

Desargues' theorem and perspectivities

A file of the [Geometrikon](#) gallery by [Paris Pamfilos](#)

The joy of suddenly learning a former secret and the joy of suddenly discovering a hitherto unknown truth are the same to me - both have the flash of enlightenment, the almost incredibly enhanced vision, and the ecstasy and euphoria of released tension.

Paul Halmos, I Want to be a Mathematician, p.3

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1 Desargues' theorem

The theorem of Desargues represents the geometric foundation of "*photography*" and "*perspectivity*", used by painters and designers in order to represent in paper objects of the space.

Two triangles are called "*perspective relative to a point*" or "*point perspective*", when we can label them ABC and $A'B'C'$ in such a way, that lines $\{AA', BB', CC'\}$ pass through a

common point P . Points $\{A, A'\}$ are then called "homologous" and similarly points $\{B, B'\}$ and $\{C, C'\}$. The point P is then called "perspectivity center" of the two triangles. The

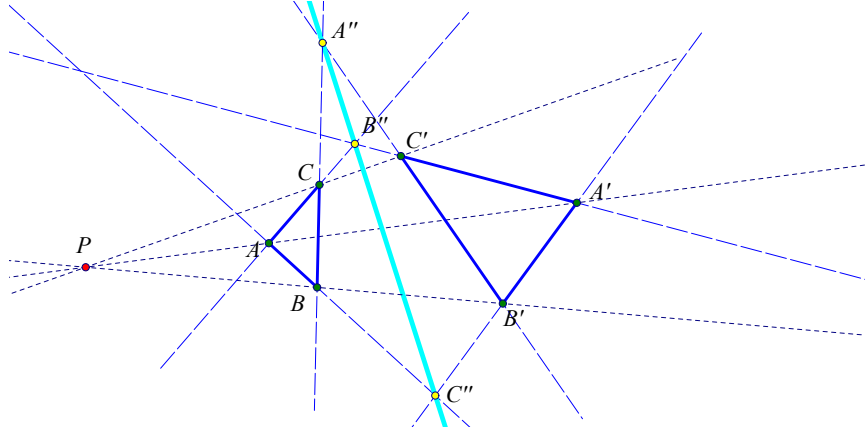


Figure 1: "Desargues' configuration": Theorem of Desargues

two triangles are called "perspective relative to a line" or "line perspective" when we can label them ABC and $A'B'C'$, in such a way (See Figure 1), that the points of intersection of their sides $C'' = (AB, A'B')$, $A'' = (BC, B'C')$ and $B'' = (CA, C'A')$ are contained in the same line ε . Sides AB and $A'B'$ are then called "homologous", and similarly the side pairs $(BC, B'C')$ and $(CA, C'A')$. Line ε is called "perspectivity axis" of the two triangles.

Theorem 1. (Desargues 1591-1661) *Two triangles are perspective relative to a point, if and only if they are perspective relative to a line.*

Proof. To prove this assume that the two triangles ABC and $A'B'C'$ are perspective relative to a point P and apply three times the theorem of Menelaus (**Menelaus' theorem**):

$$\begin{aligned}
 \text{In triangle } PAB \text{ with secant } C''B'A' : & \frac{C''A}{C''B} \cdot \frac{A'P}{A'A} \cdot \frac{B'B}{B'P} = 1. \\
 \text{In triangle } PBC \text{ with secant } A''C'B' : & \frac{A''B}{A''C} \cdot \frac{B'P}{B'B} \cdot \frac{C'C}{C'P} = 1. \\
 \text{In triangle } PCA \text{ with secant } B''C'A' : & \frac{B''C}{B''A} \cdot \frac{C'P}{C'C} \cdot \frac{A'A}{A'P} = 1. \\
 \text{Multiplying the equalities and simplifying:} & \frac{C''A}{C''B} \cdot \frac{A''B}{A''C} \cdot \frac{B''C}{B''A} = 1.
 \end{aligned}$$

According to the theorem of Menelaus, this relation means that C'' lies on the line $A''B''$.

Conversely, assume that the points A'', B'' and C'' are collinear and apply the already proved part of the theorem on triangles $C''BB'$ and $B''C'C$, which, by assumption now, are perspective relative to the point A'' . According to the proved part, the intersection points $A = C''B \cap B''C$, $A' = C''B' \cap B''C'$ and $P = BB' \cap C'C$ will be collinear. This is equivalent with the fact that the lines AA', BB' and CC' pass through the same point, in other words, the triangles ABC and $A'B'C'$ are perspective relative to a point ([Tan67]).

□

Figure 2 shows two quadrangles which are point-perspective relative to point P but they are not line-perspective. Thus, the "Desargues' configuration" ([Woo29]) is something peculiar to the triangles and not more general valid for polygons.

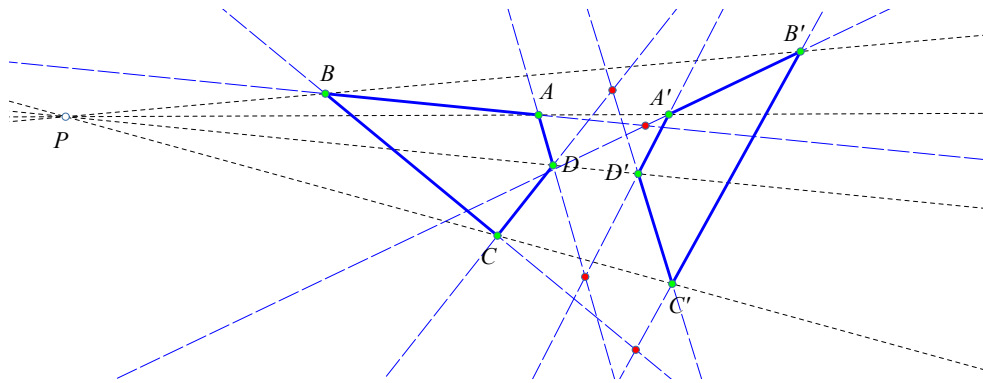


Figure 2: Point-perspective but not line-perspective

2 Perspective triangles

Theorem 2. Given two triangles $\{ABC, A'B'C'\}$ there is a similar $A''B''C''$ to $A'B'C'$, which is perspective to ABC .

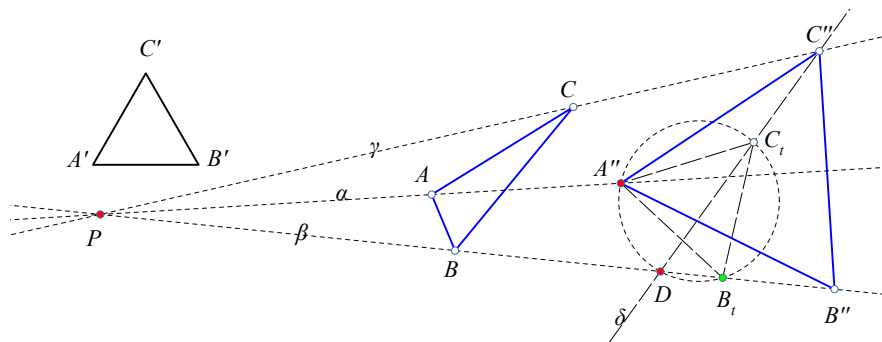


Figure 3: Perspective triangles

The proof, suggested by figure 3, starts with the triangle $\tau = ABC$ and an arbitrary point P not on the side-lines of τ . Then, we consider the lines $\{\alpha = PA, \beta = PB, \gamma = PC\}$ and an arbitrary point $A'' \in \alpha$ and a moving point B_t on β . Then, we construct a triangle $A''B_tC_t \sim A'B'C'$ and prove three things.

- 1) That the circumcircles of all triangles $\{A''B_tC_t, B_t \in \beta\}$ have their circumcircle pass through a fixed point $D \in \beta$.
- 2) That the vertex C_t of the triangles $\{A''B_tC_t\}$ move on a line δ intersecting line γ at a point C'' .
- 3) The triangle $A''B''C'' \sim A'B'C'$ has its vertex B'' on line β .

The proofs of these statements are easy exercises (see file [Similarity transformation](#)). Figure 3 shows a triangle ABC in perspective with an "equilateral" triangle. Notice that, using an appropriate "homothety" with center at P , we can find a third triangle $A_1B_1C_1$ congruent to $A'B'C'$ and in perspective to ABC . Thus, we have the following theorem.

Theorem 3. Two arbitrary triangles $\{ABC, A'B'C'\}$ can be placed in the plane so as to be perspective.

3 Desargues' theorem, special cases

A special application of the theorem of Desargues is also shown in figure 4. In this,

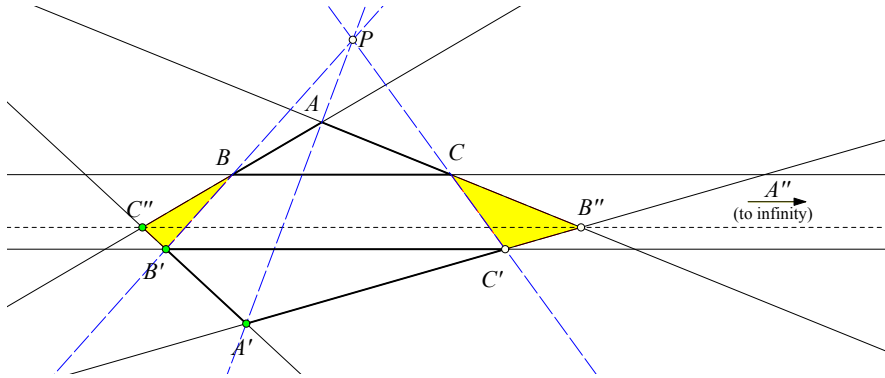


Figure 4: Desargues' theorem, $A'' \rightarrow \infty$

the two triangles ABC and $A'B'C'$ have two respective sides BC and $B'C'$ parallel to $B''C''$, where $B'' = AC \cap A'C'$ and $C'' = AB \cap A'B'$. The parallels are considered again as intersecting at a point at infinity A'' , through which passes $B''C''$. According to Desargues then, the lines $\{AA', BB', CC'\}$ will pass through a common point P .

The same figure can also be interpreted in a different way, considering as main actors the triangles $BB'C''$ and $CC'B''$. In these triangles the lines $BC, C''B''$ and $B'C''$ are parallel and can be considered as passing through the same point A'' at infinity. Then, according to Desargues, points $A = BC'' \cap CB''$, $A' = C''B' \cap B''C'$ and $P = BB' \cap CC'$ are collinear. The conclusion in this case can be proved also by applying the theorem of Thales.

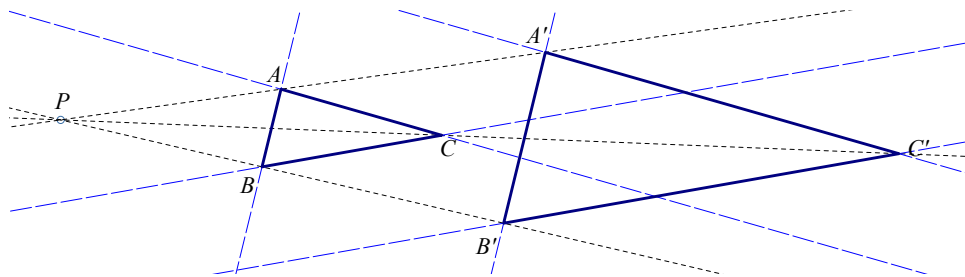


Figure 5: Desargues' theorem generalizes an homothety theorem

Figure 5 shows another special case, in which two of the points $\{A'', B'', C''\}$ on the axis of perspectivity are at infinity, hence the whole axis of perspectivity coincides with the line at infinity and the two perspective triangles have their sides parallel. Then they are "homothetic" and the perspectivity center P becomes "homothety center" (see file [Homothety transformation](#)).

Thus, the proof of the general Desargues' theorem could be reduced to the present case of homothetic triangles by sending the axis of perspectivity to the line at infinity via a (see file [Projectivity transformation](#)).

4 Sides passing through collinear points

Theorem 4. Let a variable triangle ABC have its vertices $\{B, C\}$ on two fixed lines PB', PC' respectively and its sides passing through three points $\{A'', B'', C''\}$ lying on a fixed line. Then its third vertex A varies on a line PA' passing through P .

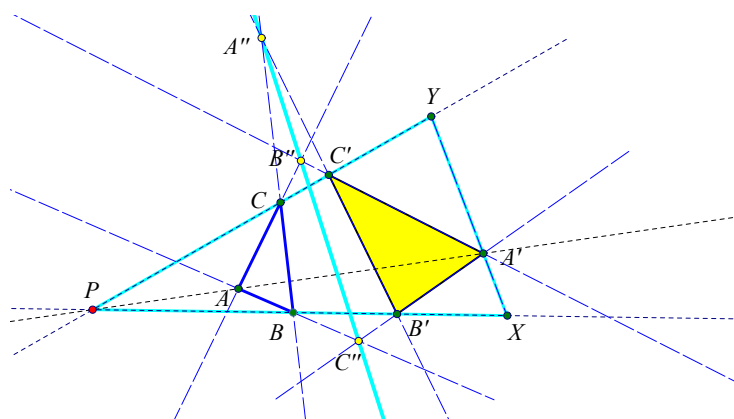


Figure 6: Sides through three collinear points

The theorem is a direct corollary of Desargues' theorem. Defining point A' as intersection of the known lines $A' = B'C'' \cap B''C'$, we have that the triangles $\{ABC, A'B'C'\}$ are line-perspective, hence also point-perspective relative to P .

Problem 1. Given the triangle PXY , inscribe to it a triangle $A'B'C'$, such that its sides pass through three given points $\{A'', B'', C''\}$ lying on a given line.

According to the previous theorem all triangles $\{A'B'C'\}$ having their vertices $\{B', C'\}$ on lines $\{PX, PY\}$ respectively and their sides passing respectively through the fixed points $\{A'', B'', C''\}$ have their third vertex A' varying on a fixed line PA (See Figure 6). Its intersection with the third side XY of the given triangle determines point A' of the desired triangle. This determines also completely triangle $A'B'C'$, since its sides must pass through the given points $\{A'', B'', C''\}$.

The line PA' is constructed through an auxiliary triangle ABC . This, in turn, is constructed by considering an arbitrary line $A''BC$ through A'' intersecting $\{PX, PY\}$ at points $\{B, C\}$ and defining point A through $A = BC'' \cap B''C$.

Remark-1 A special case of the theorem 4 occurs when line carrying $\{A'', B'', C''\}$ is the line at infinity. This means that the variable triangle $A'B'C'$ has sides always parallel to three given directions. In this special case one can see immediately, without using Desargues' argument, that the third point varies on a line. Using this special case one can even reduce to it the general Desargues' theorem by defining a projective map ("projectivity") that sends the line carrying $\{A'', B'', C''\}$ to the line at infinity.

Remark-2 Applying the theorem 4 repeatedly and proceeding inductively one can easily show its generalization :

Theorem 5. When a polygon varies so that its n sides pass through n fixed points lying on a given line ε and $n - 1$ of its vertices lie on $n - 1$ given fixed lines, then its n -th vertex describes also a fixed line.

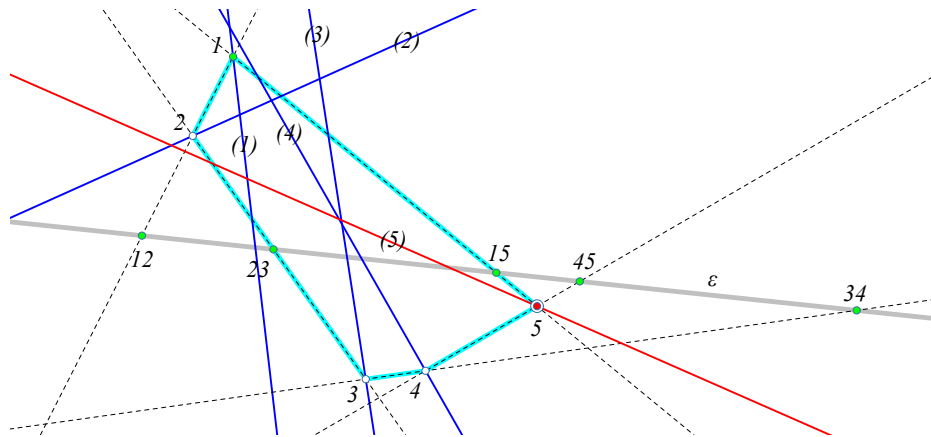


Figure 7: A polygon with sides passing through 5 fixed collinear points

Figure 7 shows a pentagon, whose 5 sides pass through 5 points of line ε and its vertices $\{1, 2, 3, 4\}$ move on respective fixed lines $\{(1), (2), (3), (4)\}$. Then its fifth vertex moves also on a line (5).

Remark-3 It is essential that points $\{A'', B'', C'', \dots\}$ through which pass the sides of the variable triangle (resp. polygon) in theorems 4 and 5 are collinear. In the more general case in which these points are not collinear the geometric locus of A , in both cases, is a conic. This is handled in the file [Maclaurin](#) ([Cha65, p.72]). There is also a special case, in which these three points are not collinear, but line $B''C''$ passes through P and the locus of A is again a line. This case is handled below.

5 A case handled with projective coordinates

Next problem can be considered as a limiting case of problem 1.

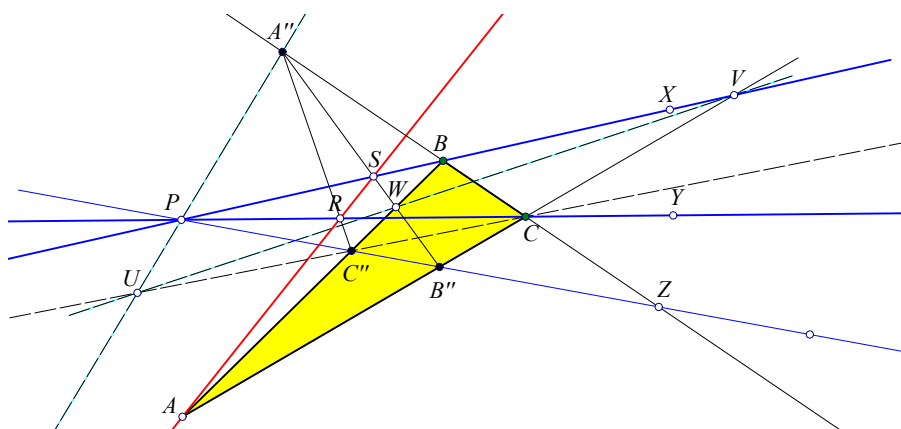


Figure 8: A limiting configuration for problem 1

Problem 2. Let a variable triangle ABC have its vertices $\{B, C\}$ respectively on two fixed lines $\{PX, PY\}$ and its sides passing through three points $\{A'', B'', C''\}$ with line $B''C''$ passing through P . Then its third vertex A varies on the line passing through points $\{R, S\}$, which are the intersection points respectively of the line-pairs $R = A''C'' \cap PY$ and $S = A''B'' \cap PX$.

To prove this consider the intersection points V, W of the line-pairs correspondingly $V = AB'' \cap OB$ and $W = A''B'', C''B$. Use a "projective base" (see file [Projective plane](#)) in which

$$A''(1, 0, 0), \quad B''(0, 1, 0), \quad C''(0, 0, 1)$$

and calculate the coordinates of A . For this consider the lines $\{PX, PY, A''BC\}$ to be described respectively by the equations

$$PX : ax + by + cz = 0, \quad PY : a'x + b'y + c'z = 0, \quad A''BC : uy + vz = 0.$$

Points $\{B, C\}$ have then coordinates

$$(bv - cu, -av, au), \quad (b'v - c'u, -a'v, a'u).$$

Lines $\{BC'', CB''\}$ have respectively the form:

$$(-av)x + (cu - bv)y = 0, \quad (-a'u)x + (b'v - c'u)z = 0.$$

Their intersection is point A with coordinates:

$$A = ((cu - bv)(b'v - c'u), av(b'v - c'u), a'u(cu - bv)).$$

Since lines $\{PX, PY\}$ intersect at the point P of line $x = 0$, we find that $b' = sb, c' = sc$ for a constant s . Thus, the coordinates of point A become:

$$(-s(cu - bv)^2, -sav(cu - bv), a'u(cu - bv)),$$

which simplifies to

$$(s(cu - bv), sav, -a'u). \tag{1}$$

On the other side the coordinates of points $\{S, R\}$ are easily calculated to be

$$S : (-b, a, 0) \quad \text{and} \quad R : (-sc, 0, a').$$

The determinant of the coordinates of these two, and the coefficients in equation 1, is then:

$$\begin{vmatrix} s(cu - bv) & sav & -a'u \\ -b & a & 0 \\ -sc & 0 & a' \end{vmatrix} = 0.$$

This proves the collinearity of points $\{A, S, R\}$. By Desargues' theorem triangles $A''WC''$ and PCV , being line-perspective, are also point-perspective with respect to a point U .

One could start from this and show the collinearity of three points $\{A, S, R\}$. The calculation though seems to be slightly more complicated than the one given above.

6 Space perspectivity

Desargues' theorem can be understood as the result of an operation on a pyramid with basis the triangle ABC on its plane α . A second plane β intersects the pyramid along a triangle $A'B'C'$ and the plane α along a line ε (See Figure 9). Then, we use a space rotation with axis the line ε . This rotation, preserving the dimensions of triangle $A'B'C'$, causes a continuous transformation of the pyramid and its vertex P until to make it coincident with a point P_0 of α and to degenerate the pyramid from a space shape to a plane shape in the form of a configuration like the one of Desargues' theorem. In this

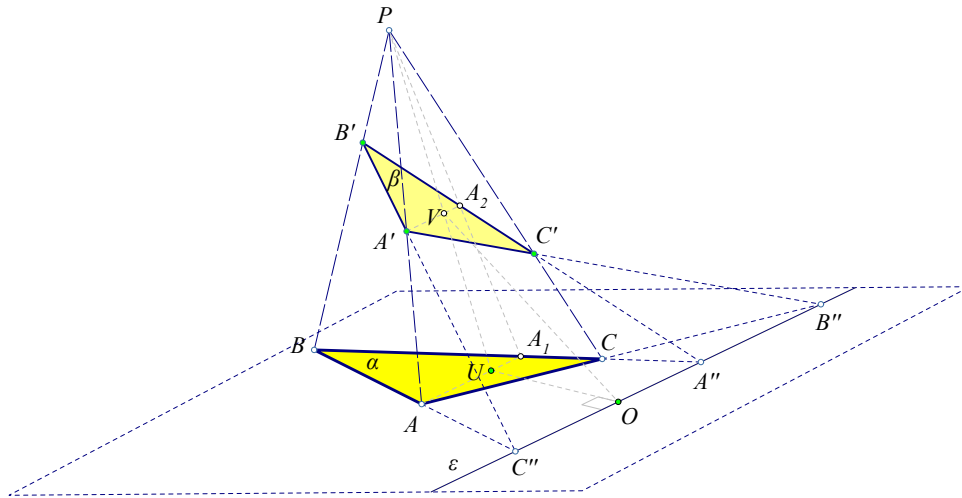


Figure 9: Pyramid intersected with a plane

limiting configuration the triangle $A'B'C'$ becomes coplanar with ABC and P_0 becomes a perspectivity center of the two triangles, whereas the line ε becomes the axis of perspectivity of the two triangles.

The procedure can be reversed and, starting from a configuration like the one of Desargues' theorem, will create a series of pyramids in space. These pyramids will have the same base ABC and the triangle $A'B'C'$ will appear as a cut of the pyramid with a plane β passing through the rotation axis ε .

Exercise 1. Show that the vertex P of the pyramid in figure 9 describes, during the rotation about ε , a circle orthogonal to ε .

Hint: Consider first the plane δ parallel to line ε and containing the edge AP of the pyramid. This plane intersects the sides $\{BC, B'C'\}$ of the triangles respectively at points $\{A_1, A_2\}$, which define the two parallel lines $\{AA_1, A'A_2\}$ and the two similar triangles $\{AA_1P, A'A_2P\}$. The bases of these two triangles preserve their length during the rotation, hence their ratio is a constant $A'A_2/AA_1 = k$.

Consider then the plane ζ orthogonal to ε , passing through the vertex of the pyramid P and intersecting $\{\varepsilon, AA_1, A'A_2\}$ respectively at points $\{O, U, V\}$. Using the similarity of the triangles $\{AA_1P, A'A_2P\}$ show that

$$\frac{PV}{VU} = \frac{k}{k-1} = k',$$

is a constant. Conclude the construction of the claimed geometric locus of P in the

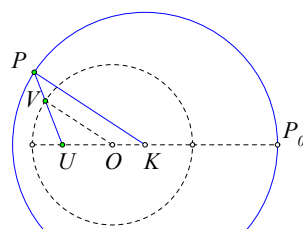


Figure 10: Locus of P in a plane orthogonal to ε

plane ζ , which is a circle with center K , such that $KO/OU = k'$ and radius KP , such that $KP/OV = UK/UO$ (See Figure 10).

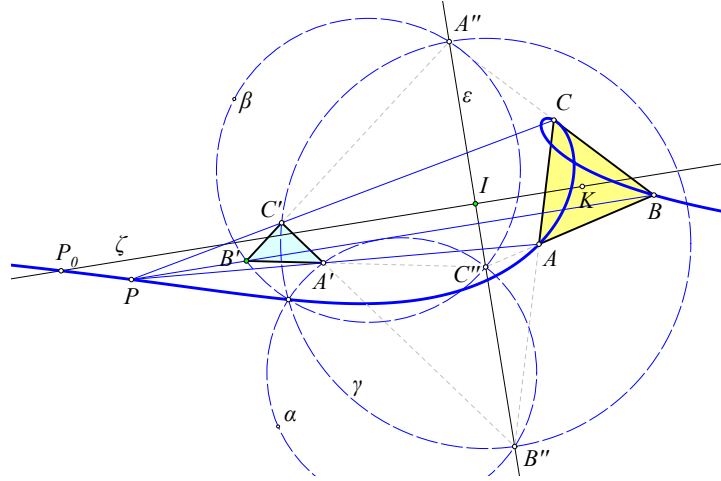


Figure 11: Revolve about ε to create a pyramid over ABC

Using the discussion of section 2 we can easily construct a pyramid over an equilateral ABC and cut it along a triangle similar to a given one $A'B'C'$. The problem becomes however somewhat more complex if we want to do that for a special pyramid, e.g. a “right pyramid over the equilateral”, meaning a pyramid with base the equilateral ABC and having its vertex P_1 orthogonally projected onto the center K of the equilateral. Figure 11 suggests a way to do that as follows:

1. We choose an arbitrary line ε to play the role of “axis”.
2. We define the intersections of $C'' = \varepsilon \cap AB$, $A'' = \varepsilon \cap BC$, $B'' = \varepsilon \cap CA$.
3. We define the circles $\{\alpha, \beta, \gamma\}$ the points of which view $\{A''B'', B''C'', C''A''\}$ under the angles $\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}$ of the given triangle $A'B'C'$ or their supplements.
4. From an arbitrary point of one of the circles, $B' \in \beta$ say, we draw the two lines $\{B'A'', B'C''\}$ intersecting a second time the circles $\{\gamma, \alpha\}$ respectively at $\{C', A'\}$.
5. As B' moves on β all these triangles are similar to the given one. Also by construction they are line-perspective with axis ε and, by Desargues, they are point-perspective from a point P .
6. Turning the plane about ε we create pyramids over ABC cut by the rotated plane along the triangle $A'B'C'$.
7. In order to obtain, by such a rotation about ε , a right pyramid over ABC the point P has to be on the line ζ orthogonal to ε through K .
8. The geometric locus of P , as B' varies on β , is a cubic passing through the vertices of ABC and B' has to be chosen so that the corresponding point P is one intersection point P_0 of ζ with the cubic.

7 Perspectivity as a projective transformation

Returning to the basic Desargues' configuration we notice some important and equal cross ratios

$$(PC_0, CC') = (PA_0, AA') = (PB_0, BB') = k.$$

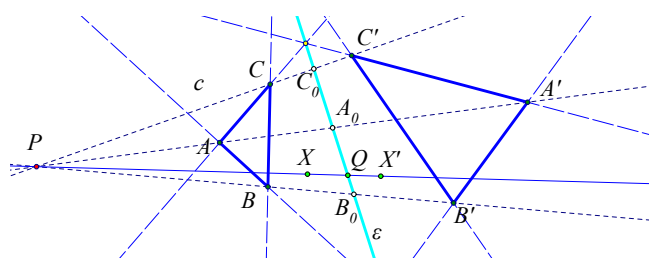


Figure 12: Equal cross ratios

This shows that the two triangles are related by a “*Perspectivity*” in the projective-geometric sense. The *perspectivities* in this sense are transformations f of the projective plane with the following characteristic properties.

1. They have a pointwise-fixed line $\varepsilon : f(X) = X$, for all $X \in \varepsilon$.
2. They have an additional fixed point $P : P \notin \varepsilon, f(P) = P$.
3. They have every line ζ through P invariant and if $Q = \varepsilon \cap \zeta$, then for every other point $X \in \varepsilon$ the image $X' = f(X)$ is the point $X' \in \zeta : (PQ, XX') = k$.
4. $\{P, \varepsilon\}$ are called respectively “*center*” and “*axis*” of the perspectivity and k is called “*coefficient*” of the perspectivity.

Traditionally “*perspectivities*” include also “*elations*”, which are transformations satisfying only the two first conditions nr-1 and nr-2, last with the property $P \in \varepsilon$. I study these in the file [Projectivities](#). Here I exclude this kind of transformations and speak of “*perspectivities*” per definition satisfying all the above three conditions. Classically, such perspectivities, are called also “*Homologies*” ([Ver71, p.60]). Homologies with ratio $k = -1$ are called “*harmonic homologies*” or “*harmonic perspectivities*”. From the properties of the *cross ratio* follows that the perspectivity with the same center and axis and coefficient $k' = k^{-1}$ is the inverse map f^{-1} of f . It follows that *harmonic homologies* are *involutive*, i.e. their inverse $f^{-1} = f \Leftrightarrow f^2 = e$.

Using this terminology, a Desargues’ configuration defines a perspectivity and conversely, given a perspectivity f , every triangle τ together with its image $f(\tau)$ defines a Desargues’ configuration. In this respect the “*harmonic perspectivities*” are characterized by the following theorem.

Theorem 6. *A perspectivity f is harmonic if and only if there is a triangle τ such that the vertices of τ and those of its image $\tau' = f(\tau)$ are six points of the same conic κ . The axis of perspectivity is in this case the polar w.r. to κ of the center of perspectivity.*

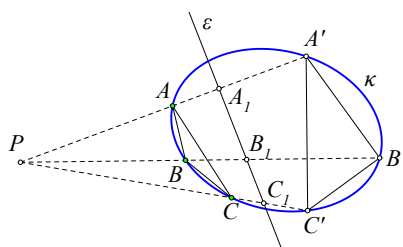


Figure 13: Perspective triangles in a harmonic perspectivity

Proof. If the triangles are perspective and there is a conic κ through their vertices (See Figure 13), then the axis must be the polar of the center P w.r. to κ , consequently the perspectivity is harmonic. In fact, consider in this case the polar of P and the corresponding harmonic perspectivity f' mapping $\{A, B, C\}$ correspondingly to $\{A', B', C'\}$. The two projectivities $\{f, f'\}$ coincide then on four points $\{P, A, B, C\}$, hence they coincide everywhere.

Conversely, if the triangles $\{\tau, \tau'\}$ are perspective w.r. to a harmonic perspectivity f , then the conic κ through the five points $\{A, B, C, A', B'\}$ has the axis ε as polar of the center P and is invariant under $f : f(\kappa) = \kappa$. Hence the sixth point C' is also on the conic. \square

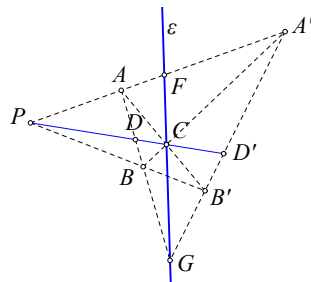


Figure 14: Harmonic perspectivity characterization

Exercise 2. Show that the perspectivity f is “harmonic” if and only if, for every pair of points $\{A, B\}$ and their images $A' = f(A)$, $B' = f(B)$ the lines $\{AB', A'B\}$ intersect at a point C of its axis ε (See Figure 14).

Hint: By the fundamental property of complete quadrilaterals $(P, C) \sim (D, D')$ are harmonic pairs. Conversely, if $C \in \varepsilon$ then $(P, F) \sim (A, A')$ are harmonic pairs etc.

8 The case of the trilinear polar

The “trilinear polar” of a point P relative to the triangle ABC is defined as the “axis” of perspectivity of ABC and the “cevian” triangle $A'B'C'$ of the point P (See Figure 15). It

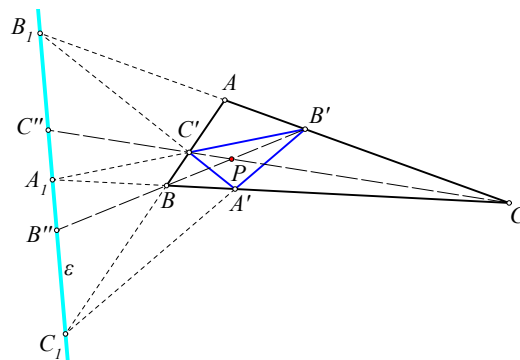


Figure 15: Trilinear polar ε of P relative to ABC

is thus, a special Desargues’ configuration and next theorem characterizes this case.

Theorem 7. The triangle ABC and the “cevian” triangle $A'B'C'$ relative to the point P are perspective w.r. to a perspectivity f with coefficient $k = -2 : A'B'C' = f(ABC)$. Conversely, if two

triangles are perspective w.r. to a perspectivity f with coefficient $k = -2 : A'B'C' = f(ABC)$ and $\{A', B', C'\}$ are on the respective sides $\{BC, CA, AB\}$ of ABC , then the perspectivity axis is the trilinear polar of either triangle relative to the perspectivity center P .

Proof. By their definition ABC and its cevian $A'B'C'$ are perspective relative to P . The "prototype" of cevian triangle is a triangle ABC and its "medial" triangle, with vertices the middles $\{A', B', C'\}$ of the sides $\{BC, CA, AB\}$. The cevians in this case are the "medians" of the triangle and P is the centroid G , the trilinear polar being the line at infinity ε_∞ . The first part of the theorem is easily verified for this special case. By the fundamental theorem of projective geometry, there is a projectivity g mapping the vertices A, B, C to themselves and the point P to G . Then, the trilinear polar ε of P maps to ε and the first part of the theorem reduces to the above special case.

For the converse and the assumed there conditions, consider the three intersection points $\{A'' = AA' \cap \varepsilon, B'' = BB' \cap \varepsilon, C'' = CC' \cap \varepsilon\}$, which, by their construction, satisfy $(PC'', CC') = (PA'', AA') = (PB'', BB') = -2$. By the first part of the theorem the same conditions satisfy also the intersections of $\{AA', BB', CC'\}$ with the trilinear polar of P , hence latter coincides with ε . \square

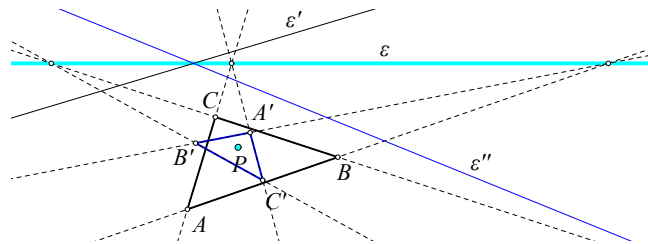


Figure 16: Trilinear polars $\{\varepsilon', \varepsilon''\}$ of $\{ABC, A'B'C'\}$

Figure 16 shows that not every triangle ABC and its image $A'B'C' = f(ABC)$ by a perspectivity f with coefficient $k = -2$ have the same trilinear polar coinciding with ε . The lines $\{\varepsilon', \varepsilon''\}$ are the trilinear polars of the respective triangles $\{ABC, A'B'C'\}$.

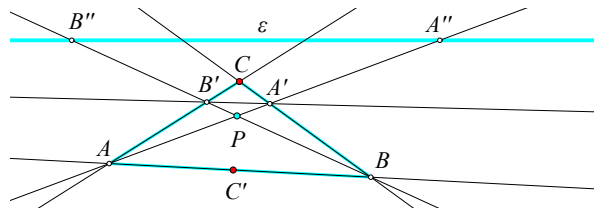


Figure 17: Triangles with prescribed pole and trilinear polar

Exercise 3. Given a line ε and a point $P \notin \varepsilon$, construct all the triangle-pairs $(ABC, A'B'C')$, such that $A'B'C'$ is the cevian of ABC relative to P and ε is their common trilinear polar.

Hint: Figure 17 suggests a way to produce such pairs showing the construction of ABC . Define first the perspectivity f with center P and axis ε and coefficient $k = -2$. Points $\{A, B\}$ can be chosen arbitrarily. Then $C = AB' \cap A'B$, where $\{A' = f(A), B' = f(B)\}$ and $C' = f(C)$.

Exercise 4. Given a line ε , a point $P \notin \varepsilon$ and the perspectivity f of the previous exercise, fix the point A and consider for each B the corresponding $C = A'B \cap AB'$ with $B' = f(B)$. Show that the map $g(B) = C$ is a harmonic perspectivity with center $A' = f(A)$ and axis ε .

Hint: $(BC, A'A_1) = -1$ in figure 17. In the prototype cevian triangle with the medians, point A' is the middle of BC and ε is the line at infinity.

9 The case of conjugate triangles

Conjugate triangles relative to the conic κ are constructed by taking an arbitrary triangle ABC and considering the “poles” $\{A', B', C'\}$ of its side-lines relative to κ . The triangle $A'B'C'$ is the “conjugate” to ABC relative to κ (See Figure 18). By the pole-polar-

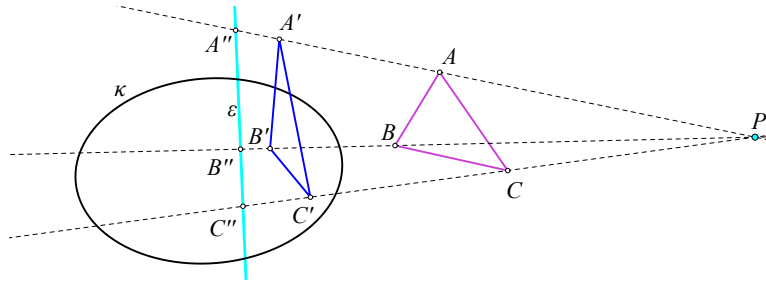


Figure 18: Conjugate triangles $\{ABC, A'B'C'\}$

reciprocity it is easily seen that the conjugate of $A'B'C'$ is ABC . Thus, we have here a map f of the set of triangles onto itself which is involutive $f^2 = e$. Next theorem shows that every such pair of triangles creates also a Desargues' configuration.

Theorem 8. *If the triangles $\{ABC, A'B'C'\}$ are conjugate relative to the conic κ , then they are perspective. The perspectivity f mapping ABC to $A'B'C'$ has axis the polar of the center w.r. to the conic. The perspectivity mapping $A'B'C'$ to ABC is the inverse of f .*

Proof. ([Sal17, p.253]) Considering the question as a problem of the projective plane (see file **Projective plane**), a conic κ is represented w.r. to a “projective coordinate system” through an invertible symmetric 3×3 matrix M and the corresponding quadratic equation of the form $x^t M x = 0$.

For a point z , the corresponding polar line $p(x) = 0$ is given then by $p(x) = z^t M x = 0$. The symmetry of this equation is the reason for the “pole-polar reciprocity”. In fact, the symmetry $w^t M v = v^t M w = 0$, means that if the polar p_v of v passes through w then also the polar p_w of w passes through v .

Let now $\{x_1, x_2, x_3\}$ represent the vertices of ABC and $\{p_1, p_2, p_3\}$ be the corresponding polars with respect to κ . Let $\{y_1, y_2, y_3\}$ represent the vertices of $A'B'C'$, y_1 being the intersection of $\{p_2, p_3\}$, y_2 being the intersection of $\{p_3, p_1\}$ etc. y_1 being on p_2 , its polar passes through x_2 . y_1 being also on p_3 , its polar passes through x_3 . Thus the polar of y_1 coincides with side BC . Analogous argument is valid for the other points $\{y_2, y_3\}$, showing that the conjugation relation is well defined and symmetric.

To show the perspectivity of the triangles define the three numbers

$$t_3 = p_1(x_2) = x_1^t M x_2 = x_2^t M x_1 = p_2(x_1) \quad (\text{by the symmetry of matrix } M).$$

Analogously define $t_1 = p_2(x_3)$, $t_2 = p_3(x_1)$. Any line through the intersection point A' of $p_2(x) = 0, p_3(x) = 0$ is represented through $p_2(x) - k p_3(x) = 0$. Such a line passing through $A(x_1)$ satisfies $p_2(x_1) - k p_3(x_1) = 0 \Rightarrow k = t_3/t_2$, hence the line being given by:

$$t_2 p_2 - t_3 p_3 = 0.$$

Analogously we obtain the equations of the other lines $\{BB', CC'\}$ through equations:

$$t_3p_3 - t_1p_1 = 0, \quad t_1p_1 - t_2p_2 = 0.$$

Obviously last equation is the negative sum of the first two, hence the three lines pass through the same point.

The claim, that the axis is the polar of P , is a consequence of the previous arguments. In fact, $A_0 = BC \cap B'C'$ being on the polars of A, A' implies that A_0 is the pole of AA' . Analogously $\{B_0 = CA \cap C'A', C_0 = AB \cap A'B'\}$ are correspondingly the poles of $\{BB', CC'\}$. Since $\{AA', BB', CC'\}$ pass through P their poles are contained in the polar of point P . \square

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