Euclidean plane Isometries (Congruences)

A file of the Geometrikon gallery by Paris Pamfilos

To properly know the truth is to be in the truth; it is to have the truth for one's life. This always costs a struggle. Any other kind of knowledge is a falsification. In short, the truth, if it is really there, is a being, a life.

Kierkegaard, Truth is the way

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1 Transformations of the plane

"Transformation" of the plane is called a process f, which assigns to every point X of the plane, with a possible exception of some special points, another point Y of the plane which we denote by $f(X)$. We write $Y = f(X)$ and we call X a *"preimage"* of the transformation and Y the *"image"* of X through the transformation. We often say that the transformation f *"maps"* X to Y. For the process f we accept that it satisfies the requirement

$$X \neq X' \quad \Rightarrow \quad f(X) \neq f(X').$$

In other words, different points also have different images. Equivalently, this means that, if for two points X, X' holds $f(X) = f(X')$, then it will also hold $X = X'$. The set of points

X, on which the transformation f is defined, is called *"domain"* of the transformation f, while the set consisting of all Y, such that $Y = f(X)$, when X varies in the domain of f, is called *"range"* of f. For transformation examples the reader may consult the beginnings of the following sections. Here we limit ourselves in a description of common characteristics of these notions.

For every shape of the plane Σ the set of images $f(X)$, where X runs through Σ, is called *"image"* of Σ and is denoted as $f(\Sigma)$.

Applying the processes one after the other, we create the notion of *"composition of transformations"*. For two given transformations f and g, we call *"composition"* of f and g, the transformation whose process results by the successive application of the processes of f and g. The composition of transformations is denoted by

$$g \circ f.$$

By definition, the process of composition $g \circ f$ first corresponds $Y = f(X)$ to X and then $Z = g(Y) = g(f(X))$ to Y. Totally then, it corresponds $Z = g(f(X))$ to X. There are some details, which we must be careful with in compositions. These have to do with the domains and ranges of the transformations, which participate in the composition. For all of it to be meaningful, the range of the first transformation (f) must be contained in the domain of the second (g). Things are considerably simplified for transformations which have domain and range the entire plane.
Since the composition $g \circ f$ is a new transformation, we may consider its composition with a third transformation h :

$$h \circ (g \circ f),$$

and more generally we can define the composition of as many transformations f_1, f_2, f_3,..., f_k we want, which, for simplicity, let us consider that they are defined on the entire plane:

$$f_k \circ f_{k-1} \circ \ldots \circ f_1.$$

The meaning of such a composition of transformations is that we apply successively the processes of the transformations which participate in sequence from right to left. f_1 maps point X_1 to $X_2 = f_1(X_1)$, f_2 next maps X_2 to $X_3 = f_2(X_2)$, and so on and so forth. This process can be denoted pictorially by the diagram

$$X_1 \overset{f_1}{\longmapsto} X_2 \overset{f_2}{\longmapsto} X_3 \overset{f_3}{\longmapsto} X_4 \ \ldots \ \overset{f_k}{\longmapsto} X_{k+1}.$$

A very simple and insignificant, regarding its action, transformation is the so called *"identity transformation"*, which we denote with e and which, to every point X corresponds X itself. This one resembles the unit in the familiar multiplication, which leaves numbers unchanged. The same way, this transformation does nothing. It leaves every point fixed. Its structural meaning however is as important as that of the multiplication unit. With its help we can define immediately the *"inverse transformation"* of a transformation f which we denote with f^{-1}. This one performs exactly the opposite process to that of f and by definition holds

$$f^{-1} \circ f = e.$$

If we confine ourselves to transformations f, g, h, ..., defined for all points on the plane, then their totality together with composition, presents a noteworthy similarity with the set of positive numbers and multiplication. I list the similarities (and one difference) in two parallel columns:

"Numbers"	*"Transformations"*
$z = x \cdot y$ (product)	$h = g \circ f$ (composition)
$x \cdot y = y \cdot x$ (commutativity)	$g \circ f \neq f \circ g$ (in general)
$x \cdot (y \cdot z) = (x \cdot y) \cdot z$ (associativity)	$h \circ (g \circ f) = (h \circ g) \circ f$
1 (unit)	e (identity transformation)
$y = x^{-1} \;\Leftrightarrow\; y \cdot x = 1$ (inverse)	$g = f^{-1} \;\Leftrightarrow\; g \circ f = e$

We will apply these rules in the next sections. I underline here the associative property $h \circ (g \circ f) = (h \circ g) \circ f$, which holds for transformations. This is due to their very nature as a correspondence process, which remains the same, any way we choose to group them (insert parentheses) in a particular composition of more than one transformations.

2 Isometries, general properties

A special category of transformations, we will deal with in the next sections, is that of *"isometries"* or *"congruences"* of the plane. With this naming we mean transformations, which are defined on the entire plane and additionally have the property of preserving distances ([Cox61, *p.* 39], [Sin95]). In other words, transformations $X' = f(X)$, such that, for every pair of points X, Y and their images X', Y' will hold

$$|X'Y'| = |XY|.$$

Theorem 1. *An isometry preserves angles.*

Proof. The short wording means that for three points X, Y, Z and their images X', Y', Z' through the transformation the angles $\widehat{YXZ}$ and $\widehat{Y'X'Z'}$ are equal. This however is a consequence of the property of the isometry of the transformation, on the basis of which $|X'Y'| = |XY|$, $|Y'Z'| = |YZ|$, $|Z'X'| = |ZX|$. In other words the triangles $X'Y'Z'$ and XYZ are congruent. From the congruence of triangles follows the equality of the angles as well. □

Exercise 1. *Show that the composition $g \circ f$ of two isometries f and g is again an isometry. Show also that the inverse transformation f^{-1} of an isometry is an isometry.*

Theorem 2. *An isometry maps a line ε onto a line ε'. If the isometry fixes two points A and B of ε, then it also fixes all the points of ε.*

Proof. The position of a point X of the line AB is completely determined by the ratio $\frac{|XA|}{|XB|}$ and the fact that $||XA| \pm |XB|| = |AB|$. The latter gives the necessary and sufficient condition so that X is on the line. If therefore the isometry f fixes points A, B then for the images X', A', B' will hold:

$$\frac{|X'A'|}{|X'B'|} = \frac{|XA|}{|XB|}, \quad \text{and} \quad |X'A'| \pm |X'B'| = |XA| \pm |XB|.$$

Consequently if X is contained in line $\varepsilon = AB$, then also X' will be contained in line $\varepsilon' = A'B'$. The second part follows immediately from the previous equalities, if we take into account that $A' = A$, $B' = B$. Then from these follows that for every point X of ε point X' coincides with X. □

Theorem 3. *An isometry, which fixes three non collinear points, coincides with the identity transformation.*

Proof. Assume that the isometry f fixes the points A, B and C. Then, according to the previous theorem, it also fixes the lines AB and AC. If X is a point not lying on these lines, we draw a line ε through X, which intersects AB and AC respectively at D and E, which are fixed by f. By the previous theorem f fixes all the points of DE therefore also X. □

Corollary 1. *Two isometries coincident at three points are coincident everywhere.*

Proof. Indeed, if f, g are the two isometries, then $g^{-1} \circ f$ will fix the three points, therefore it will coincide with the identity transformation $g^{-1} \circ f = e \iff f = g$, □

Exercise 2. *Show that an isometry f maps a circle κ onto a circle $\kappa' = f(\kappa)$ of equal radius.*

Theorem 4. *If an isometry f satisfies the relation $f \circ f = e$, then it fixes at least one point.*

Proof. Obviously the identity transformation has the property of the psoposition. Let us suppose then that f is not coincident with the identity transformation and let us denote with X' the point $f(X)$, so that, according to the hypothesis $X'' = X$. We consider now an arbitrary point X such that $X' = f(X) \neq X$. Such a point exists, for otherwise f would be the identity. Suppose M is the middle of XX'. We show that f fixes M. By hypothesis f exchanges X and X', therefore maps the line XX' to itself (Theorem 2). Also

$$|X'M'| = |XM| = |X'M| = |X''M'| = |XM'|.$$

The first and third equality hold because f is an isometry. The second because M is the middle of XX'. This means that point M' lies on the medial line of XX', but also, as we noted, it is a point of the line XX', therefore it coincides with M. □

3 Reflections or mirrorings

A line of the plane ε defines a simple transformation called *"reflection"* or *"mirroring"* relative to the line ε, which is called *"axis"* or *"mirror"* of the reflection. The process for this transformation is described as follows:

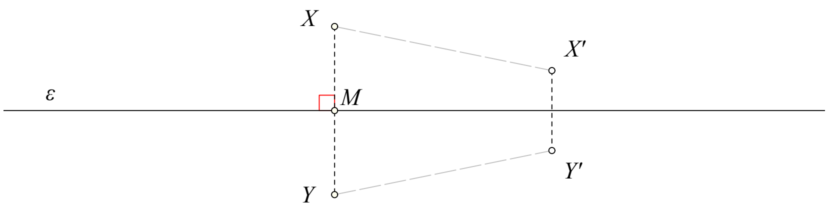


Figure 1: Mirroring or reflection relative to ε

a) To every point X not contained in the line ε corresponds the point Y, such that ε is the medial line of XY. In other words, point X is projected orthogonally to ε at its point M and XM is extended to its double towards M, until Y.
b) To every point X contained in the line ε the process corresponds the point X itself. In this case then, point X is, as we say, a *"fixed point of the transformation"*.

Thus, the reflection is well defined for every point of the plane or, in other words, its domain is the entire plane. The same happens also with its range. It also coincides with the entire plane, since for every Y there exists one X such that $f(X) = Y$. The line ε, through which a reflection is defined, consists of all the fixed points of the reflection.

Every point X not contained in ε is in correspondence with Y which is on the other side of ε than that where X is to be found. A reflection then interchanges the two sides of ε and leaves the points of ε fixed.

The reflection underlies the notion of *"axial symmetry"*: The shape Σ is axially symmetric, if there exists a reflection f, such that $f(\Sigma) = \Sigma$.

Theorem 5. *Every reflection is a plane isometry.*

Proof. By the figure 1, in which $XYY'X'$ is a trapezium, hence $|XX'| = |YY'|$. □

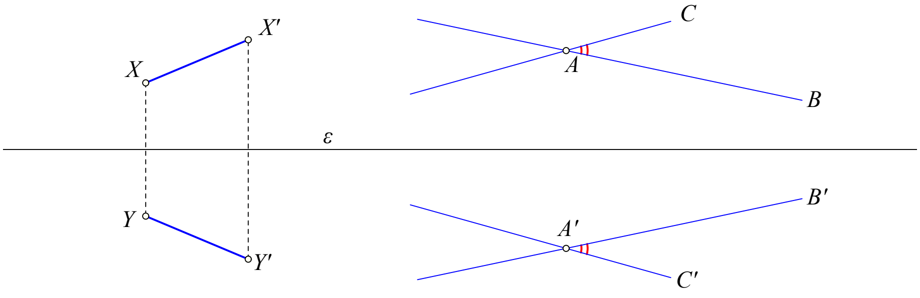


Figure 2: $|XX'| = |YY'|$ and $\widehat{BAC} = \widehat{B'A'C'}$

Theorem 6. *For every reflection f holds $f \circ f = e$, in other words the inverse of a reflection is the same transformation of the reflection.*

Proof. Indeed, if $Y = f(X)$, then point Y is the symmetric of X relative to the line ε, which defines the reflection. Then, however, point X is also the symmetric of Y relative to ε, consequently $X = f(Y)$ hence, for every X will hold $f(Y) = f(f(X) = X$, which means that the composition $f \circ f$ coincides with the identity transformation e. □

The important characteristic of reflections is that, as we say, they *"generate"* all the isometries of the plane. In other words, every isometry of the plane may be written as a composition of reflections. With a little more work we'll prove later next theorem ([Cox61, *p*. 46]):

Theorem 7. *Every isometry of the plane is either a reflection or a composition of two or three reflections.*

Figure 3: Transformation of point symmetry relative to O

Closely connected to reflections is also the other simple transformation we met, the point symmetry. A point O of the plane defines the transformation f of the *"point symmetry"* relative to O. This one for every $X \neq 0$ corresponds point $X' = f(X)$ to X, which is the symmetric of X relative to O. In other words the point X' for which O is the middle of segment XX'.

Theorem 8. *The composition $g \circ f$ of two reflections whose axes intersect orthogonally at O coincides with the point symmetry relative to O.*

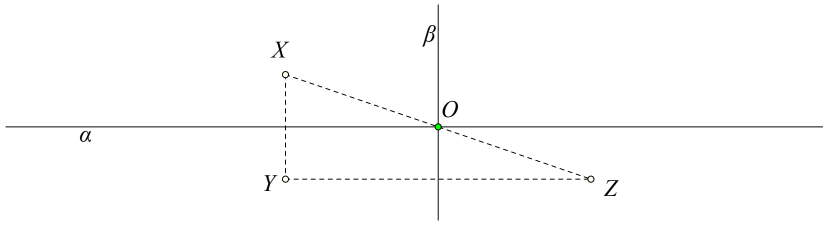


Figure 4: Point symmetry as composition of two reflections

Proof. The proof is suggested by the figure. If Y is symmetric relative to line α and Z is symmetric of Y relative to line β, which intersects line α orthogonally at O, then Z is also symmetric of X relative to point O. □

Theorem 9. *If an isometry f fixes two points A and B, then it also fixes all the points of the line AB and coincides with either the identity transformation or the reflection relative to the line AB.*

Proof. The first part of the theorem follows from Theorem 2. For the second, assume that $f \neq e$. It suffices to consider one point X off the line AB and see what is the point $X' = f(X)$. Triangles ABX and ABX' will be congruent and we see easily that they either

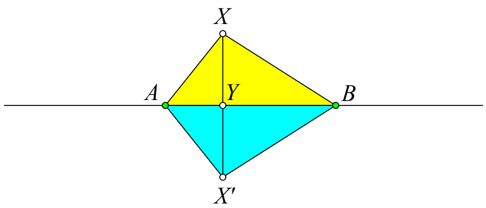


Figure 5: Isometry with two fixed points

coincide or they will be symmetric relative to AB. The first is excluded by assumption, therefore point X' will be the symmetric of X relative to AB. □

Theorem 10. *If an isometry of the plane f, different from the identity, satisfies the relation $f \circ f = e$ and has exactly one fixed point O, then it is coincident with the point symmetry relative to O.*

Proof. The proof is contained in theorem 4. We showed there that for every X with $X' = f(X) \neq X$ the middle M of the segment XX' is fixed. Then if there exists exactly one fixed point O, then all XX' will have the same middle O, which is the essence of point symmetry. □

Corollary 2. *An isometry of the plane f, different from the identity, which satisfies the relation $f \circ f = e$ is coincident with a symmetry relative to point O or with a reflection relative to line ε.*

Remark 1. For a given transformation f, the shape Σ of the plane is called *"symmetric"* relative to f, when $f(\Sigma) = \Sigma$. The shapes Σ which are symmetric relative to a point O are precisely those which satisfy $f(\Sigma) = \Sigma$, where f is the symmetry transformation relative to O. The shapes Σ which are symmetric relative to an axis ε are precisely those which satisfy $f(\Sigma) = \Sigma$, where f is the reflection relative to the line ε.

Remark 2. Transformations which coincide with their inverse ($f^{-1} = f \quad \Leftrightarrow \quad f \circ f = e$) are called *"involutions"* and play an important role in Euclidean, as well as in other Geometries ([VY10, *p.* 102 (I)]).

Exercise 3. *Show that, for two different lines of the plane ε and ε', there always exists a reflection f which transforms one to the other ($f(\varepsilon) = \varepsilon'$). In fact, depending on the position of the lines there exist exactly two or exactly one reflection which has this property. When exactly do these cases happen?*

Exercise 4. *Show that for two different line segments AB, CD of the same length, sometimes there exists a reflection which transforms one to the other $f(AB) = CD$ and sometimes there doesn't. When exactly does either case happen?*

Exercise 5. *Given two different lines (equal segments, congruent circles) examine when there exists a point symmetry which transforms one to the other.*

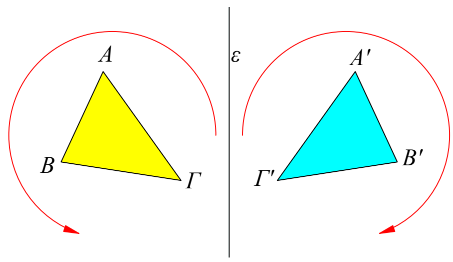


Figure 6: Reflections reverse orientation

Reflections are closely connected to the reversal of *orientation* of triangles and more generally of polygons. Every reflection f maps a triangle ABC onto a triangle $A'B'C'$, which has the opposite orientation (See Figure 6). The transformations we consider in this book

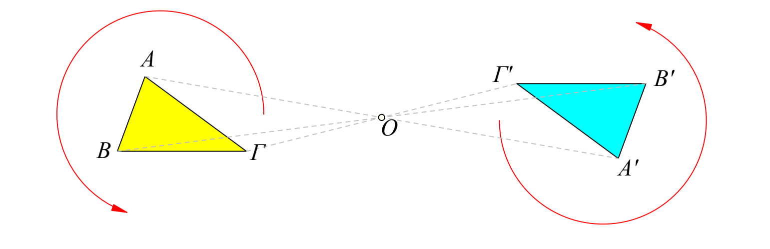


Figure 7: Point symmetries of the plane preserve orientation

(are, as it is said, *"continuous"* and) have the property that, if they preserve the orientation of a triangle, then they preserve the orientation of every other triangle. Respectively, if they reverse the orientation of a triangle, they will reverse the orientation of every other triangle. It suffices then to examine what happens to the orientation of a single triangle, for us to conclude if the specific transformation preserves or reverses the orientation.

Isometries which reverse the orientation of triangles, we say that they *reverse the orientation of the plane* and we sometimes distinguish with the name *"anti-isometries"*, while those that preserve orientation, we say that they *preserve the orientation of the plane* and we call them *"direct isometries"*. An example of a direct isometry is the point symmetry, which as we saw (Theorem 8) can be written as the composition of two reflections (See Figure 7). More generally than single reflections, every composition $f = f_k \circ \ldots \circ f_1$ of an odd number of reflections reverses the orientation of the plane, while every composition of an even number of reflections preserves the orientation.

4 Translations

"Translation" of the plane by AB is called the transformation f, which is defined by an oriented line segment AB. This transformation, to every point X of the plane, corresponds a point Y, such that the line segments XY and AB are parallel, equal and equally oriented

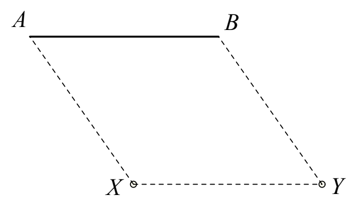


Figure 8: Translation by AB

(See Figure 8). Obviously the domain and range of this transformation is the entire plane. It is also obvious that parallel, equally oriented and equal line segments AB, CD define the same translation. From the definition follows immediately, that the inverse transformation f^{-1} is the translation by the inversely oriented line segment BA. Finally, the *"null translation"* is the identity transformation, considered as a translation by a segment whose end points coincide.

Theorem 11. *Every translation is an isometry.*

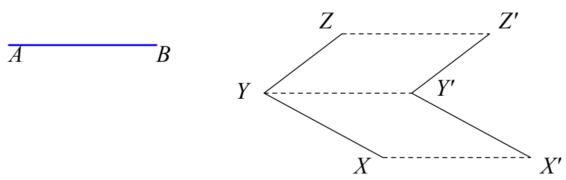


Figure 9: Translations are isometries

Proof. We must show that a translation f preserves distances. If X, Y are different points and $X' = f(X), Y' = f(Y)$ then $|XY| = |X'Y'|$. This follows immediately from the fact that $XX'Y'Y$ is a parallelogram. Segments XX' and YY' are by definition parallel, equal and equally oriented to AB, which defines the translation (See Figure 9). □

Theorem 12. *The composition $g \circ f$ of two translations by the oriented segments AB and CD is a translation by the oriented segment EZ. The segment EZ is defined as the side of the triangle EHZ which results from an arbitrary point E and from $H = f(E)$ and $Z = g(H)$.*

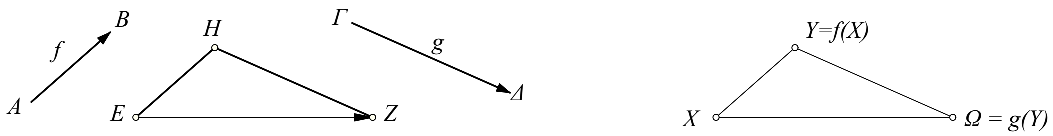


Figure 10: Composition of translations is a translation

Proof. Indeed if X is another arbitrary point different from E and $Y = f(X)$, $\Omega = g(Y)$, then the triangles $XY\Omega$ and EHZ will have respective sides parallel equally oriented and equal: $|XY| = |EH|$, $|Y\Omega| = |HZ|$, therefore they will be congruent and will also have their third sides parallel, equally oriented and equal: $|X\Omega| = |EZ|$. □

Corollary 3. *Consider the broken line with vertices $\{A_1, A_2, \ldots, A_k\}$. This defines $k-1$ translations $\{f_1, f_2, ..., f_{k-1}\}$ relative to its respective oriented sides $\{A_1A_2, A_2A_3, A_3A_4, ..., A_{k-1}A_k\}$. The composition of these translations is equal to the translation f which is defined by the oriented segment A_1A_k:*

$$f_{k-1} \circ f_{k-2} \circ \ldots \circ f_2 \circ f_1 = f.$$

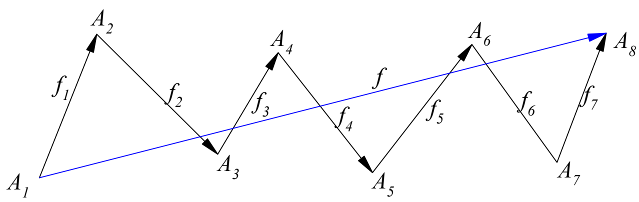


Figure 11: Composition of translations

Proof. The proof results (by induction on k) by applying the previous theorem and reducing gradually the number of the sides of the broken line. If for example, the broken line has four vertices $\{A_1, A_2, A_3, A_4\}$, then $f_2 \circ f_1 = g$ where g is the translation by A_1A_3 and $f_3 \circ g = h$, where h is the translation byA_1A_4. Totally then $f_3 \circ f_2 \circ f_1 = f_3 \circ g = h$. □

Corollary 4. *The composition of translations $f = f_k \circ f_{k-1} \circ \ldots \circ f_2 \circ f_1$ parallel to the oriented sides A_1A_2, A_2A_3, ..., $A_{k-1}A_k$, A_kA_1 of the polygon $A_1A_2...A_k$ is the identity transformation.*

Proof. In this corollary, which is a direct consequence of the previous one, we consider that the identity transformation e is a translation by a line segment whose end points coincide (*null translation*). □

Theorem 13. *The composition of two reflections relative to two parallel lines $\{\alpha,\beta\}$ lying at distance δ, is a translation by a line segment of length 2δ and direction orthogonal to that of the parallel axes, from α to β.*

Proof. Figure 12suggests the proof. If the axes α, β of the reflections are parallel at distance δ, then for every point X and its image $Y = f(X)$, $Z = g(Y)$ the distance will be $|XZ| = 2\delta$, since the middles M, N respectively of XY, YZ will be on α and β respectively. □

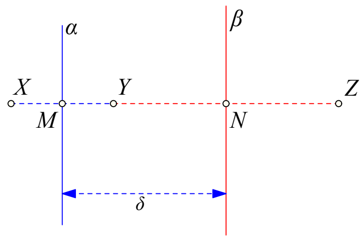
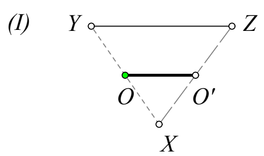


Figure 12: Translation from reflections

Remark 3. In the previous theorem the parallel axes of the two reflections can be positioned on any part of the plane. It suffices that they are orthogonal to the line segment AB, which defines the translation and their distance is equal to $\delta = \frac{|AB|}{2}$. The proof of the next theorem is not the simplest one, but demonstrates the use of composition of transformations and applies the previous remark.

Theorem 14. *The composition $g \circ f$ of a point symmetry f relative to point O and a translation g by the segment AB is the symmetry relative to a point O', where O' is the translation of O by AM, where M is the middle of AB. Similarly, the composition $f \circ g$ is a symmetry relative to the point O'', which is the translation of O by MA.*



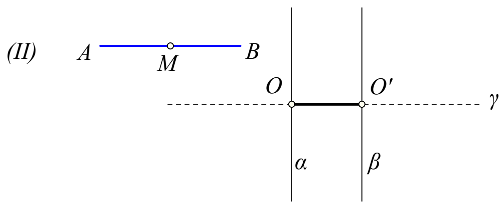


Figure 13: Translation and point symmetry ... and with compositions

Proof. Assume the composition order $g \circ f$ (the proof for the order $f \circ g$ is similar). According to Theorem 8, the symmetry f relative to a point O coincides with the composition of two reflections with axes intersecting orthogonally at O (See Figure 13-II). We therefore choose these axes γ and α, so that the first passes through O and is parallel to AB and the second passes through O and is orthogonal to AB. Then f is written as $f = f_\alpha \circ f_\gamma$, where f_α, f_γ are the reflections relative to the lines α and γ respectively. Also the translation is written as the composition $g = f_\beta \circ f_\alpha$, where f_β is the reflection relative to line β parallel of α and passing through point O', where OO' is parallel, equal and equally oriented to AM. Then, the composition that interests us is written:

$$g \circ f = (f_\beta \circ f_\alpha) \circ (f_\alpha \circ f_\gamma) = f_\beta \circ (f_\alpha \circ f_\alpha) \circ f_\gamma = f_\beta \circ e \circ f_\gamma = f_\beta \circ f_\gamma.$$

The equality between initial and final term gives the proof. □

Remark 4. On the last formula we use the fact that placement of parentheses may be arbitrary. This, because the composition of transformations is, as we say, *associative*. In other words for three transformations always holds

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

This is a direct consequence of the definition of transformation, as a process of correspondence of points. The question of how these processes are grouped, that is, where the parentheses will be, is irrelevant, as long as we don't change the order of application of these processes. The thing changes if we change the order of application of these processes. As it is seen also from the previous theorem, in general, for two transformations the order of application plays an important role, so

$$g \circ f \neq f \circ g.$$

In some cases, however, equality holds. When it holds $g \circ f = f \circ g$, we say that the transformations *"commute"*. One such case, for example, occurs in the case of a symmetry f_O relative to a point O. This, according to Theorem 8, is written as the composition of two reflections

$$f_O = f_\beta \circ f_\alpha,$$

relative to lines α and β respectively, which pass through O and are orthogonal. Besides the fact, that these lines may have an arbitrary orientation, provided they are orthogonal at O, so in this case it is easy to see that additionally holds

$$f_\beta \circ f_\alpha = f_\alpha \circ f_\beta.$$

The interesting characteristic of point symmetries and translations is that they are represented as compositions of reflections. The fact that this representation may be done in many ways is one additional characteristic, useful in many applications. The next theorem gives one such application.

Theorem 15. *The composition $f_P \circ f_O$ of two symmetries relative to two different points O and P, is a translation by the double of OP.*

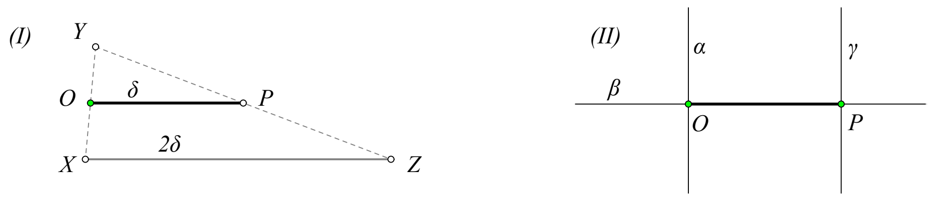


Figure 14: Two point symmetries ... simply ... and with compositions

Proof. The proof follows directly by writing $f_O = f_\beta \circ f_\alpha$ and $f_P = f_\gamma \circ f_\beta$, where f_α, f_γ are reflections relative to the lines orthogonal to OP and f_β is the reflection relative to the line $\beta = OP$ (See Figure 14-II). We have then

$$f_P \circ f_O = (f_\gamma \circ f_\beta) \circ (f_\beta \circ f_\alpha) = f_\gamma \circ (f_\beta \circ f_\beta) \circ f_\alpha = f_\gamma \circ e \circ f_\alpha = f_\gamma \circ f_\alpha.$$

The last composition, however, is exactly (Theorem 13) that one which defines the translation mentioned in the theorem. □

Corollary 5. *The composition of ν symmetries relative to ν points $A_1, A_2, ..., A_\nu$ is, for even ν a translation and for odd ν a symmetry.*

Proof. Indeed, let us denote by $f_1, f_2, ..., f_\nu$ the respective point symmetries. Then we can group their compositions in pairs

$$f = f_\nu \circ ... \circ (f_4 \circ f_3) \circ (f_2 \circ f_1).$$

If ν is even, then we have exactly $\mu = \nu/2$ pairs, representing each a translation (Theorem 13). Then their composition will also be a translation (Corollary 3). If ν is odd, then in the aforementioned composition participate $\mu = \frac{\nu-1}{2}$ pairs, which represent translations, therefore their composition will also be a translation. This translation is then composed with a symmetry f_ν and gives finally a symmetry (Theorem 14). □

Exercise 6. *Show that the composition of a reflection f_ε relative to line ε and translation f_{AB} relative to a line segment orthogonal to ε is a reflection $f_{\varepsilon'}$ relative to line ε' parallel to ε and at distance $\frac{|AB|}{2}$ from it.*

Exercise 7. *For which pairs of lines ε, ε' does there exist a translation which maps one to the other?*

Exercise 8. *Show that for two circles of equal radius there exist both a reflection and a translation which maps one to the other.*

Theorem 16. *Given ν different points $A_1, A_2, ..., A_\nu$, there exists exactly one polygon which has these points as middles of successive sides if ν is odd. If ν is even, in general, there is no such polygon. If however there exists one, then there exist infinitely many and, in fact, every point of the plane may be considered to be a vertex of such a polygon.*

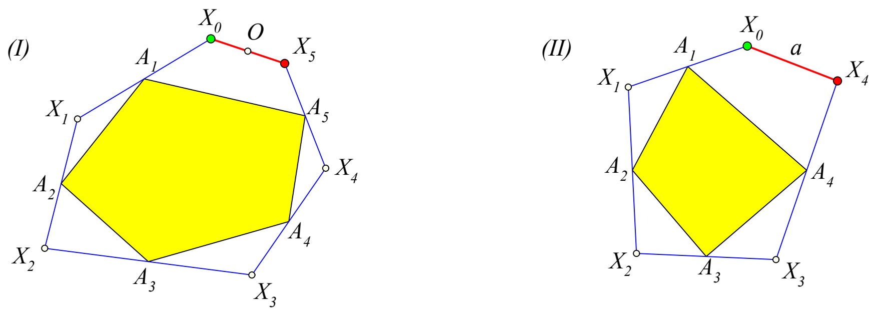


Figure 15: Polygons with given middles of sides

Proof. The proof results by tracing the orbit of an arbitrary point X_0, to which we apply successively the symmetry transformations $\{f_1, ..., f_\nu\}$ relative to the vertices $\{A_1, ..., A_\nu\}$:

$$X_1 = f_1(X_0), \quad X_2 = f_2(X_1), \ldots, \quad X_\nu = f_\nu(X_{\nu-1}).$$

According to Corollary 5, if the polygon has an odd number of vertices, then the composition $f = f_\nu \circ ... \circ f_1$ of these symmetries will be a new symmetry relative to some point O of the plane (See Figure 15-I). Consequently the last point $X_\nu = f(X_0)$ will always be the symmetric of X_0 relative to O and we will have coincidence $X_0 = X_\nu$ and therefore a closed polygon with the wanted properties, exactly then, when X_0 coincides with O. This shows the first part of the theorem.

The second part is proved by a similar argument. In this case the aforementioned theorem guarantees that f is a translation by a fixed line segment a (See Figure 15-II). Consequently, no matter which X_0 we use to start, the final X_ν will always be an image of X_0 relative to the translation by a. If, therefore, there exists one closed polygon ($X_0 = X_\nu$), then $a = X_0X_\nu$ will collapse to a point and the translation will coincide with the identity transformation e. Then, however, for every point X_0 the corresponding polygon will close and will satisfy the requirements of the theorem. □

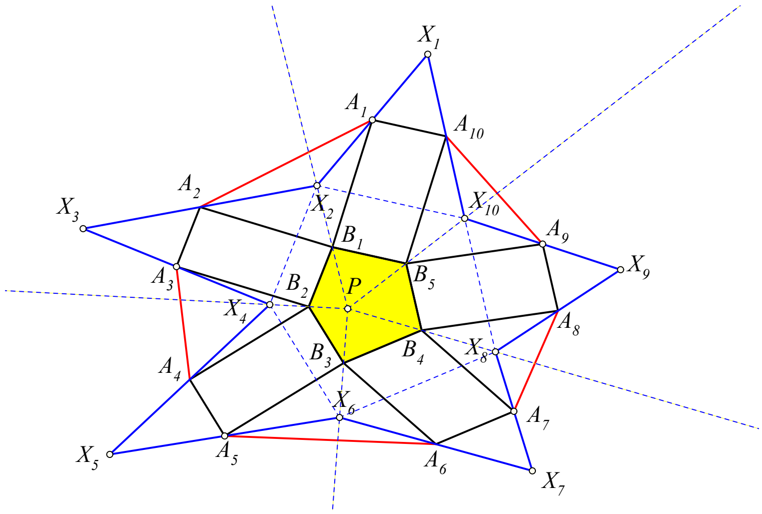


Figure 16: Polygons with an even number of sides

Next theorem points out a category of polygons of an even number of sides, for which the special case of the previous theorem applies: Every point X_0, produces a polygon with one of its vertex at X_0 and with predetermined middles of sides ([Yag62, *p*. 88, I]).

Theorem 17. *Let the polygon $a = A_1A_2...A_\nu$ for an even number $\nu = 2\mu$ be constructed from the polygon $b = B_1...B_\mu$, by attaching parallelograms to the sides of b (See Figure 16). Then for every point X_1 of the plane there is a polygon $x = X_1...X_\nu$, having for middles the vertices of a.*

Proof. Figure 16 shows one of these special polygons. Polygon a is a decagon and the respective b a pentagon. a was constructed by attaching to b parallelograms. The end points of the opposite sides of these parallelograms define the vertices of the decagon a. Let us assume then that we have one such polygon a and the respective b and let us consider an arbitrary point X_1 and the successively symmetric points relative to the vertices of a. We extend X_2B_1 towards B_1 by doubling it until point P. Because A_1 is also the middle of X_1X_2, X_1P will be parallel to and double of A_1B_1. Because A_2 is the middle of X_3X_2, X_3P will be parallel to and double of A_2B_1. Similarly X_5P will be parallel to and double of A_4B_2, and so on. This way we arrive at point $X_{\nu-1}$ (point X_9 in the figure) and we prove that $X_{\nu-1}P$ is parallel to and double of $A_{\nu-1}B_\mu$ (A_9B_5 in the figure). This implies that points X_ν, B_μ and P are collinear and point B_μ is the middle of the line segment $X_\nu P$. This, in turn, implies that the points X_ν, A_ν and X_1 are collinear and the polygon x has the desired properties. □

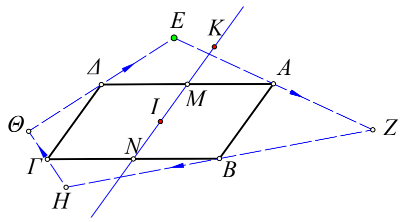


Figure 17: Composition of point symmetries

Exercise 9. *Let ABCD be a parallelogram and E a point. We define successively point Z symmetric of E relative to A, point H symmetric of Z relative to B, Θ symmetric of H relative to*

C (See Figure 17). Show that the symmetric Θ' relative to D always coincides with the initial point E. Also show that the created quadrilateral $EZH\Theta$ is a parallelogram, exactly in the case, in which the point E coincides with the symmetric K of the center I of the parallelogram, relative to the middle M of the side AD.

Exercise 10. *Show that the inverse transformation of a translation by (the oriented) segment AB is the translation by the segment BA.*

Exercise 11. *Show that a translation preserves the orientation of the plane.*

5 Rotations

In order to define the rotation we need the notions of *"oriented angle"* and of its *"signed measure"*. The oriented angle $\widehat{XOY}$ is an angle in which we distinguish the order of its sides OX, OY (See Figure 18). If the transition from OX to OY is opposite to the direction

Figure 18: Positively (+) and negatively (-) oriented angle $\widehat{XOY}$

of the clock's hands, then we consider the angle as being *"positively oriented"*, or simply a *positive angle*. If the transition is in the same direction as the clock's, we consider the angle as being *"negatively oriented"*, or simply a *negative angle*. The signed measure of an oriented angle $\widehat{XOY}$, which we denote by $(\widehat{XOY})$, coincides with $\pm|\widehat{XOY}|$, where $|\widehat{XOY}|$ is its usual measure. The sign is taken to be positive for positively oriented angles and negative for negatively oriented ones.

From the definition follows immediately, that for successive oriented angles $\widehat{XOY}$ and $\widehat{YOZ}$ the following rule is valid

$$(XOZ) = (XOY) + (YOZ).$$

"Rotation" of the plane, relative to the center O and by the (oriented) angle ω, is called the transformation f, which is defined by the rules: a) the center O of the rotation remains fixed ($f(O) = O$), b) to every other point X of the plane corresponds the point Y such that $|OY| = |OX|$ and the angle $\widehat{XOY}$ has signed measure ω.

Remark 5. As it is suggested by the figure 19, different rotations may have the same result. This way for example, for the same center O, the rotation by $\theta = \frac{\pi}{2}$ and the rotation by an angle of opposite orientation $-(2\pi - \theta) = -\frac{3\pi}{2}$ produce the same result. If we denote these rotations with f and g respectively, then $f(X) = g(X)$ for every point of the plane. The same happens also for every other positively oriented angle θ. For negatively oriented angles θ the same result can be had also with the angle $\theta' = 2\pi + \theta$. Consequently, if we are interested in the result and not in the process used to reach it, we may assume that a specific rotation f is done by an angle θ with $|\theta| \leq \pi$. The special case $|\theta| = \pi$ defines the so called *"half turn"*, which coincides with the symmetry relative to the center O of the rotation. This may be done either by rotating X by π or by rotating X by $-\pi$. In all other cases ($|\theta| \neq \pi$) we may assume that the rotation takes place by the unique angle

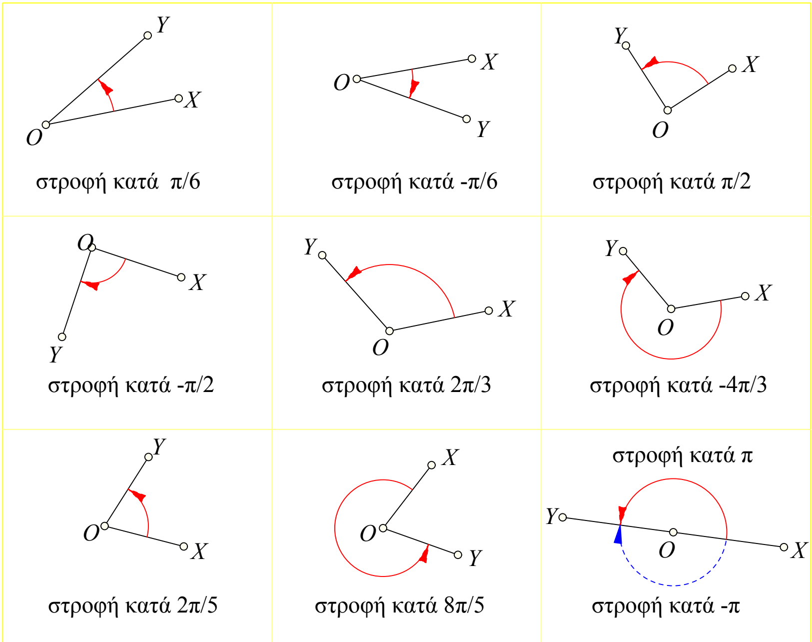


Figure 19: Some rotations

which satisfies the inequality $|\theta| < \pi$. A special case is also the identity transformation e. This may be considered as a rotation by a zero angle with center any point of the plane.

Theorem 18. *Rotations are isometries of the plane.*

Proof. If f denotes the rotation by ω relative to the center O, it suffices to show, that for two points X, Y and their images X', Y', holds $|XY| = |X'Y'|$. However this follows

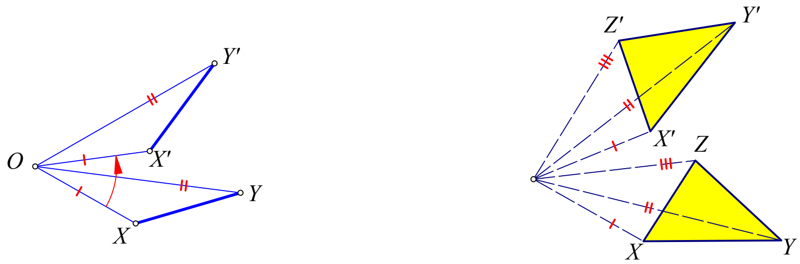


Figure 20: Rotations are isometries

from the fact that the triangles XOY and $X'OY'$ are congruent isosceli, because they have by definition of rotation $|OX| = |OX'|$, $|OY| = |OY'|$ and the angle $\widehat{XOY}$ is equal to $\widehat{X'OY'}$. Indeed $(X'OY') = (XOY') - (XOX') = (XOY) + (YOY') - (XOX') = (XOY) + \omega - \omega = (XOY)$. □

Exercise 12. *Given two points X, Y, show that there exist infinitely many rotations f with the property $f(X) = Y$. Also show that the centers of these rotations lie on the medial line of the segment XY.*

Theorem 19. *The composition of two rotations with the same center O and angles α and β is a rotation with center also O and rotation angle $\alpha + \beta$.*

Figure 21: Composition of rotations with the same center

Proof. The proof follows immediately from the definitions (See Figure 21). If point X is rotated first by α to Y, then $(XOY) = \alpha$. If, next, point Y is rotated by β to Z, then $(YOZ) = \beta$ and because the angles are successive $(XOZ) = (XOY) + (YOZ) = \alpha + \beta$. The figure to the right underlines that the relation holds also for negatively oriented angles. □

Exercise 13. *Show that the inverse transformation of a rotation f with center O and angle ω is the rotation with the same center and angle $-\omega$.*

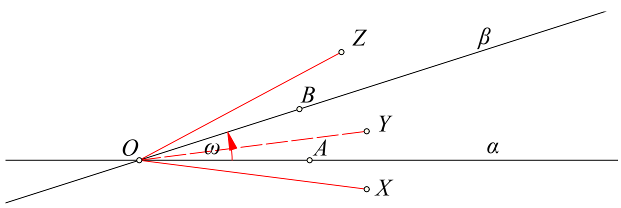


Figure 22: Rotation as the composition of two reflections

Theorem 20. *The composition of two reflections $f = f_\beta \circ f_\alpha$, whose axes intersect at point O at an angle of signed measure ω with $|\omega| \le \frac{\pi}{2}$, is a rotation with center point O and rotation angle 2ω (See Figure 22).*

Proof. The proof follows directly from the fact that, if $Y = f_\alpha(X)$, $Z = f_\beta(Y)$, then the angles $\widehat{XOY}$ and $\widehat{YOZ}$ are bisected by the axes of the reflections α and β respectively. Consequently $(XOZ) = 2\omega$. □

Remark 6. In the last theorem the order of composition of the reflections is important. In the composition $f_\beta \circ f_\alpha$ we must rotate from α to β. In the composition $f_\alpha \circ f_\beta$ we must rotate from β to α. Also from the two angles of different measure, which are formed by the two lines we consider the one which has the smaller measure. In the last figure, where we consider the composition $f_\beta \circ f_\alpha$, the angle which rotates the line α onto β is $\omega = \widehat{AOB}$, where A, B are points on α and β respectively. Here too, we can choose the smaller in absolute value oriented angle, which rotates α onto β. Besides, the case, where the two lines intersect orthogonally, this restriction determines uniquely the signed angle which does the work.

Theorem 21. *The composition of two rotations $g \circ f$ with different centers O, O' and angles respectively ϕ and ψ is a rotation when $\phi + \psi \neq 2k\pi$ (k integer), with center a point P, which is determined from the givens, and rotation angle $\phi + \psi$. When $\phi + \psi = 2k\pi$, then the composition of the rotations is a translation.*

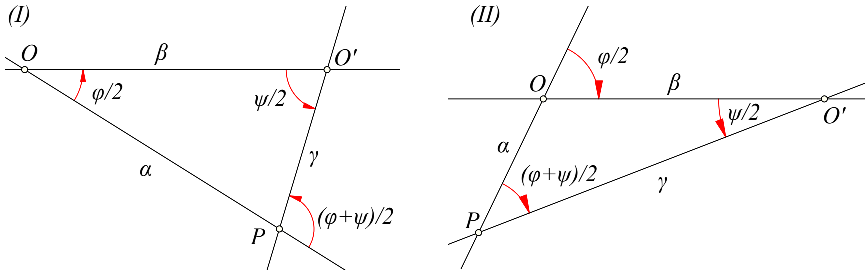


Figure 23: Composition of rotations with different centers

Proof. Let us express each rotation as a composition of two reflections. The first rotation f as a composition of two reflections relative to the lines α and β (See Figure 23-I). We can choose these lines to have any orientation we want, provided they pass through O and form there the angle $\frac{\phi}{2}$ (theorem 20). We choose them then so that β coincides with the line OO' which joins the centers of the two rotations. We consider the second rotation g as a composition of two reflections relative to two lines, the first of which coincides with β. Then the second required line for the expression of the rotation g will form with β at O' an angle equal to $\frac{\psi}{2}$. If f_α, f_β, f_γ denote the reflections relative to the corresponding lines, then we have:

$$g \circ f = (f_\gamma \circ f_\beta) \circ (f_\beta \circ f_\alpha) = f_\gamma \circ (f_\beta \circ f_\beta) \circ f_\alpha = f_\gamma \circ e \circ f_\alpha = f_\gamma \circ f_\alpha.$$

The last is a composition of two reflections, which defines a rotation, when the respective axes α and γ intersect (theorem 20). The intersection condition of these lines will be exactly $\frac{\phi+\psi}{2} \neq k\pi \quad \Leftrightarrow \quad \phi + \psi \neq 2k\pi$. In the case $\phi + \psi = 2k\pi$, the lines α and γ will be parallel and consequently the composition of the rotations $g \circ f = f_\gamma \circ f_\alpha$ will be a translation (Theorem 13). The figure 23-II underlines the case where the two rotations are negatively oriented. The sum takes account of the signed measures of the angles. The proof also gives the procedure by which the center P of the rotation $g \circ f$ (if it exists) can be constructed. □

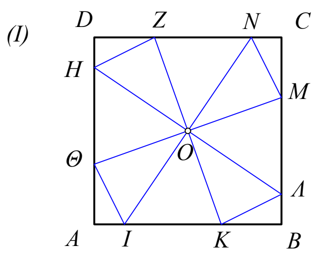
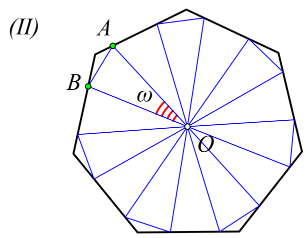


Figure 24: Square construction



Regular polygon construction

Exercise 14. *Construct a square $ABCD$ whose given is its center O and two points Z, H on the sides respectively CD and DA, with $|ZO| \neq |HO|$ (See Figure 24-I).*

Exercise 15. *Construct a regular polygon with n sides, whose given is the center O and two points $\{A, B\}$ lying on two successive sides and such that the angle $\omega = \widehat{AOB} < \frac{2\pi}{n}$ (See Figure 24-II).*

Exercise 16. *Show that a rotation preserves the orientation of the plane.*

Theorem 22. *Given two equal and non parallel line segments AB and $A'B'$ there exists exactly one rotation which maps A onto A' and B onto B'.*

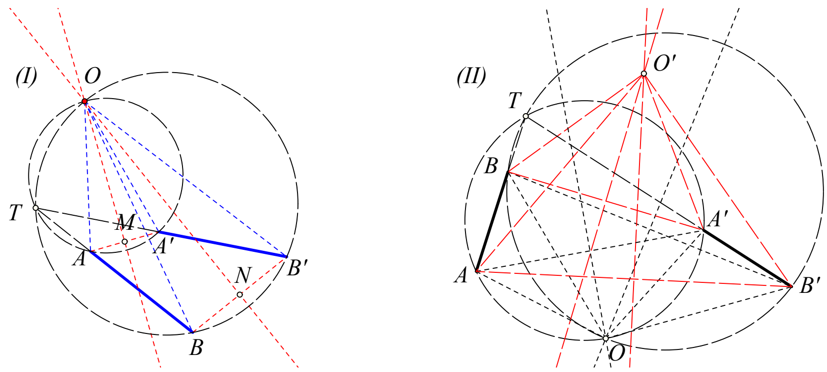


Figure 25: Rotations which map AB onto $A'B'$

Proof. Since the rotation will map A onto A' its center will be on the medial line of AA'. Similarly, its center will also be on the medial line of BB', therefore the center of the rotation will coincide with the intersection point O of these lines. The medial lines cannot be parallel, because then AB and $A'B'$ would be parallel. If the medial lines coincide, then $ABB'A'$ would be a trapezium and we take as center O the intersection of AB and $A'B'$. The hypothesis excludes the case of the trapezium being a rectangle. Because of the medial lines, the triangles OAB and $OA'B'$ are congruent and triangles OAA', OBB' are similar and the rotation angle is the one of signed measure $(AOA') = (BOB')$. □

Exercise 17. *In the previous figure show that the circles $(AA'O)$ and $(BB'O)$ intersect a second time at the intersection point T of AB and $A'B'$. Conclude that the rotation angle of the previous exercise is equal to the angle formed by the (extended) two line segments or its supplementary.*

Corollary 6. *Given two equal and non parallel line segments AB and $A'B'$, there exist exactly two rotations which map the segment AB onto $A'B'$.*

Proof. The first one is the one that maps A onto A' and B onto B' (Theorem 22) and the other the one that maps A onto B' and B onto A' (See Figure 25-II). □

Exercise 18. *The composition $g \circ f$ of a rotation f and a reflection g, whose axis passes through the center of the rotation, is a reflection.*

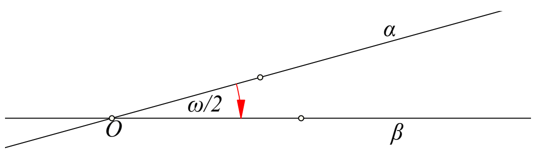


Figure 26: Composition of rotation and reflection through the center

Hint: Express the rotation as a composition $f = g \circ h$ of two reflections, one of which is g.

Exercise 19. *Show that, if an isometry of the plane fixes exactly one point O, then it coincides with a rotation with this point as the center.*

Hint: If f is the isometry and $X \neq O$, then point $X' = f(X)$ will have $|X'O| = |XO|$, in other words it will lie on the circle with center O and radius $|OX|$. Then the medial line ε of XX' passes through point O. Assume g is the reflection relative to ε. Then the composition $g \circ f$ fixes the points O and X, consequently it fixes all the points of the line OX, therefore it coincides with a reflection h with axis which passes through point O. We then have $g \circ f = h \quad \Rightarrow \quad f = g \circ h$, in other words f is the composition of two reflections with axes which intersect at O.

Exercise 20. *Using the conclusions of the three previous exercises prove Theorem 7.*

Hint: To show that every isometry f, different from the identity, is the composition of at most three reflections, we consider the fixed points of f. If it has exactly one, then (Exercise 19) it is a rotation, therefore a composition of two reflections. If it has two fixed points, then it also has a whole line consisting of fixed points and consequently coincides with a reflection (Theorem 9). If it fixes no point, consider an arbitrary point X and its image $X' = f(X)$. The reflection g with axis the medial line of XX' defines a composition $h = g \circ f$ which fixes point X. Therefore h will be either a reflection or a rotation and consequently $f = g^{-1} \circ h$ will be the composition of two or three reflections.

6 Congruence

Isometries lie at the root of the notion of *"congruence"* or *"isometry"* between shapes of the Euclidean plane, which is defined as follows:

"Two shapes Σ and Σ' of the plane are congruent or isometric, if and only if there exists an isometry f, which maps the one to the other ($f(\Sigma) = \Sigma'$)."

In the next exercises, we give specific shapes Σ, Σ' and we search for an isometry f, which satisfies the above definition. Most of these exercises have been already expressed in another form in the previous sections.

Exercise 21. *Find an isometry, which maps a line α onto a line β (identical to Exercise 3).*

Exercise 22. *Find an isometry, which maps a circle α onto a circle β of equal radius (identical to Exercise 8).*

Exercise 23. *Given are lines α, β, which intersect and one of the formed angles between them is angle ω. If also the lines α', β' intersect under the same angle, then find an isometry f, which maps line α onto α' and line β onto β'.*

Exercise 24. *Find an isometry, which maps a line segment AB onto another line segment CD of the same length.*

Hint: Theorem 22 gives the solution for the general case, it leaves however some special cases, which must be dealt with.

Exercise 25. *Given are two congruent triangles ABC and $A'B'C'$. Find an isometry which maps the one to the other.*

Hint: Again Theorem 22 applied to the line segments AB and $A'B'$, defines a rotation f (or translation) which maps the one line segment to the other and drifts the triangle ABC to a congruent $A'B'D$ ($f(A) = A'$, $f(B) = B'$, $f(C) = D$). The two congruent triangles $A'B'D$ and $A'B'C'$ have in common the side $A'B'$, therefore they will either be coincident ($D = C'$), which is exactly the case when the given triangles are similarly oriented, or D will be the mirror image of C'. In this case by composing with the reflection g relative to

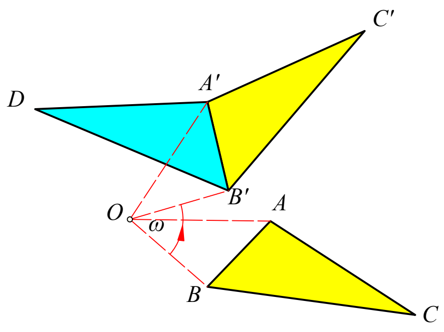


Figure 27: Rotation which maps ABC to $A'B'C'$

$A'B'$ we get the isometry $g \circ f$, which maps ABC onto (the reversely oriented) $A'B'C'$.

According to Theorem 7, every isometry is the composition of at most three reflections. We saw that compositions of two reflections give rotations or translations. Here we will examine compositions of three reflections, which lead to the so called *glide reflections*. We call *"glide reflection"* the composition $g \circ f_\varepsilon$ of a reflection f relative to line ε and a translation g by an oriented segment AB parallel to ε. We can easily see that in this definition the order of the composition is irrelevant, in other words it holds $g \circ f_\varepsilon = f_\varepsilon \circ g$.

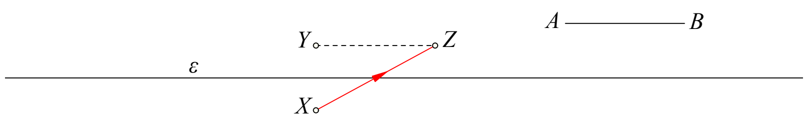


Figure 28: Glide reflection $Z = h(X)$

Remark 7. When point X is on the axis ε of the glide reflection h, then the corresponding image $Z = h(X)$ is also on the axis and XZ is equal and similarly oriented to the line segment AB, which defines the translation of the glide reflection. For every point not lying on the line ε the corresponding image $Z = h(X)$ lies on the opposite side of ε. Consequently if, for a given glide reflection h, a line ε with $h(\varepsilon) = \varepsilon$ is found (we say: ε is invariant relative to h), then this is the axis of the glide reflection, and for one of its points X and its image $Z = h(X)$, segment XZ is the translation of the glide reflection.

Theorem 23. *The composition of a translation f_{AB} by the oriented line segment AB and a reflection f_ε relative to a line ε is a glide reflection.*

Proof. The formulation leaves the order of composition of the two isometries on purpose indeterminate. The proof for both cases is the same. Let us assume then that we have the order $f_\varepsilon \circ f_{AB}$. If AB is parallel to ε, then no proof is needed. If AB is not parallel to ε, then

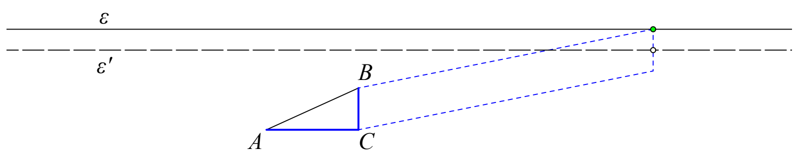


Figure 29: Composition of translation-reflection $f_\varepsilon \circ f_{AB}$

it is a hypotenuse of a right triangle ACB with AC parallel and CB orthogonal to ε. The translation f_{AB} is then written as the composition $f_{AB} = f_{CB} \circ f_{AC}$ and the composition becomes

$$f_\varepsilon \circ f_{AB} = f_\varepsilon \circ (f_{CB} \circ f_{AC}) = (f_\varepsilon \circ f_{CB}) \circ f_{AC} = f_{\varepsilon'} \circ f_{AC}.$$

The replacement of the parenthesis is done relying on Exercise 6. □

Theorem 24. *The composition of three reflections $f = f_{CA} \circ f_{BC} \circ f_{AB}$ relative to the sides of a triangle is a glide reflection.*

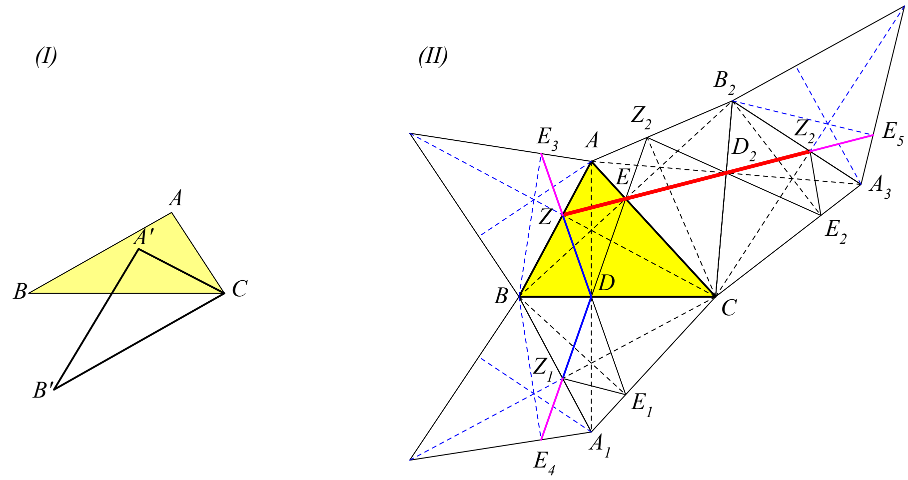


Figure 30: Composition of reflections in the sides of triangle

Proof. We write the composition as $f = f_{CA} \circ f_{BC} \circ f_{AB} = (f_{CA} \circ f_{BC}) \circ f_{AB} = g \circ f_{AB}$. The transformation $g = f_{CA} \circ f_{BC}$ is a rotation and we can rotate the angle ACB to a position $A'CB'$ so that $B'C$ is parallel to AB, while preserving $g = f_{CA'} \circ f_{B'C}$ (See Figure 30-I). Then

$$f = g \circ f_{AB} = f_{CA'} \circ f_{B'C} \circ f_{AB} = f_{CA'} \circ (f_{B'C} \circ f_{AB}) = f_{CA'} \circ h.$$

Here $h = f_{B'C} \circ f_{AB}$ is a translation by the double of their distance, because $B'C$, AB are parallel. The conclusion follows by applying the previous theorem. □

Theorem 25. *The composition of three reflections $f = f_{CA} \circ f_{BC} \circ f_{AB}$ relative to the sides of a triangle ABC is a glide reflection relative to the line, which is defined by the side EZ of the "orthic" triangle which is opposite to the vertex A. The distance of the translation is equal to the perimeter of the orthic triangle DEZ.*

Proof. From the previous theorem we know that f is a glide reflection. The proof follows from remark 7 in combination with the properties of the orthic triangle DEZ of ABC, the basic of which is, that the altitudes of ABC are bisectors of DEZ. This has implies that the side ZE of the orthic, which is opposite to vertex A, gets mapped onto an equal segment Z_2E_5 on the same line (See Figure 30-II). The proof results by considering the points Z, E and following the trajectories of their images through the successive reflections building up f. This is seen better in figure 30, instead of using a verbal description. In this figure the various triangles result by reflection of ABC and its orthic on the sides of ABC. □

Theorem 26. *Every isometry is either a reflection or a translation or a rotation or a glide reflection.*

Proof. Combination of Theorem 7, of Theorem 12, of theorem 20 and of theorem 24. □

Exercise 26. *Let f be a glide reflection, which maps a line ε to the line $\varepsilon' = f(\varepsilon)$. Show that the lines $\{\varepsilon, \varepsilon'\}$ are parallel, if and only if, the line ε is parallel or orthogonal to the axis of the glide reflection.*

7 Some compositions of isometries

By Theorem 7 the product (=composition) of more than 3 reflections reduces to a product of at most three reflections. Thus, since every isometry of the plane is a product of reflections, the product of an arbitrary number of isometries reduces also to a product of at most three reflections. Hence (Theorem 26) to a reflection or translation or rotation or a glide-reflection. The claims formulated in the next two exercises follow from this general remark. The important point though is to determine the defining elements of the composition (centers, axes etc.) from those of the factors.

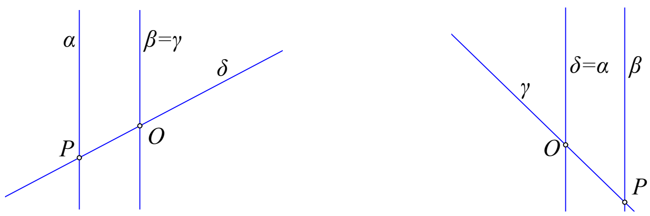


Figure 31: Composition of rotation and translation

Exercise 27. *Show that the composition of a rotation and a translation is a rotation by an angle equal in measure to the angle of the participating in the composition rotation.*

Hint: By figure 31. Analyze the factors in products of reflections as shown. The figure illustrates the two cases $\{g \circ f\ ,\ f \circ g\}$ with f a translation and g a rotation with center O. Point P is the center of the resulting rotation.

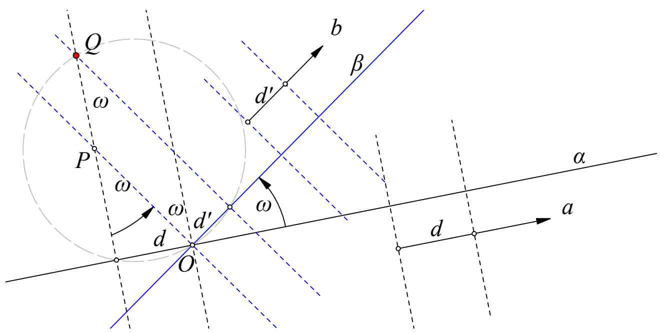


Figure 32: Composition of two glide-reflections

Exercise 28. *Show that the composition of two glide-reflections is a rotation with angle equal in measure to twice the angle of the axes of the glide-reflections.*

Hint: By deciphering figure 32 using also the previous exercise. In the figure the two glide reflections $\{f, g\}$ have respectively axes $\{\alpha, \beta\}$ and translations $\{a, b\}$. The rotation with center at Q is the composition $g \circ f$.

Exercise 29. *On a billiard table are placed two balls X, Y. Determine the trajectory of ball X, which reflected in the four walls will hit next ball Y. Examine also the case where the ball after the reflections in the four walls of the rectangle returns to its original position.*

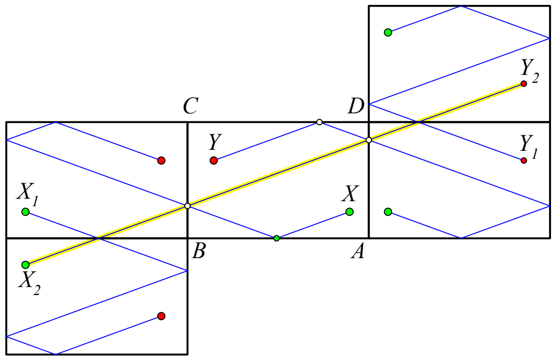


Figure 33: Billiard trajectory

Hint: ([Cat52, *p.*7]) Y_1 is the reflected of Y relative to side AD. Y_2 is the reflected of Y_1 relative to CD. X_1 is the reflected of X relative to BC. X_2 is the reflected of X_1 relative to AB. The segment X_2Y_2 determines the reflection points.

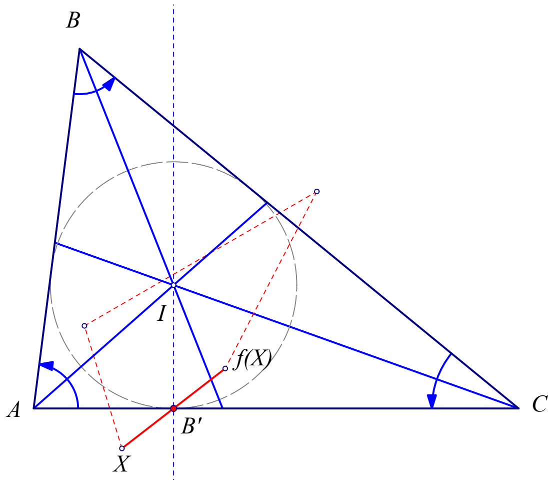


Figure 34: Rotations w.r. to triangle-angles

Exercise 30. *Consider the rotations $\{f_1, f_2, f_3\}$ about the vertices $\{A, B, C\}$ of the triangle ABC by angles equal to the respective positively oriented angles $\{\alpha, \beta, \gamma\}$ of the triangle (See Figure 34-II). Show that their composition $f = f_3 \circ f_2 \circ f_3$ is a point-symmetry with respect to the contact point B' of the inscribed circle with side AC.*

Hint: Since the angle-sum $\alpha + \beta + \gamma = \pi$, the composition f of the rotations is certainly (Proposition 21) a rotation by π or a half-turn i.e. a point symmetry. Show that B' remains constant under f, hence it is the center of symmetry.

The last exercise can be generalized, initially, for convex polygons with ν sides $A_1A_2\ldots A_\nu$. Thus, considering the angles of the polygon to be positively oriented, we can define the composition of rotations

$$f = f_\nu \circ f_{\nu-1} \circ \cdots \circ f_2 \circ f_1,$$

where each f_i is the rotation about A_i by the angle ω_i of the polygon at A_i. Since the sum of the polygon-angles $\omega_1 + \cdots + \omega_\nu = (\nu - 2)\pi$, by the general theorems for rotations, we conclude that f, for odd ν will coincide with a point symmetry and for even ν will coincide with a translation.

It is easy to see that f leaves the line $\varepsilon = A_\nu A_1$ invariant, i.e. maps ε to itself. Consequently, in the case of the point-symmetry, the center of the symmetry will be on ε and in the case of the translation, the oriented segment of translation will be parallel to ε. Next exercises discuss the simplest cases of such compositions.

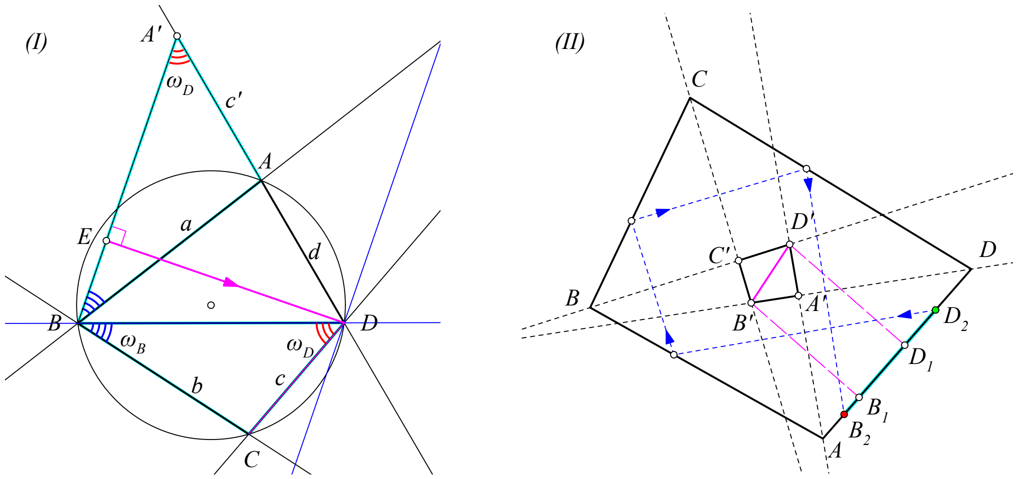


Figure 35: 4 reflections composition 4 rotations composition

Exercise 31. *Consider the cyclic quadrilateral $ABCD$, and the composition $f = f_4 \circ f_3 \circ f_2 \circ f_1$ of the reflections with respect to the sides $\{AB, BC, CD, DA\}$ having respective lengths $\{a, b, c, d\}$. Show that f is a translation parallel to the oriented segment ED (See Figure 35-I)*

$$\textit{of length}\ \ \frac{ac+bd}{R} \quad \textit{and slope to the diagonal}\ \ BD\ :\ \widehat{BDE} = \frac{\pi}{2} - \widehat{B},$$

where R is the circumradius.

Hint: Separate the composition in pairs $f = (f_4 \circ f_3) \circ (f_2 \circ f_1)$. The first two reflections $g = (f_2 \circ f_1)$ define a rotation about B by an angle equal to the double of $\widehat{B}$. This rotation can be represented also by two other reflections $g = f_2' \circ f_1'$, where f_1' is the reflection on BA' and f_2' is the reflection in the diagonal BD. For this, it suffices for BA' to make with BD the same angle $\widehat{B}$. The second pair $h = f_4 \circ f_3$ represents also a rotation, which, can be tto represented by the pair of reflections $h = f_4' \circ f_3'$, where $f_3' = f_2'$ is the reflection on DB and f_4' is the reflection on the line which forms with BD an angle equal to $\widehat{D}$. Because of the hypothesis $\widehat{B} + \widehat{D} = \pi$, the reflection-axes of $\{f_1', f_4'\}$ are parallel and the composition

$$f = (f_4 \circ f_3) \circ (f_2 \circ f_1) = (f_4' \circ f_3') \circ (f_2' \circ f_1') = f_4' \circ f_1',$$

is the translation by the double of the distance of these parallels. The rest follows from simple calculations. Noticable is the symmetry of the expression in terms of the lengths of sides, which, by Ptolemy's theorem, can be represented also through the product of the diagonals.

Exercise 32. *Let $f = f_4 \circ f_3 \circ f_2 \circ f_1$ be the composition of the rotations about the vertices, correspondingly, $\{A, B, C, D\}$ of the quadrilateral $ABCD$ by the respective positively oriented angles of the quadrilateral. Show that f is a translation by the oriented segment D_2B_2 of the side*

DA. This segment is the double of the projection D_1B_1 on DA of the diagonal of a quadrilateral $A'B'C'D'$ (See Figure 35-II). This quadrilateral is cyclic and is formed by the intersection points of the inner bisectors of the angles of the given quadrilateral $ABCD$.

Hint: The rotation f_1 about A can be represented as a composition of two reflections $f_1 = h_1 \circ g_1$. g_1 is the refelection with respect to the bisector AA' of $\widehat{A}$. h_1 is the reflection in the side AB. Similarly, the rotation f_2 about B can be represented as a composition $f_2 = h_2 \circ g_2$, where $g_2 = h_1$ is, again, the reflection on the side AB and h_2 is the reflection on the bisector BC' of $\widehat{B}$. Then the composition of the rotations

$$f_2 \circ f_1 = (h_2 \circ g_2) \circ (h_1 \circ g_1) = h_2 \circ g_1,$$

is the composition of the two reflections in the two consecutive sides $\{A'D', D'C'\}$ of the quadrilateral $A'B'C'D'$. A similar argument shows that the composition of the rotations about C and D coincides with the composition of the reflections in $B'C'$ and $C'D'$. Apply the previous exercise to the inscriptible quadrilateral $A'B'C'D'$.

Exercise 33. *Show that a convex quadrilateral $ABCD$ is circumscriptible if and only if, the composition $f = f_4 \circ f_3 \circ f_2 \circ f_1$ of the rotations by the positive oriented angles of the quadrilateral, represents the identity transformation.*

Exercise 34. *Prove that taking the composition r of successive Reflections on the sides of a generic non-cyclic quadrangle $EFGH$ produces a rotation $r(O, \phi)$, where the center O and the angle ϕ of rotation depend on the quadrangle.*

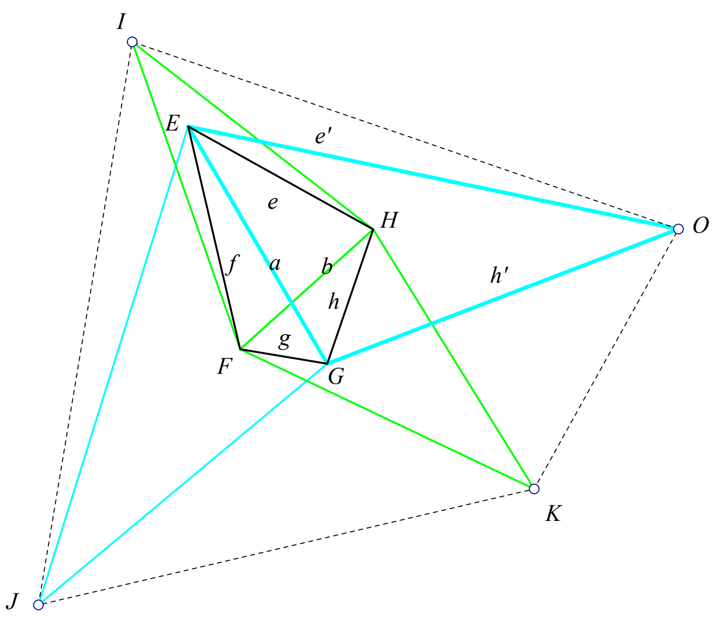


Figure 36: Reflecting on the sides of a generic quadrangle

Hint: Denote by $\{e = HE, f = EF, g = FG, h = GH, a = EG, b = FH\}$ and by $r = (hgfe)$ the order of the reflections (See Figure 36). The partial compositions (fe) and (hg) are rotations, hence they can represented by some other reflections $\{(ae'), (h'a)\}$ such that the angles $\{\widehat{ae'}, \widehat{h'a}\}$ are equal to the corresponding $\{\widehat{fe}, \widehat{hg}\}$. From this remark follows the construction of O as the intersection point of $O = e' \cap h'$, the rotation angle being the double in measure of the angle $\widehat{e'h'}$.

Remark 8. By the construction of O in the last exercise results also the measure of the oriented angle of rotation which is equal to

$$\phi \;=\; 2\widehat{e'h'} \;=\; 2(\pi - \widehat{E} - \widehat{G}) \;=\; \widehat{F} + \widehat{H} - \widehat{E} - \widehat{G}\,.$$

Remark 9. In the last exercise, considering now the order of reflections $(fehg)$ and doing the corresponding construction, we obtain the analogous triangle EGJ on the diagonal EG, which is symmetric to the EGO w.r. to that diagonal. Point J is the corresponding rotation center and the rotation angle is also ϕ. Analogously the rotation orders $\{(gfeh), (ehgf)\}$ create the rotation centers $\{I, K\}$, symmetric w.r. to FH with inversely oriented rotation angle $-\phi$. Next table summarizes these data.

center	composition	angle	rotation
O	$(hgfe)$	$+\phi$	r_O
K	$(ehgf)$	$-\phi$	r_K
J	$(fehg)$	$+\phi$	r_J
I	$(gfeh)$	$-\phi$	r_I

Exercise 35. *With the notation and conventions of the preceding exercise, the side-lines of the quadrangle $EFGH$ are orthogonal bisectors of respective sides of the quadrangle $OKJI$. More precisely*

$$e = HE \perp OK\,, \quad f = EF \perp KJ\,, \quad g = FG \perp JI\,, \quad h = GH \perp IO.$$

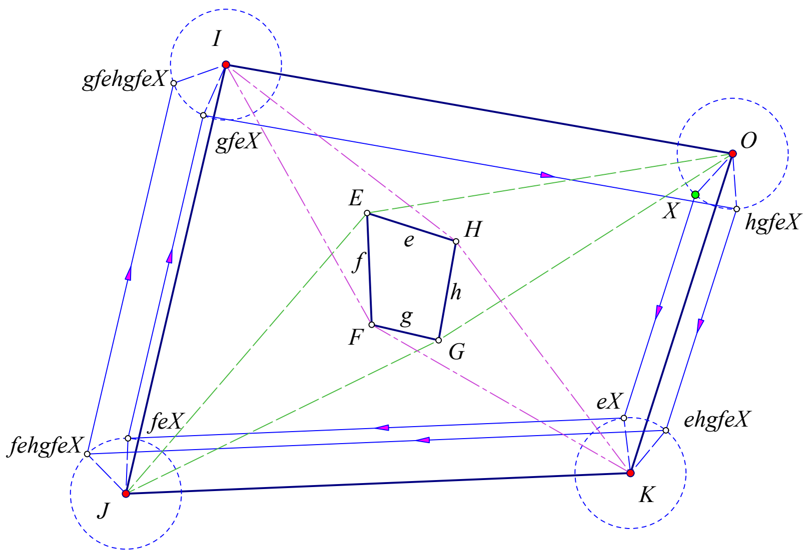


Figure 37: The quadrangle $OKJI$

Hint: Consider the orbit of a point X, i.e. the transformed points under the reflections on the side-lines of $EFGH$:

$$X\,, \quad eX\,, \quad feX\,, \quad gfeX\,, \quad hgfeX\,, \quad ehgfeX\,, \quad fehgfeX\,, \ldots$$

To see the orthogonality $e = HE \perp OK$ we consider the segments

$$X - eX \quad \text{and} \quad hgfeX - ehgfeX\,.$$

Both are orthogonal at their middle to e, hence they are parallel. Their endpoints are rotated correspondingly: X to $hgfeX$ by r_O and eX to $ehgfeX$ by r_K by opposite angles $\pm\phi$. It follows that the quadrangle with these endpoints is an isosceles trapezium and the isosceli based at its lateral sides are equal. This implies that KO is also orthogonal to e at its middle. Analogous arguments show the remaining orthogonalities.

Exercise 36. *(Hjelmslev's theorem) Prove that for a line α and its isometric image $\beta = f(\alpha)$ by an isometry f of the plane, the midpoints of all the segments XX', where $X' = f(X)$, are either collinear on the same line γ or coincident to a point O ([Cox61, p. 47]).*

Hint: From theorem 22 follows that there are two types of isometries between two lines: rotations and translations preserving the orientation, and reflections and glide reflections reversing it. Examine in each case what happens with the middles of XX'. Show that for glide reflections all these middles are on its axis.

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