Apollonian circles

A file of the Geometrikon gallery by Paris Pamfilos

> Discovery is the ability to be puzzled by simple things.
>
> *Noam Chomsky, Chicago Tribune 1/1/1993*

Contents

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1 Apollonian circles of a segment

Given a segment AB and a positive real number k, the *"Apollonian circle of the segment AB for the ratio k"* is defined to be the geometric locus of points P, such that $|PA|/|PB| = k$. If $\{PC, PD\}$ are the internal and external bissectors of the angle $\widehat{APB}$, it is a well known

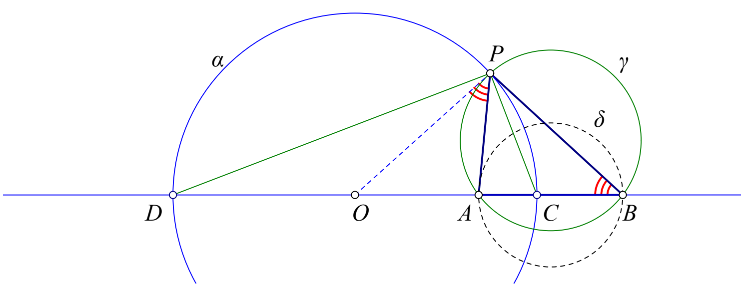

Figure 1: An Apollonian circle α

elementary theorem ([Cou80, p.15]), that $|CA|/|CB| = |DA|/|DB| = k$, hence $\{C, D\}$ which are well defined points on the line AB, belong to the locus (see figure 1) and are harmonic conjugate w.r.t. $\{A, B\}$. But the angle $\widehat{CPD}$ is a right one. Thus, every point P of the locus is viewing the segment CD under a right angle. Thus, it belongs to the circle with diameter CD.

Theorem 1. *Some further properties of figure 1 are the following*:

1. *The angles $\widehat{OPA}$ and $\widehat{PBO}$ are equal.*
2. *The triangles $\{OPA, OPB\}$ are similar.*
3. *It is $OP^2 = OA \cdot OB$, the circles $\{\alpha, \gamma\}$ are orthogonal and $\{D, C\}$ are harmonic conjugate relative to $\{A, B\}$.*
4. *For a constant AB and variable k the "Apollonian circles α for k" form a "non-intersecting pencil of circles"*

Proof. *Nr-1*. To see this, add to $\widehat{OPA}$ the angle $\widehat{APC}$ and get $\widehat{OPC}$. Add to $\widehat{PBO}$ the angle $\widehat{CPB} = \widehat{APC}$ (because PC bisector) to get $\widehat{OCP} = \widehat{OPC}$ from the isosceles OPC.
Nr-2 follows from *nr-1*. To prove the remaining properties see that triangles $\{OPA, OPB\}$ being similar implies $|OP||OP| = |OA||OB|$. This means: (i) that OP is tangent to the circumcircle γ of ABP, (ii) that the Appolonius circle α is orthogonal to the circumcircle γ of ABP, hence also to the circle δ on diameter AB. Thus, all these Apollonian circles form the pencil of circles orthogonal to δ and line AB. □

Exercise 1. *Given a circle $\alpha(O)$ and a point $A \neq O$ show that there is precisely one segment AB for which α is an Apollonian circle (see figure 1).*

Exercise 2. *Given two consecutive segments $\{AC, CB\}$ on a line, show that the geometric locus of the points X viewing the segments under equal angles is an Apollonian circle of the segment AB (see figure 1).*

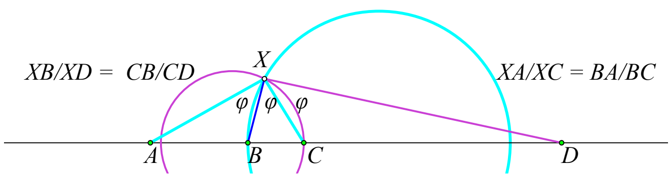

Figure 2: Point X viewing the segments $\{AB, BC, CD\}$ under equal angles

Exercise 3. *Given three consecutive segments $\{AB, BC, CD\}$ on a line, find a point X viewing these segments under equal angles (See Figure 2). Find conditions of existence of such a point.*

2 Inversion properties of Apollonian circles

As we saw in the preceding section, an Apollonian circle and the circumcircle of $\triangle ABC$ are orthogonal, consequently the **Inversion** w.r.t to either of these two circles leaves the other invariant. We notice also that the inversion w.r.t. the Apollonian circle α interchanges $\{B, C\}$ and fixes A. It follows that a circle λ tangent to the circumcircle κ of $\triangle ABC$ at A remains also invariant by the inversion w.r.t. α. Further, a circle μ through $\{A, B\}$ maps by the α-inversion to a circle μ' and the angles between the circumcircle κ and these two inverse to each other circles are equal ($\phi = \phi'$ in figure 3). This is of importance for the next property.

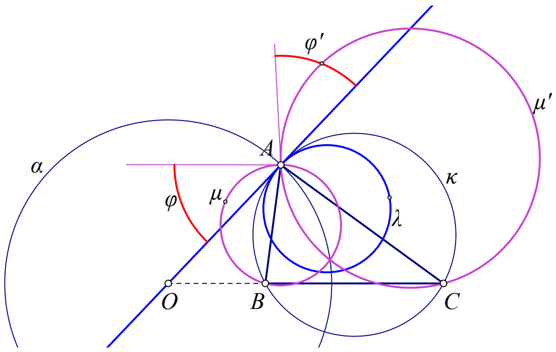

Figure 3: Circles inverted w.r.t. the Apollonian circle α

Theorem 2. *Let P' be the inverse of point P w.r.t. the Apollonian circle $\alpha(O)$ of the triangle ABC and suppose that both $\{P, P'\}$ are inside the triangle Then, the difference of the angles is preserved by the α-inversion (see figure 4)*

$$\phi_C = \widehat{APB} - \widehat{C} \; = \; \widehat{AP'C} - \hat{B} = \phi_B \; .$$

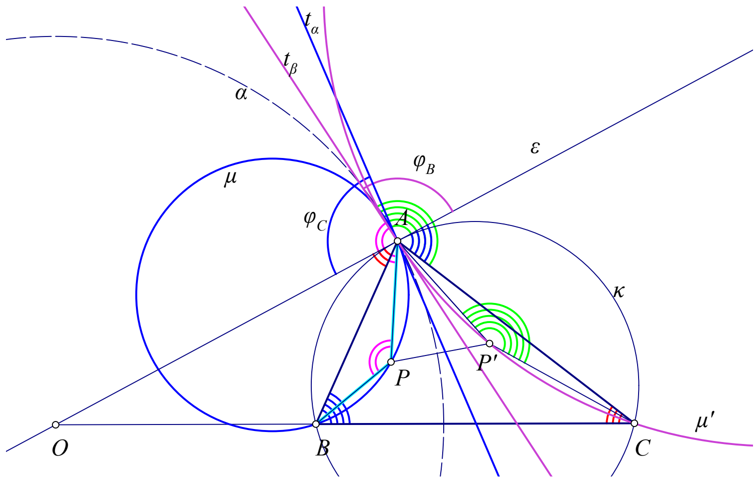

Figure 4: Preservation of the angle-difference

Proof. By the figure. Each of the four angles defines a corresponding angle between the tangents of the circles at A. The difference ϕ_C corresponds by the inversion to $\phi_B = \phi_C$ because the inversion is *"conformal"*, i.e. respects angle measures. ☐

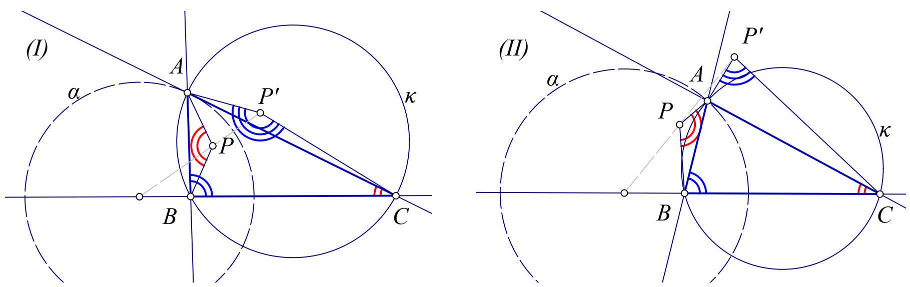

Figure 5: Angle relations by α-inversions

Remark 1. There is a multitude of similar results concerning the α-inverses P' of points P lying in the various domains defined by a triangle its side lines and its circumcircle. These results depend on the location of P and P' and take various forms, which can be unified using, instead of *"angles"* the concept of *"directed angle"* introduced by Johnson ([Joh60, p.13, p.52]). Figure 5 shows two other cases in which the angles are differently related:

$$\text{In (I)}: \quad \widehat{APB} - \widehat{C} = 2\pi - \widehat{AP'C} - \widehat{B}, \quad \text{and in (II)}: \quad \widehat{APB} + \widehat{C} = \widehat{AP'C} + \widehat{B}.$$

These and all the analogous results play a key role in the discussion on **Pedals**, in proving that the α-inverse P' of P defines a pedal triangle w.r.t. $\triangle ABC$ similar to the pedal of P.

Theorem 3. *Two points $\{I, J\}$ which are inverse w.r.t. to the Apollonian circle $\alpha(O, r)$ passing through the vertex A of $\triangle ABC$ define ratios satisfying $\frac{IB}{IC} \cdot \frac{JB}{JC} = \frac{AB^2}{AC^2}$. Conversely, if the points $\{I, J\}$ satisfy this relation and are on a circle λ passing through $\{B, C\}$ and on the same arc defined by BC, then they are inverse w.r.t. the Apollonian circle $\alpha(O, r)$ (see figure 6).*

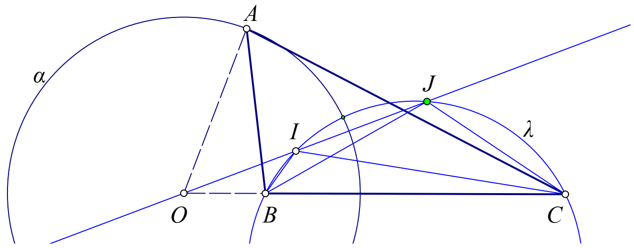

Figure 6: Characterization of inverses $\{I, J\}$ w.r.t. Apollonian circle α

Proof. In fact, if the points are inverse w.r.t. κ, then $\{(OIB \sim OCJ), (OJB \sim OIC)\}$ are pairs of similar triangles and we have:

$$\begin{aligned}\frac{IB}{IC} \cdot \frac{JB}{JC} &= \frac{IB}{JC} \cdot \frac{JB}{IC} = \frac{OB}{OJ} \cdot \frac{OB}{OI} = \frac{OB^2}{r^2} = \frac{OB}{(r^2/OB)} = \frac{OB}{OC} \\ &= \frac{OB}{AB} \cdot \frac{AB}{OC} = \frac{r}{AC} \cdot \frac{AB}{OC} = \frac{AB}{AC} \cdot \frac{r}{OC} = \frac{AB^2}{AC^2}.\end{aligned}$$

Latter because $\frac{AB}{AC} \cdot \frac{OC}{R} = 1$. For an alternative proof of the last step see exercise 4 below.

Conversely, if the points $\{I, J\}$ satisfy the stated conditions, consider the inverse I' of I w.r.t. the Apollonian circle. By the first part of the theorem I' will satisfy the stated condition too and will lie on the same Apollonian circle with J and also on the same circle-arc through $\{B, C\}$, hence it will coincide with J. □

Remark 2. The Apollonian circles of the segment AB form one of the main categories of *"circle pencils"*, namely that of *"non-intersecting type"*. All types of circle pencils are studied in the file **Circle Pencils**.

3 The Apollonian pencil of a segment

Consider a segment AB and the Apollonian circles dividing it in various ratios. All these circles are members of a (coaxal) *"hyperbolic pencil"* of circles, which I call the *"Apollonian pencil"* of the segment. This is a pencil of *"non-intersecting type"*, i.e. two member-circles do not intersect. The common radical axis of the member-circles of the pencil is the per-

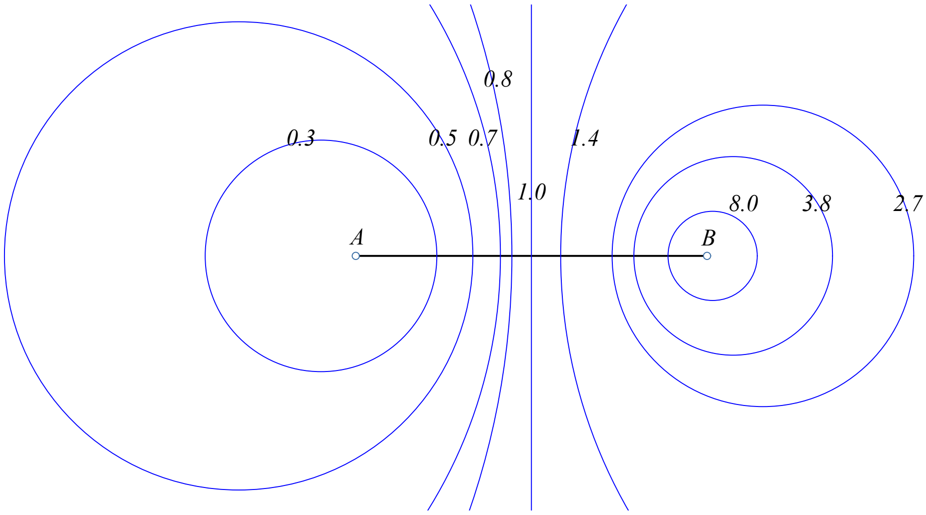

Figure 7: Apollonian pencil of segment AB

pendicular bisector line of the segment. As is with every pencil, the Apollonian pencil is generated, i.e. completely defined, from two particular members, one of which can be the perpendicular bisector ε of the segment. The points $\{A, B\}$, called the *"limit points"* of the pencil are considered as member-circles with radius zero. Figure 7 shows some circles of such a pencil labeled by their characteristic ratio satisfying $|XA|/|XB| = k$ for all points of the corresponding circle. The perpendicular bisector line is the particular case for $k = 1$.

Theorem 4. *In the following formularium are listed the formulas for various distances concerning the Apollonian circle α of AB relative to the ratio $k > 0$ (see figure 8).*

1. $\frac{|XA|}{|XB|} = k$, *for* $X \in \alpha$ *implies the following:* $CA = -\frac{k}{1+k}AB$, $CB = \frac{1}{1+k}AB$,
2. $DA = \frac{k}{1-k}AB$, $DB = \frac{1}{1-k}AB$, $CM = \frac{1-k}{2(1+k)}AB$, $DM = \frac{1+k}{2(1-k)}AB$,
3. $OA = \frac{k^2}{1-k^2}AB$, $OB = \frac{1}{1-k^2}AB$, $OM = \frac{1+k^2}{2(1-k^2)}AB$, $|OZ| = \frac{k}{|1-k^2|}|AB|$.

The Apollonian circles of AB relative to the ratios k and $\frac{1}{k}$ have equal radii and lie symmetrically relative to the medial line of AB (see figure 8).

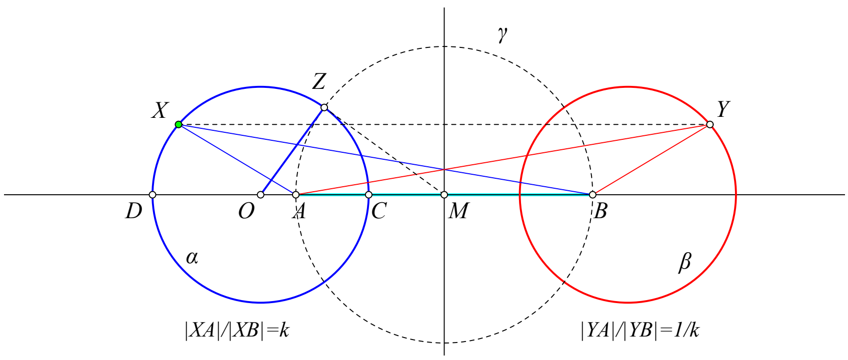

Figure 8: Symmetric lying Apollonian circles of AB

Proof. The formulas are easily computed. The geometric property by which the circles corresponding to ratios k and $\frac{1}{k}$ lie symmetrically relative to the orthogonal bisector of the segment AB, results from the fact that for every point X which satisfies $\frac{|XA|}{|XB|} = k$ its symmetric Y relative to the perpendicular bisector of AB will satisfy $\frac{|YA|}{|YB|} = \frac{1}{k}$. □

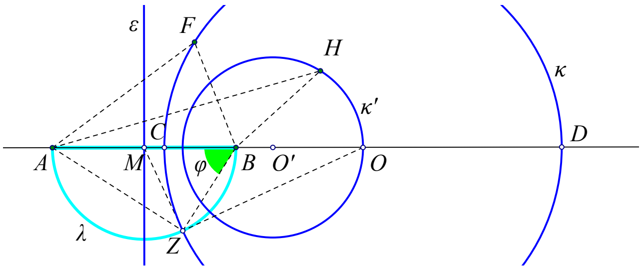

Figure 9: A member circle $\kappa(O)$ and the member κ' passing through O

A half-circle λ with diameter the segment AB gives another parametrization of the pencil, in the sense, that every member-circle κ cuts λ in exactly one point Z and satisfies $k = |ZA|/|ZB| = \tan(\phi)$. This leads to another way to construct the member-circle corresponding to ratio k :

1. Find $\phi = \arctan(k)$ and locate point Z on the half-circle.
2. Draw ZO orthogonal to ZM at Z. This determines the center O and the radius $|OZ| = |AB||\tan(2\phi)|/2$ of the $k-$ratio member-circle.

Notice that λ is part of the minimal member-circle of the pencil $\mathcal{A}$ whose member-circles are orthogonal to the Apollonian circles of AB. Figure 9 shows also the member-circle κ' passing through the center O of κ, for which the following property holds.

Exercise 4. *If $\kappa(O)$ is an Apollonian circle of the segment AB characterized by the ratio k, then the Appolonian circle passing through the center O of κ is characterized by the ratio κ^2.*

4 Related exercises

Exercise 5. *Construct a triangle ABC from its sides $\{b = |AC|, c = |AB|\}$ and its bisector $d = |AD|$ from A.*

Hint: Consider an arbitrary segment $B'C'$ of length a'. Construct its Apollonian circle κ_1, for the ratio $\lambda = b/c$, which intersects $B'C'$ at its inner point D'. Construct then the Apollonian circle κ_2 of the segment $D'C'$ for the ratio $\mu = d/b$. If A' is an intersection point of the circles $\{\kappa_1, \kappa_2\}$, then the triangle $A'B'C'$ is similar to the wanted, with a known similarity ratio.

Exercise 6. *Given the triangle ABC, to determine a point X, which has the two ratios of distances $|XA|/|XB| = \kappa$ and $|XB|/|XC| = \lambda$. Show that for given $\{\kappa, \lambda\}$, there exist in general two points $\{X, X'\}$, whose line XX' passes through the center K of the circumcircle $\mu(K, r_\mu)$ of the triangle ABC. Show also that if $\kappa\lambda = \sigma$ is constant, then the points $\{X\}$ are on a fixed circle.*

Hint: X is one of the intersection points of the two Apollonian circles $\{\alpha\beta, \beta\gamma\}$, whose points satisfy respectively $|XA|/|XB| = \kappa$ and $|XB|/|XC| = \lambda$ (See Figure 10). Then it will also hold $|XA|/|XC| = \kappa \cdot \lambda$ and the corresponding Apollonian circle for AC relative to the ratio $\kappa\lambda$ will also pass through points $\{X, X'\}$. The three circles are orthogonal to the circumcircle μ, hence the center K of μ is contained in the common radical axis of $\{\alpha\beta, \beta\gamma, \gamma\alpha\}$. From the fact that the power of K relative to $\alpha\beta$ is equal to r_μ^2, follows

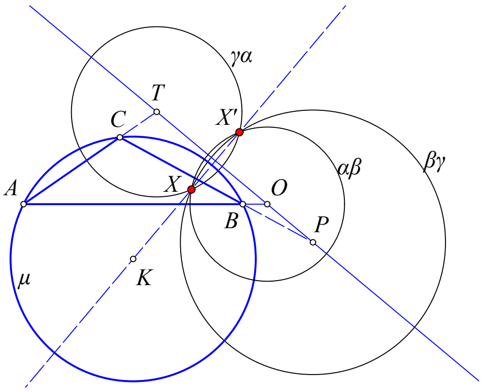

Figure 10: Three circles forming a coaxal pencil of circles

$|KX| \cdot |KX'| = r_{\mu}^2$. This shows that the two points $\{X, X'\}$ are coincident only in the case when the circles $\{\alpha\beta, \beta\gamma, \gamma\alpha\}$ are tangent at a point X contained in the circumcircle μ of ABC and then the common tangent to these circles coincides with XK.

Remark 3. The preceding exercise generalizes the construction of the *"three Apollonian circles of a triangle"* (see file **Isodynamic points of the triangle**).

Exercise 7. *Given is a line η, a point H not lying on it and a constant $\kappa > 0$. Point A moves along a given line α. There exist at most two positions of A on line α, for which holds $|AH| = \kappa|AX|$, where X is the projection of A onto η.*

Hint: Assume that the line α intersects line η at point C (see figure 11-(I)). Then, for every point A of α the ratio $\lambda = \frac{|AX|}{|AC|}$ will be fixed. Consequently, for the wanted positions of A, the ratio $\mu = \frac{|AH|}{|AC|} = \frac{|AH|}{|AX|} \cdot \frac{|AX|}{|AC|} = \kappa\lambda$ will be known. It follows that the wanted points A coincide with the intersection points of line α and the Apollonian circle of segment HC relative to ratio μ. In the case where α is parallel to η, segment $|AX|$ is a known fixed length and consequently segment $|AH| = \kappa|AX|$ is also known (see figure 11-(II)). In this case then, the wanted points are again intersections of a circle and a line.

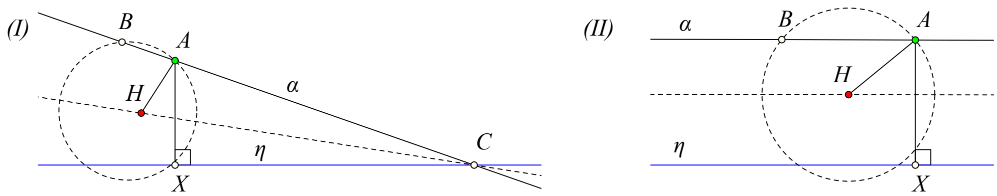

Figure 11: Intersections of a line and a conic

Remark 4. The last proposition gives the geometric proof of the fact that a line (α) intersects a conic in at most two points ([IS01, p.42]).

Exercise 8. *Construct a triangle, for which are given the side $a = |BC|$, the altitude from the opposite side v_A and the trace of the bisector (internal or external) from A.*

Exercise 9. *Construct a triangle, for which are given the side $a = |BC|$, the angle $\alpha = \widehat{BAC}$ of the opposite vertex and the trace of the bisector (internal or external) from A.*

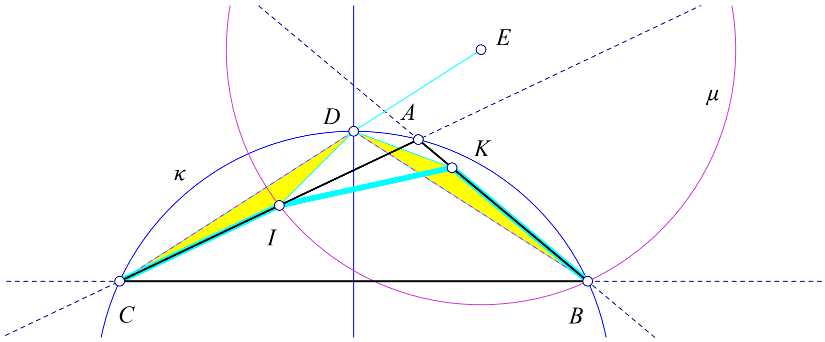

Figure 12: Construct equal segments $CI = IK = KB$

Exercise 10. *Construct a triangle, for which are given the side $a = |BC|$, and the traces of the altitude and the bisector on that side.*

Exercise 11. *Find points $\{I, K\}$ on the sides of a triangle ABC, such that segments $\{CI, IK, KB\}$ are equal in length.*

Hint: ([Yag62, I,p.133]) Since $\{IC, KB\}$ are assumed to be equal, there is a rotation about some point D that brings the one to the other. Point D can be determined without knowing the exact lengths of $\{CI, IK, KB\}$. In fact, the center of the rotation has to be on the medial line of BC and triangles $\{DCI, DBK\}$ must be equal, the angles at D being $\widehat{CDB} = \widehat{IDK} = \widehat{CAB}$. Thus D is one of the intersection points of the circumcircle κ of ABC with the perpendicular bisector of BC. Now the isosceles DCB with a known angle at D has a known ratio of side-lengths (depending on $\widehat{CAB}$)

$$k = CI/ID = IK/ID = CB/DC = CB/DB.$$

Hence I lies on the Apollonian circle μ of the segment CD, relative to the ratio k.

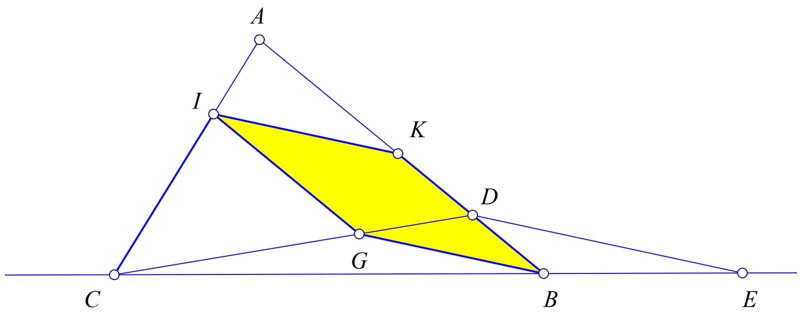

Figure 13: Construct equal segments $CI = IK = KB$ (II)

Figure 13 suggests another way to solve the problem. In fact, start by assuming that $\{I, K\}$ have been constructed, AC is shorter than AB and the ratio $r = CI/CA$ is known. Define D on AB, so that $AD = AC$. The homothety $f_{C,r}$ (center at C, ratio r) maps D to a point G, so that $IG = IC$. Thus, $BGIK$ is a parallelogram and $IK = IC = IG$, implying that it is a rhombus. Draw a parallel to BG through D intersecting BC at E. Because $DA = DE$, this is a constructible point and maps through $f_{C,r}$ on B. Thus, $r = BC/BD$ can be determined from the data. Having the homothety $f_{C,r}$ the construction is obvious.

Exercise 12. *To construct a triangle ABC from the angle A and the sums of its side-lengths $a + c$ and $a + b$.*

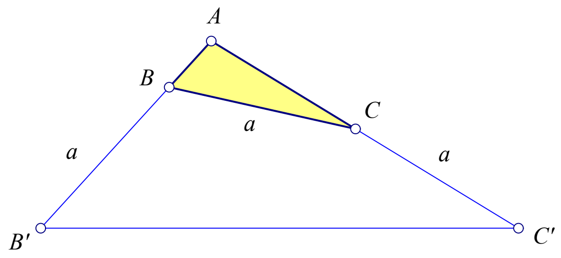

Figure 14: Triangle construction from $\{\widehat{A}, a+b, a+c\}$

Hint: Use the previous exercise starting from the constructible $AB'C'$ and locating $\{B, C\}$.

Bibliography

[Cou80] Nathan Altshiller Court. *College Geometry*. Dover Publications Inc., New York, 1980.

[IS01] Italo D' Ignazio and Ercole Suppa. *Il Problema Geometrico, dal compasso al cabri*. interlinea editrice, Teramo, 2001.

[Joh60] Roger Johnson. *Advanced Euclidean Geometry*. Dover Publications, New York, 1960.

[Yag62] I. M. Yaglom. *Geometric Transformations I, II, III, IV*. The Mathematical Association of America, 1962.

Related material

1. **Circle Pencils**
2. **Inversion**
3. **Isodynamic points of the triangle**
4. **Pedals**

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