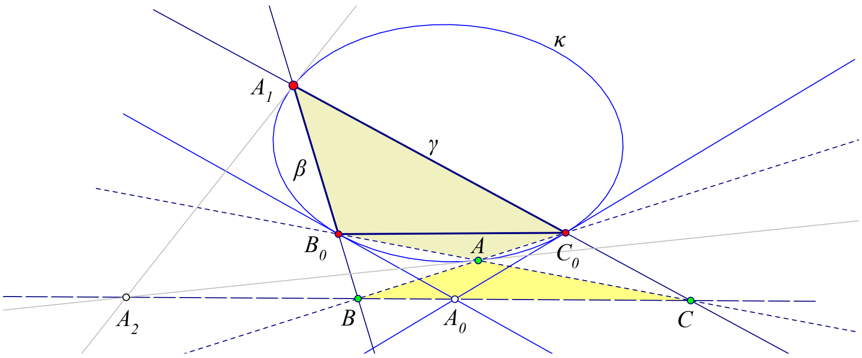

Figure 50: Special case of Maclaurin's theorem

Proof. In fact, if the variable point $A \in \kappa$ is on the line A_1A_0,, then A_2 becomes the pol of A_1A_0 (see figure 51). Then, since A_0 is on the polar of A_2, point A_2 is also on the polar of A_0 i.e. A_2 is on B_0C_0 and the lines $\{A_1A_2, A_1A_0\}$ are harmonic conjugate w.r.t. $\{A_1B_0, A_1C_0\}$, thereby proving the claim. □

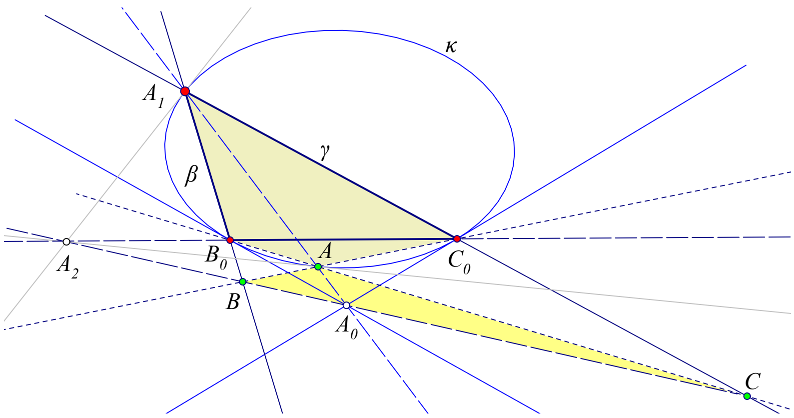

Figure 51: The harmonic quadruple $(A_0A_2; BC) = -1$

Corollary 21. *For every quadrangle inscribed in a conic, the intersection points $\{B, C\}$ of opposite sides and the intersection points $\{A_0, A_2\}$ of the tangents at opposite vertices are on a line and form a harmonic quadruple $(A_0A_2; BC) = -1$.*

12 Double tangency of two conics

We say that two conics are *"doubly tangent"* when they are *"bitangent"*, i.e. when they are different but have two common points and the same tangent at each of these points. As an application of the abridged notation we prove the following theorem.

Theorem 13. *Given three fixed chords $\{AA', BB', CC'\}$ of a conic κ, a fourth chord DD', for which the cross ratios are equal $(AB; CD) = (A'B'; C'D')$, is always tangent to a conic κ' having double tangency with κ.*

Proof. Figure 52 displays such an example. The chords $\{AA', BB', CC'\}$ are fixed and point D is free to move on the conic κ. Then, point D' is calculated so that the cross ratios are equal $(AB; CD) = (A'B'; C'D')$. These cross ratios are calculated using the line coordinates of the perspective projections of the points from an arbitrary point $E \in \kappa$. In the

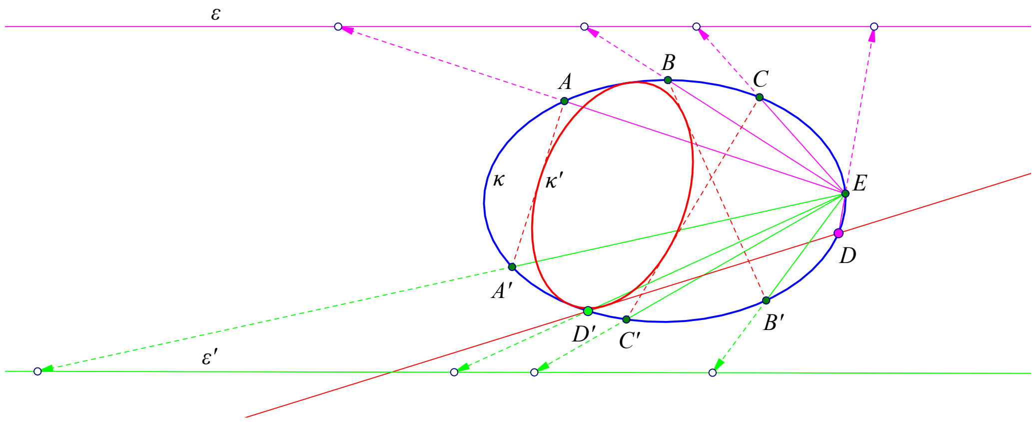

Figure 52: Condition for two conics to be *"bitangent"*

figure the points of the conic are projected on two lines $\{\varepsilon, \varepsilon'\}$ but the arguments below could use also one line only. The proof (after Salmon p. 253) is easy.

We adopt a coordinate system like the one of section 6 (figure 15 consisting of the three lines $\{\alpha, \gamma, \beta\}$, the two first being tangent to κ and the third joining the contact points of the two first. Assuming that A is given by a number a i.e. it is the intersection of lines

$$A(a): \quad \alpha - a\beta = 0, \quad \beta - a\gamma = 0,$$

and using analogously small letters to represent the other points, the equality of the cross ratios translates, after theorem 6, to:

$$\frac{a-c}{b-c} : \frac{a-d}{b-d} = \frac{a'-c'}{b'-c'} : \frac{a'-d'}{b'-d'}.$$

This, considering $\{d, d'\}$ as variables, obtains the form of a *"homographic relation"* (see file **Homographic relation**).

$$pdd' + qd + rd' + s = 0, \quad \text{for constants} \quad (p, q, r, s) \quad \text{depending on} \quad a, b, c, \ldots \tag{32}$$

Solving for d' and substituting into equation (27) of the chord DD', gives the expression

$$\gamma d(qd + s) + \beta(d(pd + r) - (qd + s)) - \alpha(pd + r) = 0. \tag{33}$$

As expected, this is a one-parameter family of lines w.r. to the parameter d, which can be written in the form

$$d^2(q\gamma + p\beta) + d(s\gamma + (r - q)\beta - p\alpha) - (s\beta + r\alpha) = 0. \tag{34}$$

From theorem 2 follows that these lines, depending on the parameter d, envelope the conic κ' with equation:

$$(s\gamma + (r - q)\beta - p\alpha)^2 + 4(q\gamma + p\beta)(r\alpha + s\beta) = 0. \tag{35}$$

The nice thing is that the equation of this conic can be put in the form:

$$4(qr - ps)(\alpha\gamma - \beta^2) + (s\gamma + (q + r)\beta + p\alpha)^2 = 0. \tag{36}$$

Since $\alpha\gamma - \beta^2 = 0$ is our conic κ and $s\gamma + (q + r)\beta + p\alpha = 0$ is a line, the conic 36 belongs to the family generated by the conic κ and this (double) line. This is a *"bitangent"* family of conics , all members of which are tangent to the conic $\kappa : \alpha\gamma - \beta^2 = 0$ at the points where the line $s\gamma + (q + r)\beta + p\alpha = 0$ intersects κ. Note that the intersection points can be imaginary as, for example, is the case with a family of concentric circles, which is also a *"bitangent"* family all members of which are tangent at the same two imaginary points at the same two imaginary lines. □

Remark-1 The homographic relation becomes involutive (see file **Homographic relation**) when $q = r$ and equation 32 takes the form

$$pdd' + q(d + d') + s = 0,$$

showing that all chords $\{DD'\}$ pass through a common point (see equation 28). Thus, we should exclude this case from the beginning, since it shows a totally different behaviour.

Remark-2 Theorem 13 is equivalent to the well known fact, that a *"homography"* on a conic $f : \kappa \to \kappa$ is completely determined by prescribing the images $\{A', B', C'\}$ at three arbitrary points $\{A, B, C\}$ of it. Then, the chords $\{DD',\ D' = f(D),\ D \in \kappa\}$ envelope a conic κ' bitangent to κ.

From this point of view, remark 1 is equivalent with the condition that the three chords $\{AA', BB', CC'\}$ have no common point. In the contrary case, we have still a homography but not a genuine enveloping conic κ'. All chords $\{DD'\}$ pass, in that case, through the common point of $\{AA', BB', CC'\}$ and the homography becomes an *"involution"* of the conic κ, characterized by the functional condition $f^2 = e$, where e the identity transformation of the conic.

Theorem 14. *A variable triangle ABC circumscribes a conic f and has its two vertices $\{A, B\}$ gliding respectively on two fixed lines $\{\delta, \varepsilon\}$. Then the third vertex C describes a conic κ bitangent to f (see figure 53).*

Proof. Consider the coordinate system with lines $\{\alpha, \gamma\}$ the tangents to f from the intersection point of the given fixed lines $O = \delta \cap \varepsilon$ and the line of contacts β and assume the equation of the conic in the form $f = \alpha\gamma - \beta^2$.

The tangents at two points $\{t, t'\}$ of the conic f are given by equation 29:

$$t^2\gamma - 2t\beta + \alpha = 0 \quad \text{and} \quad t'^2\gamma - 2t'\beta + \alpha = 0\,.$$

The line joining their intersection with the base point $O(\alpha = \gamma = 0)$ is found by eliminating β (see section 7 *nr-11*): $\alpha - tt'\gamma = 0$. If we consider these tangents to be the lines $\{AB, BC\}$, then last line is by assumption a fixed one ε through $O : \alpha - k\gamma = 0$, implying that $tt' = k$ is constant. Analogously the line $\delta = \alpha - k'\gamma = 0$ satisfies $k' = tt''$ where t'' is the point of contact of AC with f. Consequently the point C is the intersection point of the tangents at the points of f :

$$t' = \frac{k}{t} \quad \text{and} \quad t'' = \frac{k'}{t}\,.$$

Thus, these tangents satisfy

$$\left.\begin{aligned} k^2\gamma - 2k\beta t + t^2\alpha = 0 \\ k'^2\gamma - 2k'\beta t + t^2\alpha = 0 \end{aligned}\right\} \quad \Rightarrow \quad \alpha \cdot \gamma - \frac{4kk'}{(k + k')^2}\beta^2 = 0\,,$$

latter equation resulting by eliminating t and representing a conic bitangent to the conic $f = \alpha\gamma - \beta^2 = 0$. □

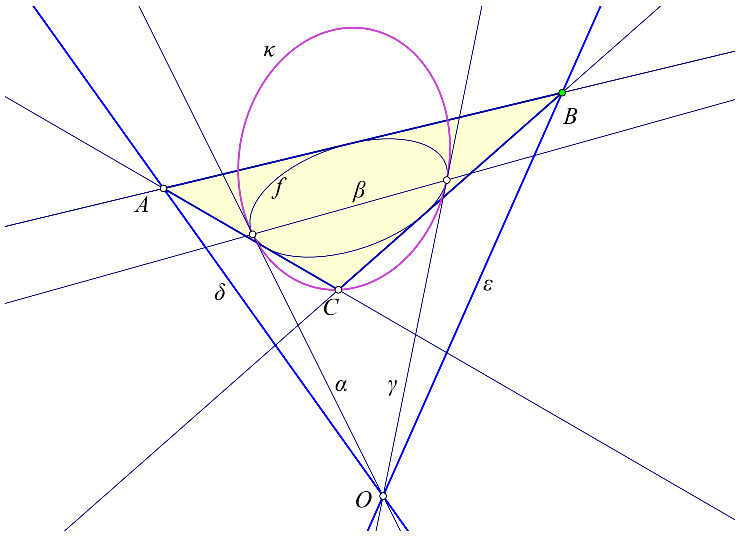

Figure 53: Variable triangle ABC circumscribing the conic f

Next theorem could be considered as a definition of a conic analogous to the one of the Chasles-Steiner method of section 9, now considering the conic in its *"dual"* aspect as envelope of its tangents.

Theorem 15 (Chasles-Steiner dual)**.** *Given a homographic relation $Y = f(X)$ between the points of two lines $\{\alpha, \beta\}$, the lines $\{XY\}$ joining corresponding points envelope a conic tangent to the two lines. Conversely, fixing two tangents $\{\alpha, \beta\}$ of a conic κ, then any other tangent γ of the conic intersects these two in points respectively $\{X, Y\}$ related homographically (see figure 54).*

Proof. This can be reduced to theorem 13 by considering the two lines $\{\alpha, \beta\}$ as a degenerate conic. Then, taking three arbitrary pairwise different points $\{X_1, X_2, X_3 \in \alpha\}$ and their images $\{Y_1, Y_2, Y_3 \in \beta\}$, the hypothesis and the basic property of homographies, to preserve the cross ratio, implies, that for a fourth point $X \in \alpha$ and its image $Y \in \beta$ the cross ratios are equal: $(X_1X_2; X_3X) = (Y_1Y_2; Y_3Y)$. By theorem 13 this implies that XY envelopes a conic as stated.

Conversely, considering three fixed tangents of the conic κ and a variable fourth tangent, we know by corollary 5, that the cross ratios defined by the intersections of the four lines with the two tangents $\{\alpha, \beta\}$ are equal:

$$(X_1X_2; X_3X) = (Y_1Y_2; Y_3Y) \quad \text{with} \quad X_1, X_2, X_3, X \in \alpha \quad \text{and} \quad Y_1, Y_2, Y_3, Y \in \beta.$$

But this equation defines a homographic relation $Y = f(X)$ between the lines $\{\alpha, \beta\}$ as claimed. □

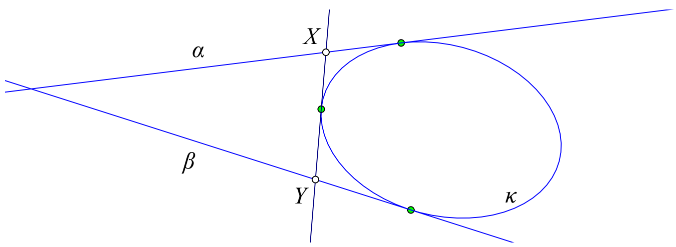

Figure 54: Conic defined by a homography $Y = f(X)$ between two lines $\{\alpha, \beta\}$

13 Polygons inscribed and circumscribed about conics

Here we discuss another application of the abridged method concerning polygons inscribed in a fixed conic and having their sides pass through fixed points.

Theorem 16. *If a polygon of n sides is inscribed in a conic κ and its $(n-1)$ sides pass through corresponding pairwise different fixed points, then its $n-$th side envelopes another conic κ' bitangent to κ.*

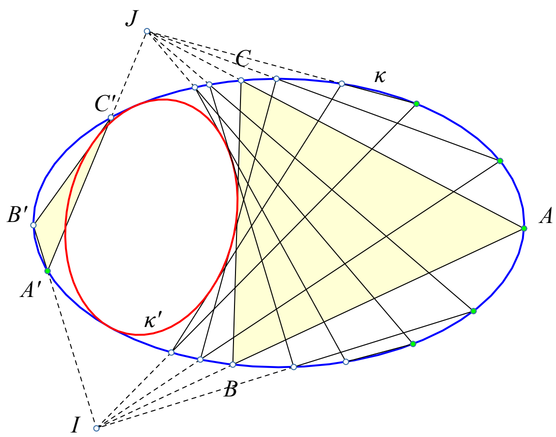

Figure 55: Triangles inscribed in κ with sides $\{AB, AC\}$ passing through $\{I, J\}$

Proof. Figure 55 displays such an example for triangles. Triangle ABC moves having all the time its vertices on the conic κ and two of its sides passing through two corresponding fixed points $\{I, J\}$. Then the third side BC envelopes another conic κ' which is bitangent to κ. The proof reduces to the theorem 13 by taking three different positions of the triangle and defining the corresponding chords $\{B'C', B''C'', B'''C'''\}$. Then, for the moving fourth triangle ABC we have the preservation of cross ratio $(B'B''; B'''B) = (C'C''; C'''C)$.

This, because the central correspondence (involution at) from I, $f_I : B \to A$ preserves the cross ratio. This is due to the *"homographic relation"* (see equation 27)

$$ptt' + (t + t')q + s = 0 \quad \text{with constants} \quad p, q, s$$

and the simple fact that such relations preserve the cross ratio. The same is true for the correspondence from J, $f_J : A \to C$. Thus, composing the two correspondences we have the map $f = f_J \circ f_I : B \to C$ preserving the cross ratio. Hence, theorem 13 applies for the moving side BC. The proof is easily generalized for any n by considering the analogous composition $f = f_{I_{n-1}} \circ \ldots \circ f_{I_1}$ of involutions defined by the given points $\{I_k, k = 1 \ldots n-1\}$. □

Next proposition generalizes theorem 14, its proof suggesting also another aspect of the subject.

Theorem 17. *A polygon $ABCD\ldots$ is circumscribed about a conic κ and has its $n-1$ vertices gliding respectively on $n-1$ lines $\{\alpha, \beta, \gamma, \ldots\}$. Then its n-th vertex describes a conic κ' bitangent to κ.*

Proof. Figure 56 shows the case of a quadrangle (n=4). The theorem can be deduced from theorem 16. We discuss this case, the general one being easily proved using analogous

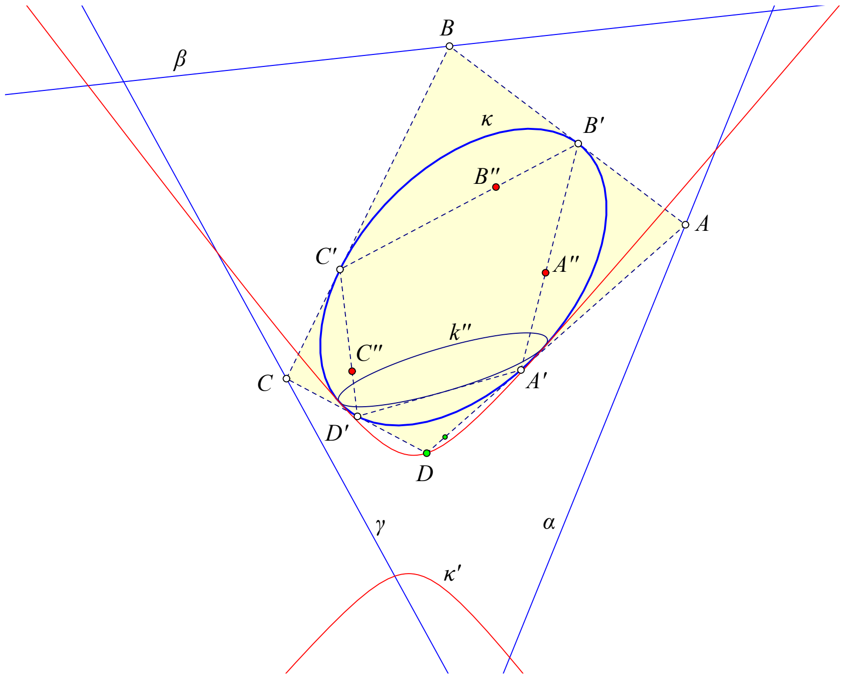

Figure 56: Quadrangle $ABCD$ circumscribed to conic κ

arguments. Since the points $\{A, B, C\}$ move on fixed lines $\{\alpha, \beta, \gamma\}$ their tangents define the corresponding polars $\{A'B', B'C', C'D'\}$, which pass through respective fixed points $\{A'', B'', C''\}$. Thus, for every position of the quadrangle $ABCD$ we have a corresponding (dual) inscribed in κ quadrangle $A'B'C'D'$ whose three sides pass respectively through the fixed points $\{A'', B'', C''\}$. From theorem 16 we know that its fourth side $D'A'$ is tangent to a conic κ'' bitangent to κ.

This implies that the fourth point D of the quadrangle is the pol of a variable tangent $D'A'$ of the conic κ''. By theorem 4 point D describes a conic κ' belonging to the bitangent pencil gnerated by κ and κ''. □

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Related material

1. **Cross ratio**
2. **Homographic relation**
3. **Projective line**
4. **Projective plane**
5. **Quadratic equation**

Any correction, suggestion or proposal from the reader, to improve/extend the exposition, is welcome and could be send by e-mail to: pamfilos@uoc.gr