Ceva's theorem

A file of the Geometrikon gallery by Paris Pamfilos

To see what is in front of one's nose
requires a constant struggle.

G. Orwell, In Front of Your Nose

Contents

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1 Ceva's theorem

Ceva's theorem, and its older brother Menelaus' theorem, deal with *"signed ratios"* of segments, which are properly defined in *"affine geometry"*. In euclidean geometry the theorem has the following formulation.

Theorem 1. *A necessary and sufficent condition, that the three points $\{A', B', C'\}$ on respective sides $\{BC, CA, AB\}$ of the triangle ABC, define three lines $\{AA', BB', CC'\}$ intersecting at the same point P not lying on the side-lines of ABC, is (see figure 1)*

$$\frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B} = -1. \tag{1}$$

Proof. Here let us first observe that, for every point P of the plane, not lying on the side-lines of the triangle, the intersections $\{A', B', C'\}$ of $\{AP, BP, CP\}$ with the respective opposite sides either all are contained in the interiors of the sides or exactly one of them is contained in the interior and all others are in the exterior, hence the sign -1.

Now, for the proof, paying attention to the correct signs. Draw a parallel from one vertex, for example from A to the base BC. This creates pairs of similar triangles:

$$(PBC, PYZ), \qquad (B'BC, B'YA), \qquad (C'BC, C'AZ).$$

From the side proportions of these similar triangles we have the equalities:

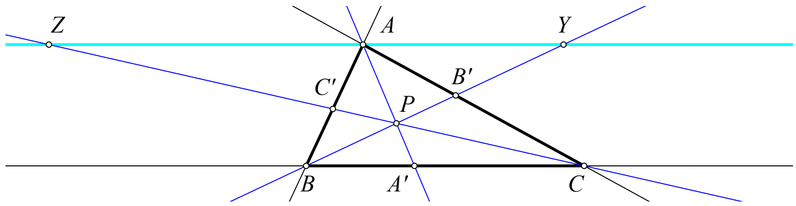

Figure 1: Ceva's theorem

$$\frac{A'B}{A'C} = -\frac{|AY|}{|AZ|}, \quad \frac{B'C}{B'A} = -\frac{|BC|}{|AY|}, \quad \frac{C'A}{C'B} = -\frac{|AZ|}{|BC|}.$$

The claimed relation follows by multiplying these by parts and simplifying.

For the converse, let us assume that the two lines $\{AA', BB'\}$ intersect at point P and also assume that C'' is the intersection point of CP with AB. Then, according to the proved part of the theorem, will hold

$$\frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C''A}{C''B} = -1, \quad \text{and by assumption} \quad \frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B} = -1.$$

From these two follows immediately

$$\frac{C''A}{C''B} = \frac{C'A}{C'B},$$

which shows that $C'' = C'$. □

2 Cevians

Because of Ceva's theorem the term *"cevian"* has been established to mean a line segment starting at a vertex of a triangle, ending on the opposite side and passing through a point

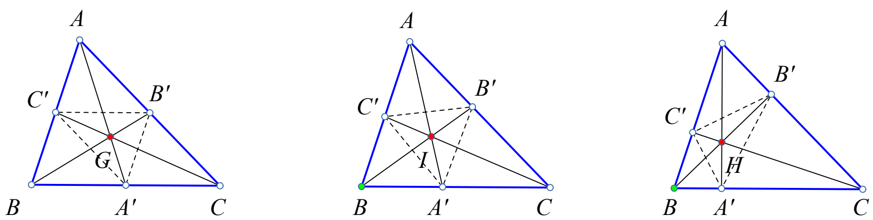

Figure 2: Medians Bisectors Altitudes

P not lying on the side-lines of the triangle. The most prominent cevians are the *"medians"*, the *"bisectors"*, the *"altitudes"* and the *"symmedians"* of the triangle. The medians pass through the middles of the opposite sides. The bisectors divide the opposite sides in the signed ratios $A'B/A'C = -|AB|/|AC|, \ldots$ and the altitudes are orthogonal to respective opposite sides. Figure 2 shows these lines together with the corresponding *"cevian triangles"* defined by the traces of the corresponding cevians on the opposite sides. In the case of the medians the triangle $A'B'C'$ is the *"medial"* triangle of ABC. In the case of the bisectors $A'B'C'$ is the *"incentral"* triangle of ABC. In the case of the altitudes $A'B'C'$ is the *"orthic"* triangle of ABC. The proof that these lines pass really through the corresponding points $\{G, I, H\}$ is an easy exercise in the use of Ceva's theorem. These three points are respectively the *"centroid, incenter"* and *"orthocenter"* of the triangle.

There are lots of analogous *"centers, cevians"* and *"cevian triangles"* of the triangle ABC, examined in the frame of *"triangle geometry"* (see [Kim97] and [Kim18]). Figure 3 shows

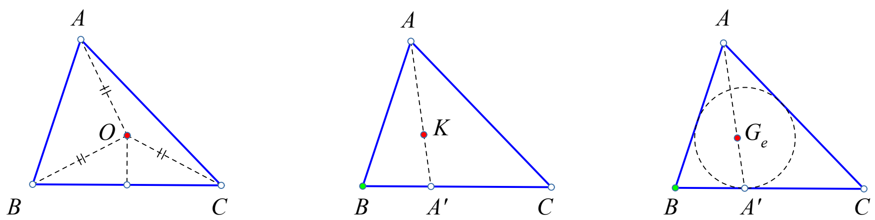

Figure 3: Circumcenter Symmedian point Gergonne point

three other remarkable *"triangle centers"*: the center of the circumcircle: *"circumcenter"*, the *"symmedian point"*, defined as intersection of the *"symmedian lines"* and the *"Gergonne point"*, defined as intersection of the lines joining the vertices with the contact points of the *"incircle"* with the opposite sides. The two last points are discussed in short in the file **Symmedian center of the triangle**. Figure 4 shows three other remarkable *"triangle centers"*. The first is the *"Nagel point"*, through which pass the lines $\{AA', \ldots\}$ joining the vertices with the opposite side contact point of the corresponding *"excircle"* or *"tritangent circle"* (see files **Nagel center of the triangle** and **Tritangent circles of the triangle**). The second is the *"Fermat"* point, from which each side is seen under $120°$. The third is the *"1st isodynamic point"*, which together with the *"2nd isodynamic point"* are the common intersection points of the three *"Apollonian circles"* handled in the file **Apollonian circles** (see also file **Isodynamic points of the triangle**).

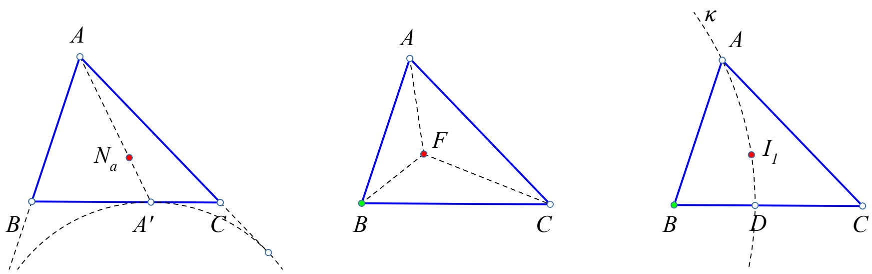

Figure 4: Nagel point Fermat point 1st Isodynamic point

3 Relation of Ceva's and Menelaus' theorems

Theorem 2. *Given is a triangle ABC and three points $\{A', B', C'\}$ on its sides and different from its vertices. Then the relation of ratios of lengths:*

$$\frac{|A'B|}{|A'C|} \cdot \frac{|B'C|}{|B'A|} \cdot \frac{|C'A|}{|C'B|} = 1$$

implies exactly one of the next two propositions:

1. *Points $\{A', B', C'\}$ are collinear.*
2. *The lines $\{AA', BB', CC'\}$ pass through a common point.*
   Also, if C'' is the harmonic conjugate of C' relative to $\{A, B\}$, then (see figure 5),
3. *when (1) occurs, the lines $\{AA', BB', CC''\}$, pass through a common point.*
4. *when (2) occurs, points $\{A', B', C''\}$, are collinear.*

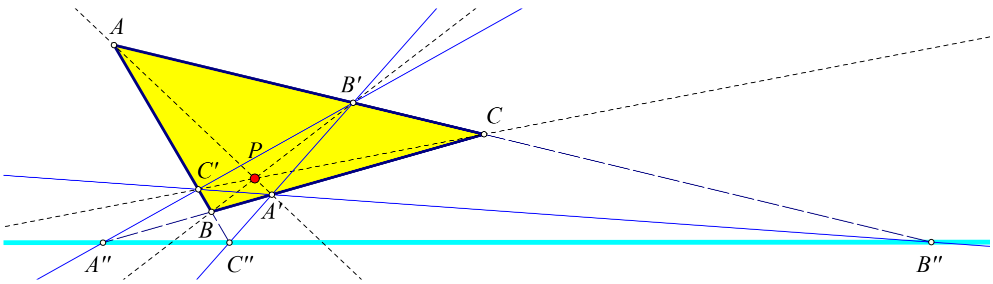

Figure 5: The *"trilinear polar"* of P relative to ABC

Proof. The proof follows directly from the previous theorem and the theorem of Menelaus (see file **Menelaus' theorem**). If the product of ratios of lengths is equal to 1, then the corresponding product of signed ratios will equal 1 or -1. In the first case, by Menelaus' theorem, we have (1). Because of the relation between harmonic conjugates

$$(C', C'') \sim (A, B) \quad \Rightarrow \quad \frac{C'A}{C'B} = -\frac{C''A}{C''B},$$

(3) will also be valid. Similarly in the second case holds simultaneously (2) and (4).

Note that similar properties will be valid also for the other corresponding harmonic conjugates $(A'', A') \sim (B, C)$ and $(B'', B') \sim (A, C)$. □

Remark 1. The last corollary reveals that the two properties, expressed by the theorems of Menelaus and Ceva, are intimately related. To see them in a unifying spirit and to include the symmetry implied in this relationship, we must, along with the three points $\{A', B', C'\}$ on the sides of triangle ABC, consider also their three harmonic conjugates $\{A'', B'', C''\}$ relative to the end points respectively on the sides $\{BC, CA, AB\}$. There results then the interesting figure 5, in which all the coincidences, besides those of $\{A'', B'', C''\}$ on a line, are consequences of the previous propositions. The line which contains the points $\{A'', B'', C''\}$ is called *"trilinear polar"* of P relative to triangle ABC and point P is called *"trilinear pole"* of the line $A''B''$ relative to the triangle ABC. These two notions occupy an important position in the so called *"Geometry of the triangle"*, ([Yiu13], [Gal13]). The fact, that points $\{A'', B'', C''\}$ are collinear is a direct consequence of *"Desargues' theorem"*. If fact, by definition the triangles $\{ABC, A'B'C'\}$ are point-perspective, hence, by Desargues' theorem, they are also line perspective and the trilinear polar is *"their axis of perspectivity"* (see file **Desargues' theorem**).

4 A limit case

Ceva's theorem is valid also in the limit case, in which the point P goes to infinity.

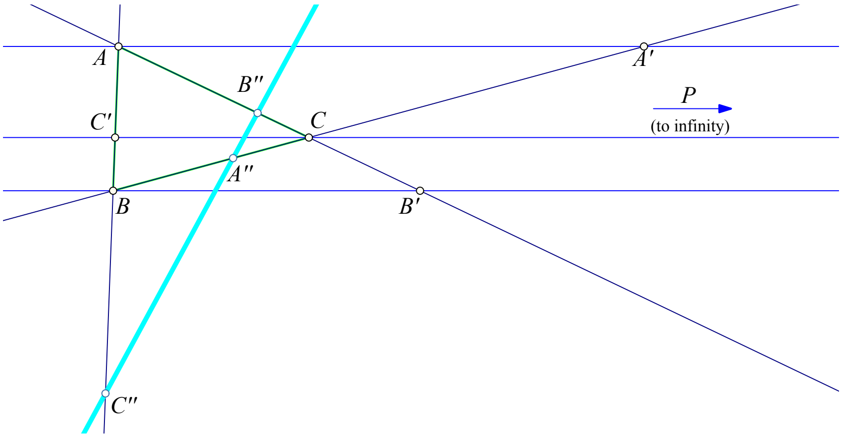

Figure 6: Theorem of Ceva for parallels $\{AA', BB', CC'\}$

In that case the three lines $\{AA', BB', CC'\}$ are three parallels, which we consider as concurring to a point P at infinity. The theorem of Ceva follows, then, from that of Thales

$$\frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B} = -1.$$

Figure 6 shows that case, displaying also the corresponding to P *trilinear polar*. Changing P i.e. changing the direction of parallels, changes also the location of the corresponding trilinear polar. Figure 7 displays the *"envelope"* of all these trilinear polars. It is an ellipse

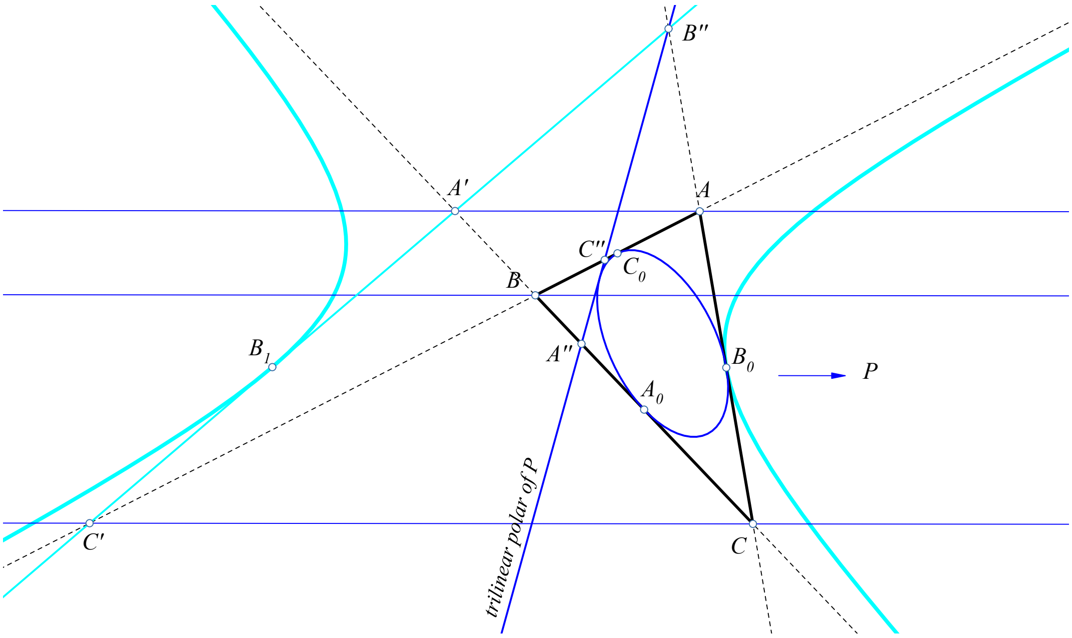

Figure 7: The Steiner inellipse enveloping the trilinear polars

inscribed in the triangle and tangent to its sides at their middles. This is the maximal ellipse that can be inscribed in the triangle, called *"Steiner inellipse"* of the triangle. The figure displays also the envelope of lines $C'A'B''$, which is a hyperbola touching these lines at the middle B_1 of the segment $A'C'$. The center of the hyperbola is the vertex B and the sides $\{BA, BC\}$ are its *"asymptotes"*. There are two other analogous hyperbolas enveloping correspondingly all lines $A'B'C''$ and all lines $B'C'A''$ with centers correspondingly at $\{C, A\}$.

5 A second version of Ceva's theorem

Another form of Ceva's theorem is obtained by introducing two angles at each vertex. The two angles are defined through the respective *"cevian"* and the sides of the triangle. In figure 8 the oriented angles are such that

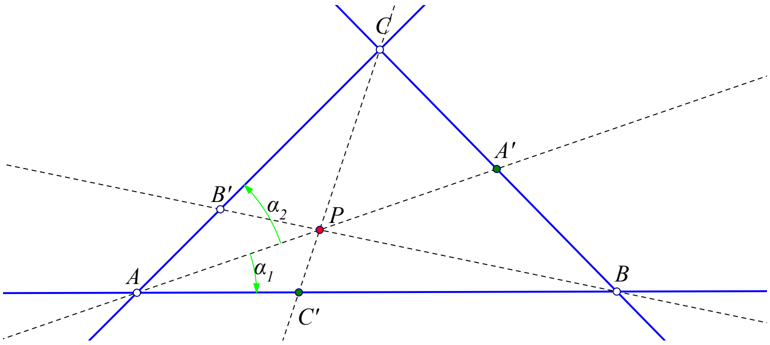

Figure 8: Theorem of Ceva expressed through angles

$$\widehat{A} = \widehat{\alpha_2} - \widehat{\alpha_1}, \qquad \hat{B} = \widehat{\beta_2} - \widehat{\beta_1}, \qquad \widehat{C} = \widehat{\gamma_2} - \widehat{\gamma_1}.$$

Then Ceva's condition is equivalent with ([Ask03, p.41]).

$$\frac{\sin(\alpha_2)}{\sin(\alpha_1)} \cdot \frac{\sin(\beta_2)}{\sin(\beta_1)} \cdot \frac{\sin(\gamma_2)}{\sin(\gamma_1)} = -1. \tag{2}$$

This follows by observing that from the sinus theorem for triangles we have

$$\frac{A'C}{\sin(\alpha_2)} = \frac{A'A}{\sin(\widehat{C})}, \qquad \frac{A'B}{\sin(\alpha_1)} = \frac{A'A}{\sin(\hat{B})}, \qquad \Rightarrow \qquad \frac{A'C}{A'B} : \frac{\sin(\alpha_2)}{\sin(\alpha_1)} = \frac{\sin(\hat{B})}{\sin(\widehat{C})}. \tag{3}$$

Analogous equations to the last one are valid also for the other vertices and multiplying the three corresponding equations and simplifying we get at equation 2.

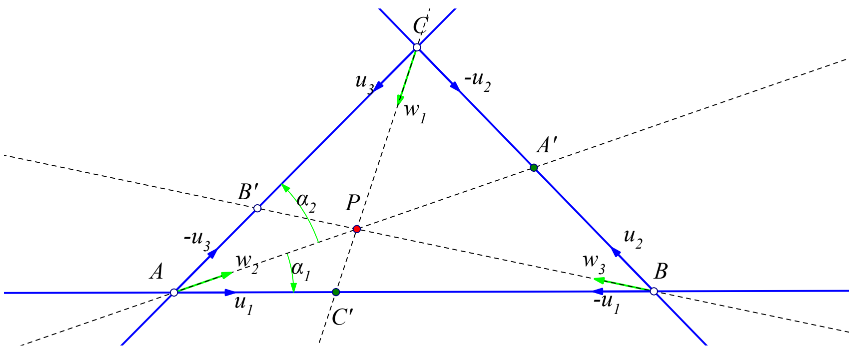

Figure 9: Theorem of Ceva in vectorial form

6 Vectorial form of Ceva's theorem

Another version of Ceva's condition is obtained by introducing unit vectors $\{u_1, u_2, u_3\}$ respectively along the sides $\{AB, BC, CA\}$ and $\{w_2, w_3, w_1\}$ along the cevians $\{AP, BP, CP\}$. Then, denoting by $\langle \ldots, \ldots \rangle$ the usual inner product and by $J(X)$ the transformation that turns every vector by $\pi/2$, we have the equivalent to Ceva's condition:

$$\frac{\langle u_1, J(w_2)\rangle}{\langle u_3, J(w_2)\rangle} \cdot \frac{\langle u_2, J(w_3)\rangle}{\langle u_1, J(w_3)\rangle} \cdot \frac{\langle u_3, J(w_1)\rangle}{\langle u_2, J(w_1)\rangle} = -1. \tag{4}$$

This follows from the second version of Ceva's condition (equation 2) by observing that the sinus of angles can be expressed by inner products:

$$\sin(\alpha_2) = \langle -u_3, J(w_2)\rangle, \qquad \sin(\alpha_1) = \langle u_1, -J(w_2)\rangle \quad \Rightarrow \quad \frac{\sin(\alpha_2)}{\sin(\alpha_1)} = \frac{\langle u_3, J(w_2)\rangle}{\langle u_1, J(w_2)\rangle}.$$

Analogous formulas are valid also for the other angles and the condition follows by substitution into the condition of the previous section.

7 Projective version of Ceva's theorem

A fourth version of Ceva's condition is obtained by intersecting the sides of the triangle and the cevians with an arbitrary line ε as in the figure 10. This defines on each side-line

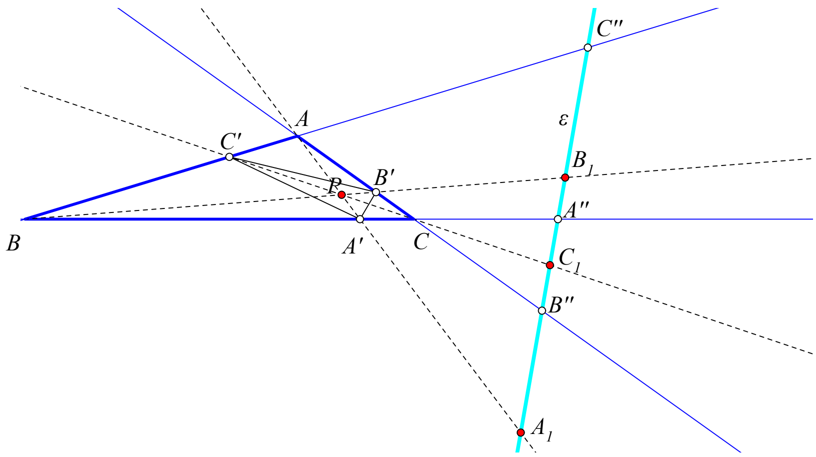

Figure 10: Ceva's theorem, projective version

of the triangle one cross ratio (see file **Cross Ratio**) and the condition of concurrence is that the product of these cross ratios is -1 ([Gre54, p.354]).

$$(BC; A'A'') \cdot (CA; B'B'') \cdot (AB; C'C'') = -1. \tag{5}$$

The proof of this *"projective"* version of Ceva's theorem results from general principles of *"projective geometry"*, according to which *for any two sets of four lines in general position, there is a "projectivity" transforming the first set onto the second*. In particular, taking the first three lines to be the sides of the triangle, the fourth line of the first set to be ε and the corresponding line in the second set to be the *"line at infinity"*, we construct a map sending the given triangle and the line ε to a triangle and the line at infinity.

Then, we use the fact that *"projectivities" preserve the cross ratio* and the fact that *cross ratios with one point at infinity reduce to simple ratios*. Thus, the projective case is reduced to the affine one handled in section 1.

8 Projective version using an arbitrary line

From the previous version of Ceva's theorem results also a fifth version, which can be read on line ε using the six traces of the lines (sides + cevians). This results from the following relation (see figure 10) :

$$(BC;A'A'') = (B_1C_1;A_1A''), \quad (CA;B'B'') = (C_1A_1;B_1B''), \quad (AB;CC'') = (A_1B_1;C_1C'').$$

These, in turn, result by considering the pencils of lines at P. The first equality, for example, results by considering the pencil $P(B,C,A',A'')$ and the two lines $\{BC,\varepsilon\}$ intersecting it and using the fact that *any line intersecting a pencil, defines through its four intersection points a cross ratio independent of its particular position, hence the same for all lines* (see file **Cross Ratio**). Thus, using these equalities and equation 5 we arive at the equivalent to Ceva's theorem condition

$$(B_1C_1;A_1A'') \cdot (C_1A_1;B_1B'') \cdot (A_1B_1;C_1C'') = -1. \tag{6}$$

9 Triangle's ratio coordinates

The ratio coordinates w.r. to triangle ABC locate a point P by the signed ratios $r_i = p_i/q_i$ defined by the cevians through P on the sides of the triangle. Each one of the signed ratios

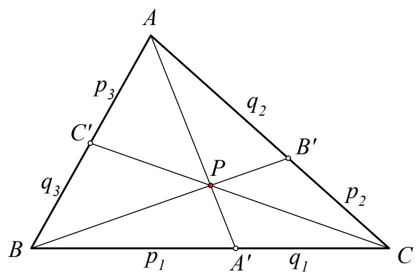

Figure 11: Ratio coordinates $\{r_i = \frac{p_i}{q_i}\}$ of P

determines the corresponding *"trace"* of P on the side of the triangle:

$$r_1 = \frac{p_1}{q_1} = \frac{A'B}{A'C}, \qquad r_2 = \frac{p_2}{q_2} = \frac{B'C}{B'A}, \qquad r_3 = \frac{p_3}{q_3} = \frac{C'A}{C'B}.$$

By Ceva's theorem $r_1r_2r_3 = -1$. Thus, the three numbers are not independent and two of them determine the third. Two of them can be given arbitrarily, the third determined then by the preceding equation. Next theorem leads to the relation of this kind of coordinates with the *"barycentric coordinates"* or *"barycentrics"* (b_1,b_2,b_3) of the point P (see file **Barycentric coordinates**).

Theorem 3. *Let the points $\{B',C'\}$ on the sides $\{AC,AB\}$ of the triangle ABC divide the sides correspondingly in the ratios $\{B'C/B'A = p_2/q_2,\ C'A/C'B = p_3/q_3\}$, then the following relations are valid (see figure 12).*

1. $p_1/q_1 = A'B/A'C = -(p_2/q_2)^{-1}(p_3/q_3)^{-1}$,
2. $QB/QC = -p_1/q_1$,
3. $PA/PA' = p_3/q_3 + q_2/p_2$,

In particular, the ratios p_1/q_1 and PA/PA' do not depend on the lengths of the sides but only on the ratios $\{p_2/q_2, p_3/q_3\}$.

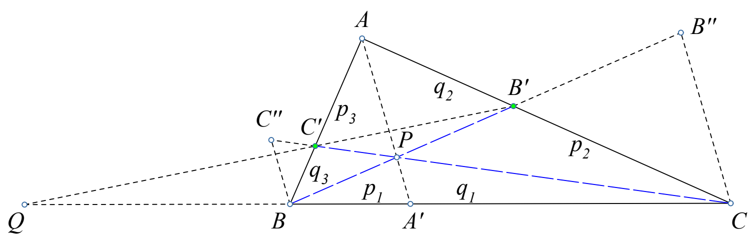

Figure 12: The ratio PA/PA'

Proof. *Nr-1* follows from Ceva's theorem.

Nr-2 follows from the fact that $(Q,A') \sim (B,C)$ are harmonic pairs.

Nr-3. Start with the right side:

$$\frac{p_3}{q_3} = \frac{PA}{C''B'}, \qquad \frac{q_2}{p_2} = \frac{PA}{B''C} \quad \Rightarrow$$
$$\frac{p_3}{q_3} + \frac{q_2}{p_2} = PA\left(\frac{1}{B''C} + \frac{1}{C''B}\right) = \frac{PA}{PA'}\left(\frac{PA'}{B''C} + \frac{PA'}{C''B}\right) = \frac{PA}{PA'}. \qquad \square$$

Corollary 1. *For a point P not lying on a side-line of the triangle ABC its absolute barycentrics (b_1, b_2, b_3) w.r. to the triangle ABC are related to the ratio coordinates (r_1, r_2, r_3) of P through equations:*

$$b_1 = \frac{1}{1 - r_3 + r_1 r_3}, \qquad b_2 = \frac{1}{1 - r_1 + r_2 r_1}, \qquad b_3 = \frac{1}{1 - r_2 + r_3 r_2}.$$

Proof. By the preceding theorem, denoting by (XYZ) the signed area of triangle XYZ, we have (see figure 12)

$$r_3 + r_2^{-1} = \frac{PA}{PA'} = 1 - \frac{AA'}{PA'} = 1 - \frac{(ABC)}{(PBC)} = 1 - b_1^{-1} \quad \Rightarrow$$
$$b_1 = \frac{1}{1 - r_3 - r_2^{-1}} = \frac{r_2}{r_2 - r_3 r_2 - 1} = \frac{r_2}{r_2 - r_3 r_2 + r_1 r_2 r_3} = \frac{1}{1 - r_3 + r_1 r_3},$$

the other relations obtained from this by cyclically permuting the indices. $\square$

Remark 2. For points lying on the side-lines of the triangle the relation of ratio-coordinates to barycentrics can be found directly. In fact, from the proper definition of the barycentrics as a quotient of areas (see file **Barycentric coordinates**) we see that for a point A' on side BC with ratio $r_1 = A'B/A'C$ the corresponding barycentrics are ((see figure 12)):

$$BC \ni A'(r_1) \; : \; \left(0\,,\, \frac{1}{1 - r_1}\,,\, \frac{-r_1}{1 - r_1}\right) \quad \text{and analogously} \tag{7}$$
$$CA \ni B'(r_2) \; : \; \left(\frac{-r_2}{1 - r_2}\,,\, 0\,,\, \frac{1}{1 - r_2}\right), \tag{8}$$
$$AB \ni C'(r_3) \; : \; \left(\frac{1}{1 - r_3}\,,\, \frac{-r_3}{1 - r_3}\,,\, 0\right). \tag{9}$$

As an example application of this case, using the well known area formula in barycentrics

$$(A'B'C') = (ABC)\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \tag{10}$$

in which the columns of the determinant represent the absolute barycentrics of $\{A', B', C'\}$, we can compute the area of the *"cevian triangle"* $A'B'C'$ of the point P in dependence of the ratio coordinates of P,

$$(A'B'C') = (ABC)\begin{vmatrix} 0 & \frac{1}{1-r_1} & \frac{-r_1}{1-r_1} \\ \frac{-r_2}{1-r_2} & 0 & \frac{1}{1-r_2} \\ \frac{1}{1-r_3} & \frac{-r_3}{1-r_3} & 0 \end{vmatrix} = \frac{1 - r_1 r_2 r_3}{(1-r_1)(1-r_2)(1-r_3)} \cdot (ABC). \tag{11}$$

Taking into account that $r_1 r_2 r_3 = -1$ this reduces to

$$(A'B'C') = \frac{2}{(1-r_1)(1-r_2)(1-r_3)} \cdot (ABC). \tag{12}$$

10 Dividing the sides of a triangle

Here, using the theorem of Ceva, we study some properties of figures created by points on the sides of a triangle and the corresponding ratios they define on them.

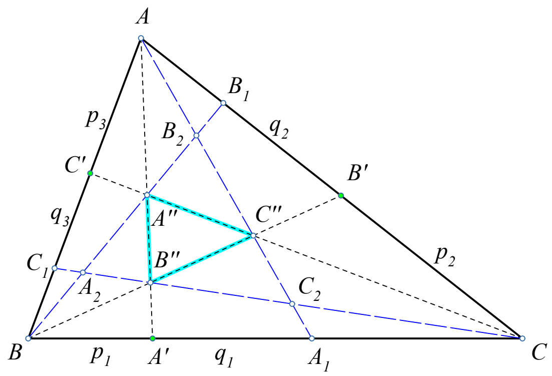

Figure 13: Location of intersection points

Figure 13 illustrates a typical problem related to Ceva's theorem:
To determine the ratio coordinates of the various intersection points of three given cevians $\{AA', BB', CC'\}$, for which are given the ratios $\{r_1 = p_1/q_1, r_2 = p_2/q_2, r_3 = p_3/q_3\}$. By applying Ceva's theorem we find:

$$\frac{A_1B}{A_1C} = -(r_2r_3)^{-1}, \qquad \frac{B_1C}{B_1A} = -(r_3r_1)^{-1}, \qquad \frac{C_1A}{C_1B} = -(r_1r_2)^{-1}.$$

This leads to the ratio coordinates of the points $\{A'', B'', C''\}$:

$$A''(r_1, -(r_3r_1)^{-1}, r_3), \quad B''(r_1, r_2, -(r_1r_2)^{-1}), \quad C''(-(r_2r_3)^{-1}, r_2, r_3).$$

Using these one can solve a classical problem concerning the determination of the area of the triangle $A''B''C''$ ([Ste71, (I) p.163]). For this we can use again the area formula in barycentrics

$$(A''B''C'') = (ABC)\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

in which the columns of the determinant represent the absolute barycentrics correspondingly of the points $\{A'', B'', C''\}$. Using the preceding expressions and the corrolary 1 we find these barycentrics

$$
\begin{aligned}
(a_1,\ a_2,\ a_3) &= \frac{1}{1-r_3+r_3r_1}(\ 1,\ \ -r_3,\ \ r_3r_1\ )\ ,\\
(b_1,\ b_2,\ b_3) &= \frac{1}{1-r_1+r_1r_2}(\ r_1r_2,\ \ 1,\ \ -r_1\ )\ ,\\
(c_1,\ c_2,\ c_3) &= \frac{1}{1-r_2+r_2r_3}(\ -r_2,\ \ r_2r_3,\ \ 1\ )\ .
\end{aligned}
$$

Introducing these expressions into the preceding determinant we find that

$$
(A''B''C'') \ = \ \frac{(r_1r_2r_3+1)^2}{(1-r_1+r_1r_2)(1-r_2+r_2r_3)(1-r_3+r_3r_1)}\cdot(ABC)\ . \qquad (13)
$$

This shows once again that the three cevians $\{AA', BB', CC'\}$ pass through the same point precisely when

$$
0 \ = \ (A''B''C'') \quad \Leftrightarrow \quad r_1r_2r_3 \ = \ -1\ .
$$

Exercise 1. *Show that the ratio coordinates of the points $\{A_2, B_2, C_2\}$ are correspondingly:*

$$
\begin{aligned}
(a_1,\ a_2,\ a_3) &= (-r_1^2r_2r_3\ ,\ -(r_3r_1)^{-1}\ ,\ -(r_1r_2)^{-1}\ )\ ,\\
(b_1,\ b_2,\ b_3) &= (-(r_2r_3)^{-1}\ ,\ -(r_3r_1)^{-1}\ ,\ -(r_2r_3^2r_1)\ )\ ,\\
(c_1,\ c_2,\ c_3) &= (-(r_2r_3)^{-1}\ ,\ -(r_1r_2^2r_3)\ ,\ -(r_1r_2)^{-1}\ )\ ,
\end{aligned}
$$

and the signed area of $A_2B_2C_2$ is

$$
(A_2B_2C_2) = -\frac{(r_1r_2r_3-1)^2(r_1r_2r_3+1)^2}{(r_1^2r_2r_3+r_1r_2+1)(r_1r_2^2r_3+r_2r_3+1)(r_1r_2r_3^2+r_1r_3+1)}\cdot(ABC). \qquad (14)
$$

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Related material

1. Apollonian circles
2. Barycentric coordinates
3. Cross Ratio
4. Desargues' theorem
5. Isodynamic points of the triangle
6. Menelaus' theorem
7. Nagel center of the triangle
8. Projective line
9. Symmedian center of the triangle
10. Tritangent circles of the triangle

Any correction, suggestion or proposal from the reader, to improve/extend the exposition, is welcome and could be send by e-mail to: pamfilos@uoc.gr