Nagel point of the triangle

by Paris Pamfilos

A file of the Geometrikon gallery by Paris Pamfilos

The bottom line for mathematicians is that the architecture has to be right. In all the mathematics that I did, the essential point was to find the right architecture. It's like building a bridge. Once the main lines of the structure are right, then the details miraculously fit. The problem is the overall design.

C.L. Dodgson, College Math. J. 25(1994)

Contents

(Last update: 08-03-2023)

1 Nagel point of the triangle

This is defined as the intersection point N_a of the lines joining the vertices $\{A, B, C\}$ with the contact points $\{A'', B'', C''\}$ of the opposite sides with the corresponding *"excircles"* of the triangle $t = ABC$. That this point exists can be easily proved by applying *"Ceva's theorem"*. For this we use the basic relations between the segments defined by the contact points of the side-lines of the triangle with the incircle and the excircle (see figure 1), both inscribed in the same angle of the triangle (see file **Ceva's theorem**). In fact, denoting the side-lengths by

$$a = |BC|\ ,\ b = |CA|\ ,\ c = |AB| \quad \text{and the semi-perimeter} \quad s = \frac{1}{2}(a+b+c) \quad \Rightarrow$$
$$|AB| + |BA'| \ = \ |AB'| \ = \ |AC'| = |AC| + |CA''| \ = \ s \quad \Rightarrow$$
$$|BA''| = s - c\ ,\ |BA'| = s - b\ ,\ |CA'| = s - b\ ,\ |CA''| = s - c\ ,$$

and analogous formulas for the excircles contained in the other angles of the triangle. Thus, a characteristic property of the Nagel Cevian AA'' is that its trace A'' on BC separates the perimeter of the triangle in two halves $|AB| + |BA''| = |A''C| + |CA|$. Analogous properties hold also for the other Nagel Cevians.

Theorem 1. *The lines* $\{AA'', BB'', CC''\}$ *intersect at a point.*

This follows by measuring the ratio and applying Ceva's theorem

$$\frac{A''B}{A''C} = -\frac{BB'}{CC'} = -\frac{s-c}{s-b} \quad \Rightarrow \quad \frac{A''B}{A''C} \cdot \frac{B''C}{B''A} \cdot \frac{C''A}{C''B} = -\frac{(s-c)(s-a)(s-b)}{(s-b)(s-c)(s-a)} = -1.$$

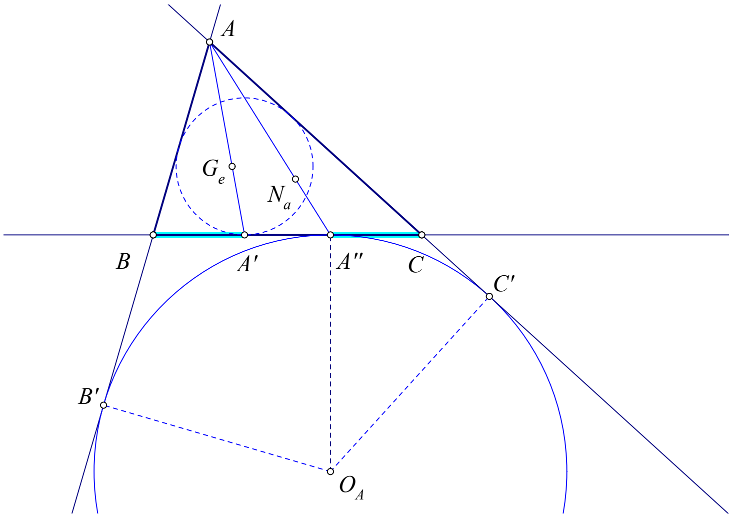

Figure 1: Nagel's point characteristic ratios $\frac{A''B}{A''C} = -\frac{s-c}{s-b}$

Remark 1. We notice the equality of the segments $|BA'| = |A''C| = s - b$, implying a simple relation of the Nagel point N_a and the *"Gergonne point"* G_e of the triangle, defined as the intersection of the Cevians joining the vertices with the contacts of the incircle on the opposite side. This relation implies that the $\{G_e, N_a\}$ are *"isotomic conjugate"* points of the triangle, i.e. the intersections of their Cevians with each side, like the points $\{A', A''\}$ in the figure, lie symmetrically w.r.t. to the middle of that side.

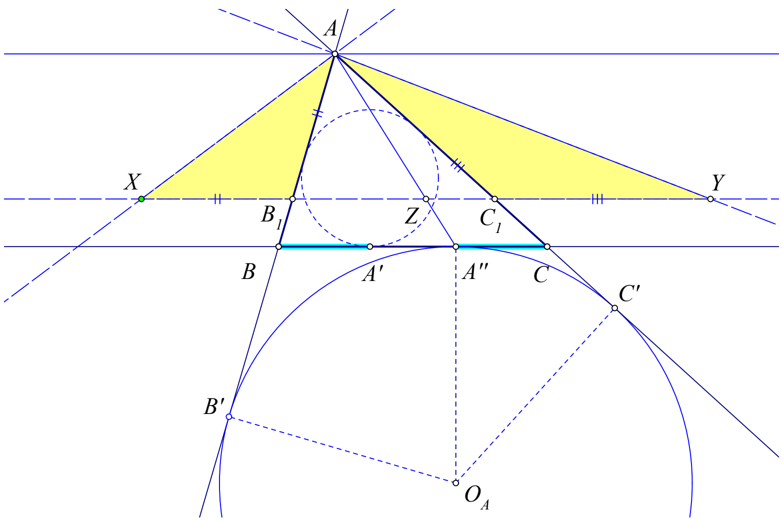

Figure 2: Nagel Cevian of $\triangle ABC$ as median of $\triangle AXY$

Theorem 2. *From the vertex A of the triangle ABC draw the parallel to BC and the bisectors of the angles it forms there with the sides $\{AB, AC\}$. From any point X on such a bisector draw the parallel to BC intersecting the other bisector in Y (see figure 2). The middle Z of XY lies on the Nagel Cevian of ABC through A.*

Proof by picture. The triangles $\{AXB_1, AC_1Y\}$ are isosceli. And

$$|AC_1| + |C_1Z| = |YZ| = |AB_1| + |B_1Z| = |XZ| .$$

Thus point Z has the characteristic property of the Nagel Cevian from A w.r.t. to the triangle AB_1C_1 to bisect its perimeter. Because of the similarities of triangles $\{AB_1C_1, ABC\}$ this is also a Nagel Cevian for the triangle ABC.

2 Barycentric coordinates of the Nagel point

For the calculation of the barycentrics (see file **Barycentric coordinates**) of N_a we use the known signed ratio

$$r = \frac{A''B}{A''C} = -\frac{s-c}{s-b} \quad \Rightarrow \quad A'' = \frac{1}{1-r}(B - rC) \quad \Rightarrow \quad A'' = \frac{1}{a}((s-b)B + (s-c)C).$$

Analogously we obtain $B'' = \frac{1}{b}((s-c)C + (s-a)A)$. The Nagel point is the intersection of the lines $N_a = AA'' \cap BB''$, the coefficients of which are expressed by the vector products

$$\begin{aligned} AA'' &: A \times A'' = (0 : -(s-c) : (s-b)), \\ BB'' &: B \times B'' = ((s-c) : 0 : -(s-a)). \end{aligned}$$

The barycentrics of N_a result by taking again the vector product

$$AA'' \times BB'' = (s-a : s-b : s-c).$$

Remark 2. From the isotomic relation between points, which in barycentrics is expressed by the reflexive relation

$$X(p : q : r) \leftrightarrow X^*\left(\frac{1}{p} : \frac{1}{q} : \frac{1}{r}\right) ,$$

we see that the barycentrics of the Gergonne point alluded to in remark 1 are

$$G_e\left(\frac{1}{s-a} : \frac{1}{s-b} : \frac{1}{s-c}\right) .$$

In figure 3 we consider a triangle ABC and draw parallels to the side BC. Shown are the *"double"* triangle AB_2C_2 and the triangle AB_1C_1, in which B_1C_1 is a common tangent to the incircles of the triangles $\{BB_2M', CM'C_2\}$, M' being the middle of B_2C_2. The incircle of the triangle B_3C_3 created in the same way is the excircle of the triangle AB_1C_1. By the similarity of all these triangles, created by parallels to BC, their orthocenters, incenters, the Gergonne and Nagel points move on four lines through A labeled accordingly in the figure. Next theorem formulates some properties of this figure.

Theorem 3. *With the preceding definitions and the labels shown in figure 3 hold the following properties.*

1. *The the Nagel Cevian through the vertex A meets B_1C_1 at the line bisector ZM' of B_1C_1.*
2. *The Gergonne Cevian through the vertex A passes through the intersection point A_1 of B_1C_1 with $M'Y$.*

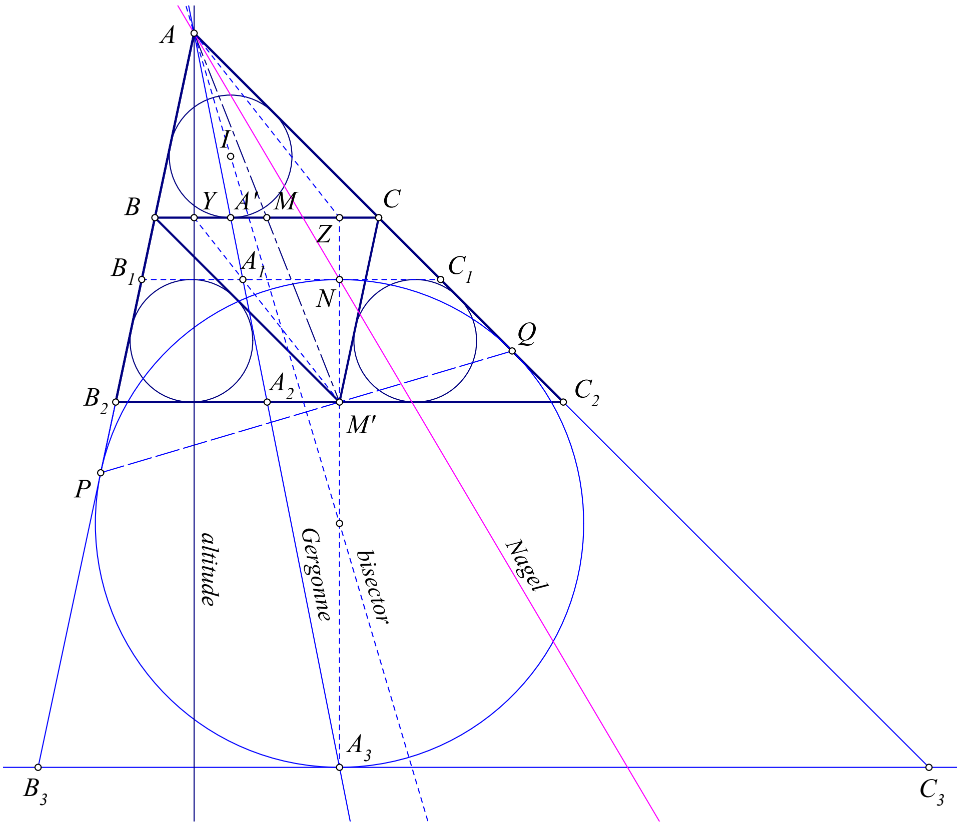

Figure 3: Atlitude, Gergonne, bisector and Nagel Cevian lines from A

3. *The semi-perimeter of the triangle* AB_1C_1 *is* $\tau = b + c$.
4. *The line* PQ *is orthogonal to the bisector* AI *and passes through* M'.
5. *The altitude line is harmonic conjugate of the bisector line w.r.t. to the pair of Gergonne and Nagel lines.*

nr-1. In fact, the triangles $\{ABC, M'C'B\}$ are equal and $\{AY, M'Z\}$ are their corresponding equal altitudes. It follows that $AYM'Z$ is a parallelogram and their diagonals bisect each other at M which is also the middle of BC. Hence M' is the middle of B_2C_2.

nr-2. From the equality of $\{BY, ZC\}$ and the equality of the triangles follows easily that the intersection A_1 of B_1C_1 and $M'Y$ satisfies $A_1B_1 = NC_1$. Hence A_1 is on the Gergonne Cevian through A.

nr-3. The similarity ratio of AB_1C_1 to ABC is $\lambda = (2(h_A - r))/h_A$, where $h_A = |AY|$ the altitude of ABC from A and r its inradius. But with the area of $ABC : E = rs = (h_A \cdot a)/2$ we have

$$\lambda = \frac{2(h_A - r)}{h_A} = 2 - \frac{2}{h_A}\frac{E}{s} = 2 - \frac{2}{h_A}\frac{h_A a}{2s} = \frac{2s - a}{s} = \frac{b + c}{s}.$$

Hence, by the similarity, the semi-perimeter of AB_1C_1 will be equal to $\tau = \lambda \cdot s = b + c$.

nr-4. The bisectors from $\{B, C\}$ of the triangles $\{B_2BM', M'CC_2\}$ are parallel to the bisector AI. Thus, the orthogonal from M' to AI cuts on AB the segment $BP = BM' = AC$ and on AC the segment $CQ = CM' = AB$. It follows that $AP = AQ = b + c$ and the result follows from *nr-3*.

nr-5 follows at once from the fact that A_3N is a diameter of the excircle of AB_1C_1 parallel to the altitude line and the bisector line passes through its center whereas the Gergonne and Nagel lines pass through its extremities.

3 The Nagel line of the triangle

This is the line containing the *incenter*, the *centroid* and the *Nagel point* and, as Bottema says, it is *"a counterpart of the Euler line"* [Bot07, p.83]. The justification for this is given by the following theorem.

Theorem 4. *The Nagel point* N_a *is the incenter of the "anticomplementary" triangle* $t' = A'B'C'$ *of* ABC*. The points* $\{I, G, N_a\}$ *are collinear and* $|GN_a| = 2|IG|$.

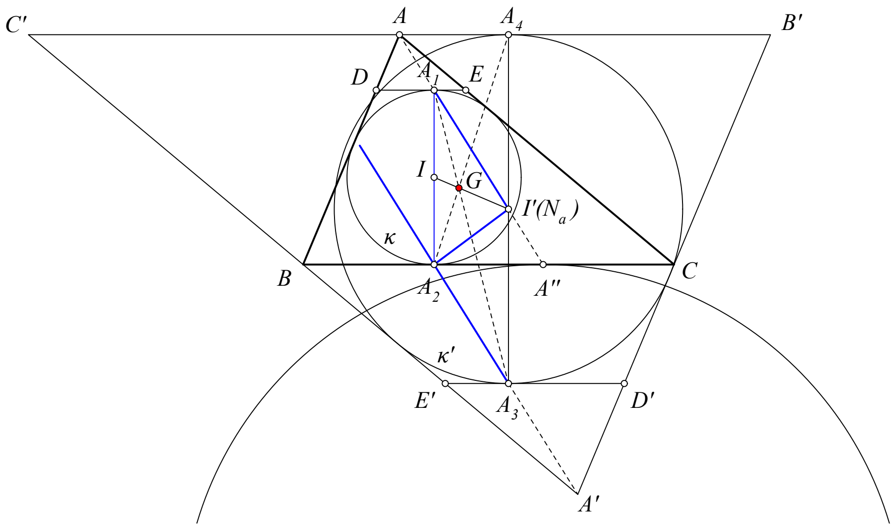

Figure 4: The cevians of the Nagel point

Consider the homothety f with center G and ratio -2. This maps the triangle t onto t' and the incircle κ of t onto the incircle κ' of t'. The proof amounts to show that the line A_1I' passes through the vertex A (See Figure 4). Here $\{I, I', A_2, A_1\}$ denote respectively the incenters of t, t' and the contact points of κ with $\{BC, DE\}$, where DE is the parallel to BC tangent to κ.

The triangle $A_1I'A_2$ has G for centroid. Hence the line $A_2A_3 = f(I'A_1)$. Because of $BA_2 = CA'' = s - b$, next ratios in the similar triangles $\{ADE, A'CB, A'D'E'\}$ are equal:

$$\frac{A_1E}{A_1D} = \frac{A_2B}{A_2C} = \frac{A_3E'}{A_3D'}.$$

Hence A_2A_3 passes through A' and consequently its homothetic $I'A_1$ passes through A. By the similarity of triangles $\{ADE, ABC\}$ line AI' passes also through the contact point A'' of the excircle with BC. Thus, the cevian AA'' containing the Nagel point passes through I' and analogously the other cevians do the same. Hence I' coincides with the Nagel point N_a of ABC.

Next theorem is a continuation of theorem 3. In this triangle ABC is extended to its *"double"* AB_2C_2 and we consider the incircles and some lines related to the created triangles $\{ABC, AB_2C_2, BB_2M', CM'C_2\}$ (see figure 5)

Theorem 5. *Referring to figure 5, the following properties hold.*

1. *The incenter* I' *of* $\triangle AB_2C_2$ *and points* $\{M', A'\}$ *are collinear.*
2. *The exterior common tangent* B_1C_1 *of the circles* $\{\kappa_B, \kappa_C\}$ *and the analogous exterior common tangents of the other pairs of circles meet at the incenter* I' *of* $\triangle AB_2C_2$.
3. *The internal common tangents of the circles* $\{\kappa_B, \kappa_C\}$ *intersect at a point* J *of the line* $M'A'$.

4. *Line* $A'M'$ *is Nagel Cevian for the triangles* $\{M'CB, A'PP', I'NN'\}$ *and is also parallel to the Nagel Cevian from* A *of the triangle* ABC.
5. *Lines* $\{A'M', AN\}$ *intersect at a point* K *on the parallel to* BC *from* A.

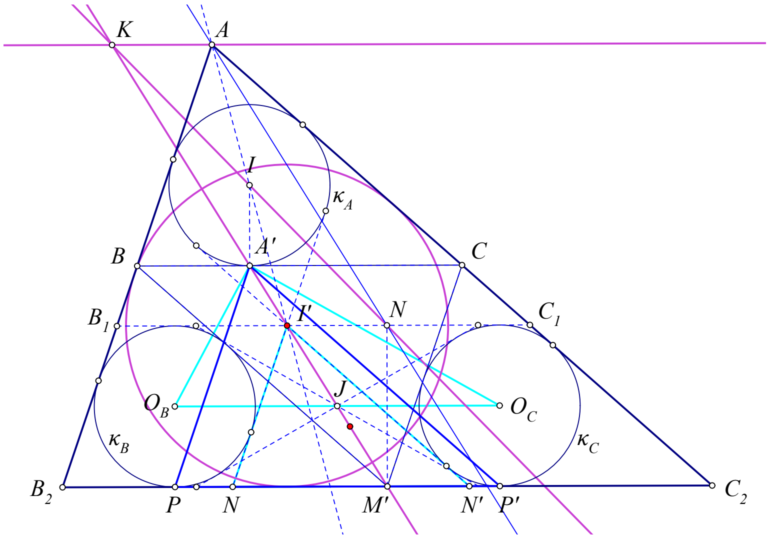

Figure 5: Relations between Nagel Cevians of triangles similar to ABC

nr-1. In fact, triangle AB_2C_2 is the anticomplementary of $M'BC$ and by the preceding theorem the incenter I' of AB_2C_2 is the Nagel point of $M'BC$. Also A' is on the Nagel Cevian from M' of the triangle $M'BC$ since $|BA'| = s - b$, as is necessary for this Cevian.

nr-2 follows from the fact that these exterior tangents, together with the sides of the triangle AB_2C_2 form rhombi, like the one at the corner B_2 : $B_2NI'B_1$, showing that I' is on the bisector of $\widehat{B_2}$ and analogously I' is on the bisector of the other angles.

nr-3 follows from theorem 2, since J is the middle of O_BO_C, hence, by that theorem, J is on the Nagel Cevian from I' of the triangle $I'NN'$. By the similarity of triangles $\{PA'P', NI'N'\}$ the line $I'M$ is also Nagel Cevian for the triangle $A'PP'$.

nr-4 follows from *nr-3* and the fact that $\triangle A'PP'$ is homothetic to AB_1C_1 whose Nagel Cevina is AN.

nr-5. The intersection point of the lines $\{A'M', JN\}$ defines two homothetic triangles $\{KJA', KNM'\}$ with homothety ratio 2, since $|NM'| = 2|JA'|$ and this implies the claim.

4 Alternative construction of the Nagel point

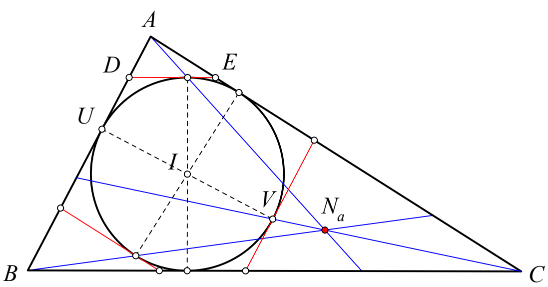

Figure 6: Alternative definition of the Nagel point

The following method ([Ask03, p.13]), rediscovered in [Hoe07]) gives another way to construct the Nagel point using only the incircle and not the excircles. For this draw tangents to the incircle parallel to the sides. Then join the contact points of these parallels with the opposite vertices. The three cevians thus created concure at the Nagel point (see figure 6).

The proof follows directly from the similarity of triangles $\{ADE, ABC\}$ alluded to also in the previous section.

5 Other Nagel-like points

Below is drawn an extension of figure 1 defining the Nagel point (see figure 7). In this there are seen three additional Nagel-like points $\{N_1, N_2, N_3\}$, resulting as intersections of three cevians to the contact points with the *"tritangent circles"* of the triangle.

1. If point U is the contact of incircle with side AB, then its antipode V is on the line CN_a.
2. The extension of CB_1 passes through the antipode of A_1.

Nr-1 is a consequence of the homothety of the incircle to the excircle opposite to C. The homothety has center at C and this implies that the end-points of parallel radii of the two circles are aligned with C. This is the case with $\{C', V\}$.

Nr-2 holds for, essentially, the same reason. This time C is the (anti) homothety center of the two excircles with centers A″, B″, hence again end-points of anti-parallel radii are collinear with C. This is the case with $\{B_1, B_3\}$.

Analogous properties, of course, hold also for the other vertices of the triangle and the corresponding excircles and contacts.

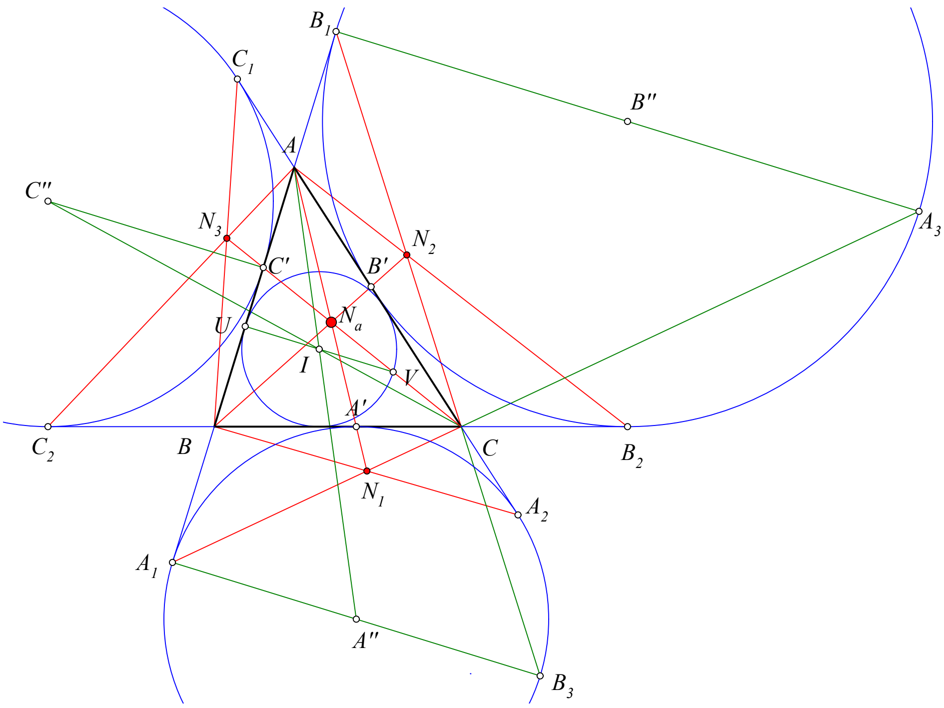

Figure 7: Other Nagel-like points $\{N_1, N_2, N_3\}$

6 Connection with the de Longchamps point

The *"de Longchamps"* point of the triangle is the symmetric of the orthocenter w.r.t. to the circumcenter of the triangle. Next theorem relates it to the Nagel-like points of the triangle.

Theorem 6. *The three Nagel-like points* $\{N_1, N_2, N_3\}$ *joined to respective excenters* $\{A'', B'', C''\}$ *define three concurring lines at the "de Longchamps" point* D_e *of the triangle.*

Thus D_e is the center of perspectivity of the two triangles $\{ABC, N_1N_2N_3\}$ and coincides with the *"De Longchamps point"*. (See Figure 8).

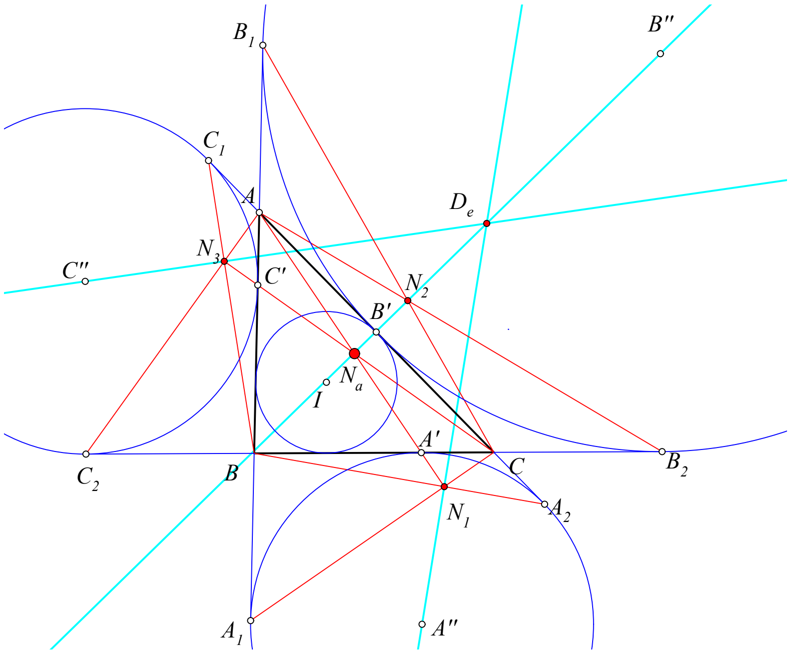

Figure 8: de Longchamps point related to $\{N_1, N_2, N_3\}$

As we did in section 2, we compute here also the barycentrics of the contact points from the known ratios, the symbol $\cong$ denoting here the equality of vectors up to a non-zero scalar factor:

$$\begin{aligned}
\frac{A_1A}{A_1B} &= r = \frac{s}{s-c} \quad \Rightarrow \quad A_1 = \frac{1}{1-r}(A - rB) \cong ((c-s)A + sB),\\
\frac{A_2C}{A_2A} &= r = \frac{s-b}{s} \quad \Rightarrow \quad A_2 = \frac{1}{1-C}(A - rA) \cong (sC + (b-c)A),\\
&A_1C: \; A_1 \times C = (-s : c-s : 0),\\
&A_2B: \; A_2 \times B = (s : 0 : s-b) \quad \Rightarrow\\
&N_1 = ((s-b)(c-s) : s(s-b) : s(s-c)).
\end{aligned}$$

The collinearity of $\{A, N_a, N_1\}$ and the similar to it triples, suggested by figure 8, results from the obviously vanishing determinant

$$\begin{vmatrix} 1 & 0 & 0 \\ s-a & s-b & s-c \\ (s-b)(c-s) & s(s-b) & s(s-c) \end{vmatrix} = 0.$$

Similar to the previous arguments lead to the barycentrics

$$A'' = (-a : b : c), \qquad B'' = (a : -b : c), \qquad C'' = (a : b : -c).$$

Taking into account that the de Longchamps point has barycentrics

$$D_e = (-3a^4 + 2a^2(b^2 + c^2) + (b^2 + c^2)^2, \dots)$$

and computing the determinant of the barycentrics vectors of the points $\{D_e, N_1, A''\}$ we find that this vanishes, hence the three points are collinear. This proves that the three lines $\{A''N_1, B''N_2, C''N_3\}$ pass through the de Longchamps point D_e of the triangle ABC.

Bibliography

[Ask03] E.H. Askwith. *A course of pure Geometry*. Cambridge University Press, Cambridge, 1903.

[Bot07] Oene Bottema. *Topics in Elementary Geometry*. Springer Verlag, Heidelberg, 2007.

[Hoe07] L. Hoehn. A New Characterization of the Nagel Point. *Missouri J. Math. Sci.*, 19:45–48, 2007.

Related topics

1. **Barycentric coordinates**
2. **Ceva's theorem**
3. **Tritangent circles of the triangle**
4. **Menelaus' theorem**

Any correction, suggestion or proposal from the reader, to improve/extend the exposition, is welcome and could be send by e-mail to: pamfilos@uoc.gr