Lahire's triangle construction problem

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Abstract

Here we study the problem of constructing a triangle from the data $\{\alpha, b+c, h_A\}$. The key-point is the detection of a circumstance where $b+c$ appears in explicit form.

1 The problem

Denoting, as usual, by $\{a = |BC|, b = |CA|, c = |AB|\}$ the side-lengths, by $\{\alpha, \beta, \gamma\}$ the angles of the triangle ABC and by h_A the altitude from A, the problem of Lahire is, to construct the triangle from the given data $\{\alpha, b+c, h_A\}$.

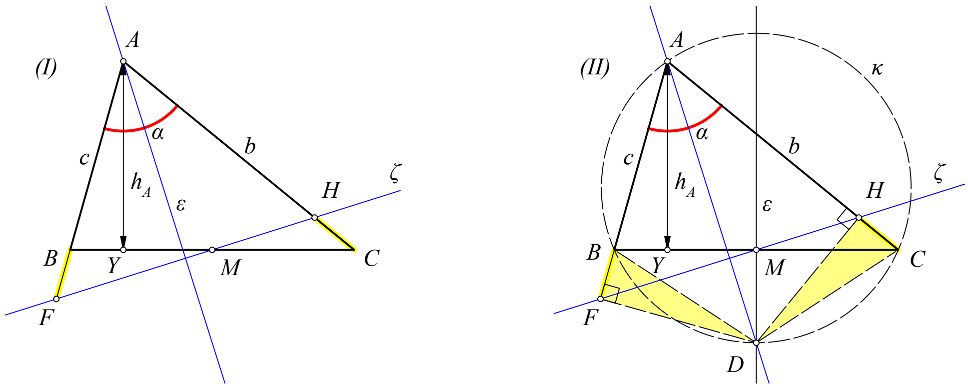

Figure 1: Representing the sides-sum $b+c$

A key-point, to solve the problem geometrically, is to realize that the sides-sum $b+c$ appears in an isosceles triangle AFH with appex A (See Figure 1-I). This isosceles is constructed by drawing from the middle M of BC the line ζ orthogonal to the bisector ε of angle $\widehat{A}$. This claim results from the following lemma.

Lemma 1. *If, from the midle M of BC, we draw a line ζ orthogonal to the bisector ε of the angle $\widehat{A}$, then this line intersects the sides $\{AB, AC\}$ at points, correspondingly $\{F, H\}$, so that $|BF| = |CH|$ and, consequently $|AF| + |AH| = b + c$.*

Proof. The proof follows by noticing that the bisector ε of $\widehat{A}$ passes from the middle D of the arc $\widehat{BDC}$ of the circumcircle κ of ABC (See Figure 1-II). In addition, $\{DF, DM, DH\}$ are the verticals from D on the sides and consequently line ζ is the Simson line of point D. Points $\{F, M, H\}$ are on this Simson line and the right-angled triangles $\{BFD, CHD\}$ are easily seen to be equal. □

2 The quadratic equation

The solution to Lahire's problem follows by showing that $x = |DM|$ satisfies a quadratic equation depending on the given data. The derivation of the equation exposed below follows closely the one given by Altshiller-Court [Cou80, p.144].

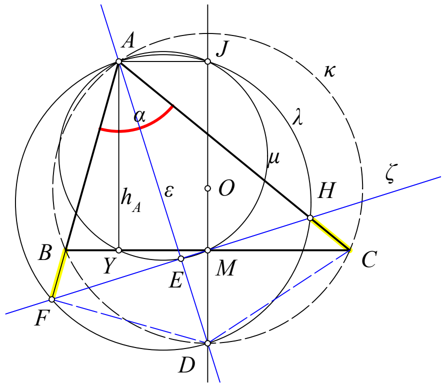

Figure 2: The circles $\{\lambda, \mu\}$

Notice first that the circumcircle λ of the isosceles AFH passes through D, having AD as a diameter (See Figure 2). If E denotes the intersection of lines $\{\varepsilon, \zeta\}$ and J is the projection of A on line DM, then the five points $\{A, Y, E, M, J\}$ are on a circle μ with diameter AM. Here Y is the foot of the altitude on BC. This follows easily from the fact, that all three points $\{Y, E, J\}$ see the segment AM under a right angle. Point J is also on the circle λ, since it is viewing its diameter AD under a right angle. Thus J is the second intersection point of circles $\{\lambda, \mu\}$ and $AJMY$ is a rectangle. Using these facts, we can now calculate the difference of squares:

$$\begin{aligned}
|FD|^2 - |DM|^2 &= |FE|^2 - |EM|^2 = (|FE| + |EM|)(|FE| - |EM|) \\
&= |FM||MH| = |MD||MH| = |MD||AY| \quad \Rightarrow \\
|FD|^2 - x^2 &= x \cdot h_A, \quad \text{while} \quad |FD| = |AF| \tan\left(\frac{\alpha}{2}\right) = \frac{b+c}{2} \tan\left(\frac{\alpha}{2}\right).
\end{aligned}$$

3 The solution

From the data $\{\alpha, b+c\}$ construct the isosceles AFH and determine D on the bisector of angle $\widehat{A}$, hence the length $|FD|$. Solving the previous quadratic, determine the length $x = |DM|$. The line BC is a common tangent to the circles with centers at $\{A, D\}$ and respective radii $\{h_A, |DM|\}$.

REMARK There is also another method to represent the sum $b + c$ using the respective altitudes $\{h_B, h_C\}$. This is described by the following lemma.

Lemma 2. *Given the measure α of the angle $\widehat{A}$, the lengths $\{b + c, h_b + h_c\}$ are respectively hypotenuse and vertical side of a right-angled triangle with an angle equal to α (or its complement).*

Proof. Extend h_B by the length h_C and draw from the resulting point D a parallel DA' to side AC (See Figure 3-I). Then $A'AC$ is isosceles, since it has equal altitudes from $\{C, A'\}$ and $A'BD$ is a right-angled triangle with the stated properties. Figure 3-II shows the case of an obtuse-angled triangle. □

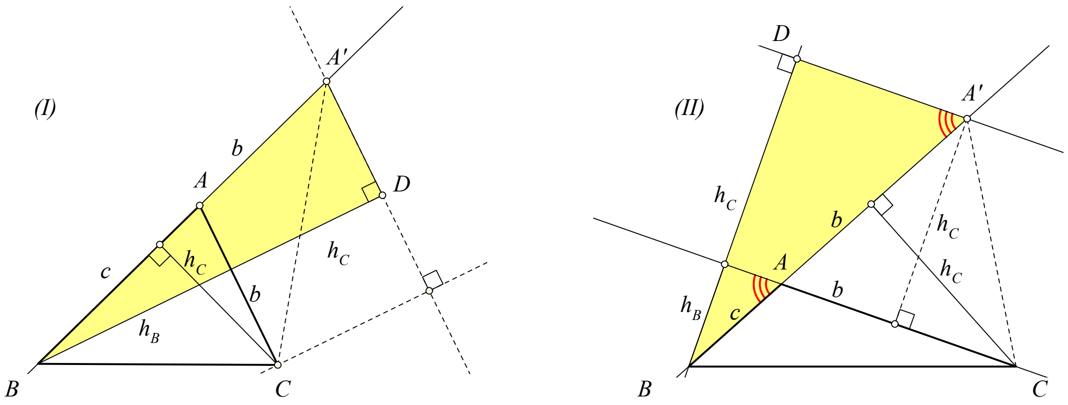

Figure 3: The sum $h_B + h_C$ related to $b + c$

Analogous property holds also for the lengths $\{|h_B - h_C|, |b - c|\}$. Applying the lemma, one can easily construct the triangle ABC, given the data: (1) $\{a, \alpha, h_B + h_C\}$, (2) $\{\alpha, |b - c|, h_B + h_C\}$, (3) $\{a, \gamma, h_B + h_C\}$ and (4) $\{a, \gamma, |h_B - h_C|\}$. For the case of the difference of lengths see the figure 4.

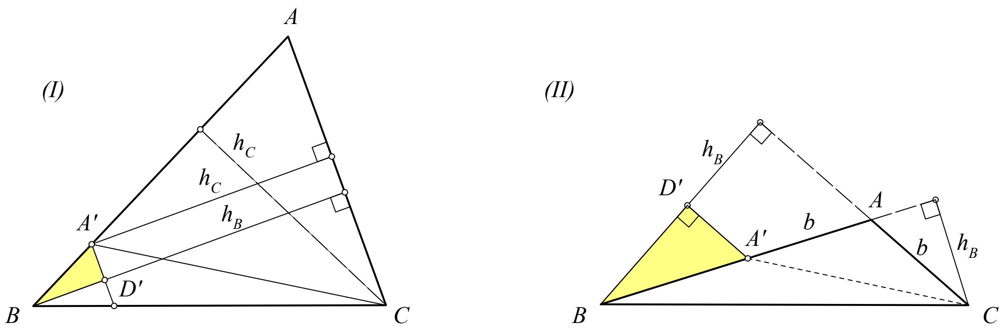

Figure 4: The difference $|h_B - h_C|$ related to $|b - c|$

4 A similar problem

A similar problem to the one of Lahire is to construct the triangle ABC from its elements $\{\alpha, |b - c|, h_A\}$. The preceding method applies, with slight modifications, to deliver a solution also for this problem.

In fact, draw from the middle M of BC the line ζ' orthogonal to the external bisector ε' of the angle $\widehat{A}$ (See Figure 5). Then show that ζ' intersects the sides $\{AB, AC\}$ at points correspondingly $\{F', H'\}$ such that $|AF'| = |AH'| = |b - c|$. Hence the isosceles triangle $F'AH'$ and point D' is again constructible from the given data. Line ζ' is again the Simson line relative to the point D', which is on the circumcircle κ of ABC. A similar to the previous calculation leads also to a quadratic equation for $x = |D'M|$:

$$
\begin{aligned}
|D'M|^2 - |F'D'|^2 &= |E'M|^2 - |E'F'|^2 = (|E'M| - |E'F'|)(|E'M| + |E'F'|) \\
&= |MH'||MF'| = |MJ||MD'| = |AY||MD'| \quad \Rightarrow \\
x^2 - |F'D'|^2 &= h_A \cdot x, \ \text{ with } |F'D'| = |AF'| \tan\left(\frac{\pi - \alpha}{2}\right) = \frac{b+c}{2} \tan\left(\frac{\pi - \alpha}{2}\right).
\end{aligned}
$$

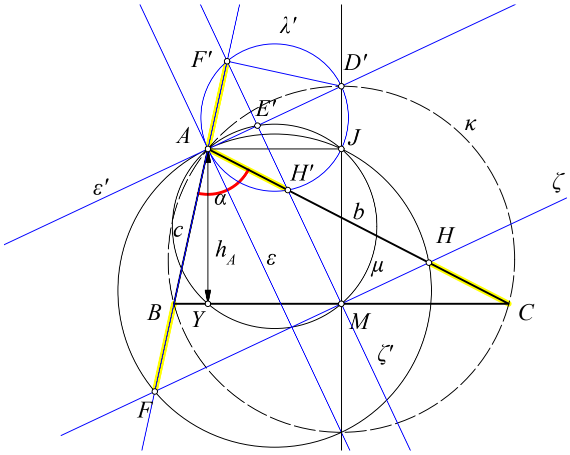

Figure 5: Triangle from $\{\alpha, |b-c|, h_A\}$

This shows that $|D'M|$ is constructible from the given data and then, line BC is constructed as a common tangent to the circles with centers at $\{A, D'\}$ and corresponding radii $\{h_A, |D'M|\}$.

References

[Cou80] Nathan Altshiller Court. *College Geometry*. Dover Publications Inc., New York, 1980.