Tritangent circles

A file of the Geometrikon gallery by Paris Pamfilos

As the whole of nature is akin, and the soul has learned everything, nothing prevents a man, after recalling one thing only - a process men call learning - discovering everything else for himself, if he is brave and does not tire of the search, for searching and learning are, as a whole, recollection.

Plato, Meno 81 d

Contents

(Last update: 28-11-2021)

1 Inscribed, Escribed, Incenter and Excenters

"Tritangent" circles are circles simultaneously tangent to three lines *"in general position"*, i.e. lines not all passing through a point. Such a triple of lines defines a triangle ABC, six *"outer domains"* and four *tritangent circles* (see figure 1). The *"inscribed"* in the triangle and the three *"escribed"* lying in respective *outer domains*. Often in the literature by *tritangent* is meant only one of the *escribed* circles. The center of the *inscribed* is the *"incenter"* of the triangle and the centers of the *escribed* are the *"excenters"* of the triangle.

The configuration of the four tritangent circles has strong ties with the *"bisectors"* of the triangle ABC, since these carry the centers of the circles and are, each, a symmetry axis of two of the circles. In this file we study the tritangent circles configuration and some of the related to it topics. Complementary material can be found in [Cou80, p.72].

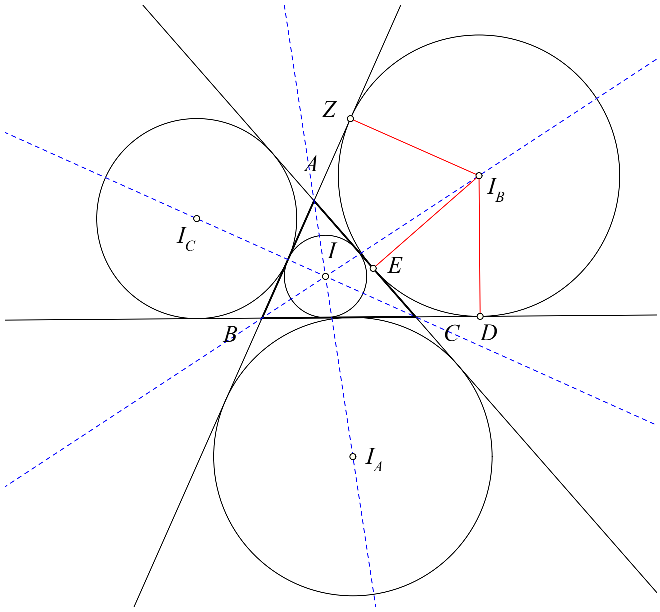


Figure 1: The tritangent circles of three lines forming a triangle

Theorem 1. *The internal bisector of one triangle angle and the external bisectors of the other two angles pass through a common point. The same is true for the three internal bisectors.*

Proof. We will show that the internal bisector at B and the external bisectors at A and C intersect at the same point, which we denote by I_B. Similar things will hold also for the other angles, which will define the points I_A and I_C (See Figure 1). The proof for the three internal bisectors is the same, defining the incenter I of the triangle. Let I_B be the intersection point of two out from the three bisectors of the triangle and specifically of the external bisectors of the angles A and C. We will show that the internal bisector of angle $\hat{B}$ also passes through I_B. Indeed, the distances of I_B from the sides of angle $\widehat{A}$ are equal $|I_BE| = |I_BD|$. Similarly, also the distances of I_B from the sides of angle $\widehat{C}$ are equal $|I_BE| = |I_BD|$. Consequently the three distances will all be equal $|I_BZ| = |I_BE| = |I_BD|$, therefore they are radii of a circle with center I_B and radius $r_B = |I_BD|$. The equality of the distances $|I_BZ| = |I_BD|$ from the sides of angle $\hat{B}$ show that I_B is to be found also on the bisector of angle $\hat{B}$. The fact that the sides are orthogonal to these radii of this circle, shows that the circle is tangent to all three sides of the triangle. □

Exercise 1. *Let $\{I, I_A, I_B, I_C\}$ be respectively the incenter and the excenters of the triangle ABC (see figure 1). Show that:*

1. *Each of the triples $\{(A, I, I_A),\ (B, I, I_B),\ (C, I, I_C)\}$ consists of collinear points.*
2. *The line defined by each of these triples is an altitude of the triangle $I_A I_B I_C$.*
3. *Point I is the "orthocenter" of the triangle $I_A I_B I_C$.*
4. *Triangle ABC is the "orthic" of triangle $I_A I_B I_C$.*

2 Tangents from the vertices

Theorem 2. *The length of the tangent BD, from the vertex B to the corresponding escribed circle with center I_B, is equal to half the perimeter τ of the triangle ABC (see figure 1).*

$$|BD| = \frac{1}{2}(a+b+c) = \tau.$$

Proof. Here, as usual, with $\{a, b, c\}$ we denote the lengths of the sides of the triangle. The proof follows directly from the equality of the tangents from B : $|BZ| = |BD|$, as well as from A and C : $|AZ| = |AE|,\ |CE| = |CD|$. The perimeter therefore is written as

$$\begin{aligned} a+b+c &= (|BC| + |CE|) + (|AE| + |BA|) \\ &= (|BC| + |CD|) + (|BA| + |AZ|) \\ &= 2(|BC| + |CD|) = 2|BD|. \end{aligned}$$ □

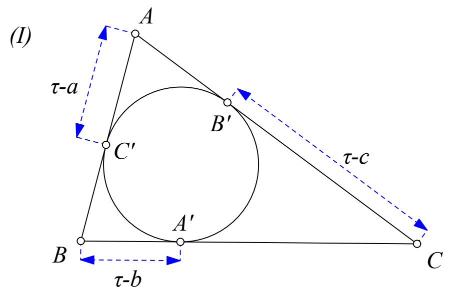
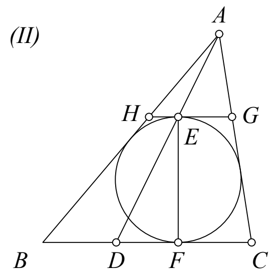


Figure 2: Tangents from the vertices



Tangent parallel to base

Theorem 3. *The tangents $\{AB', AC'\}$ from the vertex A of triangle ABC to its inscribed circle have length (see figure 2-(I)):*

$$|AC'| = |AB'| = \tau - a,$$

where $\tau = \frac{1}{2}(a+b+c)$ is the half perimeter of the triangle.

Proof. As the proof of the previous proposition, so this one is also relying on the equality of the tangents from one point to a circle: $|AB'| = |AC'|,\ |BC'| = |BA'|,\ |CA'| = |CB'|$. It suffices therefore to write the perimeter as

$$a+b+c = 2\tau = 2(|AC'| + |BA'| + |A'C|) = 2(|AC'| + a),$$

from which the wanted equality follows directly. □

Exercise 2. *In the triangle ABC, with $|AC| \geq |AB|$, the circle κ is tangent to the sides $\{AB, AC\}$ and passes through the point A' of the base BC (see figure 2-(I)). Show that κ coincides with the inscribed circle of the triangle ABC, if and only if, it holds*

$$|A'C| - |A'B| = |AC| - |AB|.$$

Then point A' coincides with the contact point of the circle with the base BC.

Hint: For $\{x = |A'B|,\ \ y = |A'C|\}$, the above relation is equivalent with the two relations $\{y - x = b - c,\ \ y + x = a\}$.

Exercise 3. *Let E be the contact point of the tangent HG of the inscribed circle, which is parallel to the base BC of the triangle ABC. Show that the line AE intersects BC at a point D, such that $|AB| + |BD| = |DC| + |CA| = \tau$. Conclude that $|BD| = |FC|$, where F is the contact point of the incircle with BC (see figure 2-(II)).*

Hint: $\{AHJ, ABC\}$ are similar and $AH + HE = AJ + JE$.

Theorem 4. *Next table gives the centers of the inscribed and escribed circles of triangle ABC, as well as their respective projections on the sides AB, BC and CA (see figure 3).*

	AB	BC	CA
I	D	H	Z
I_A	M	J	N
I_B	R	L	O
I_C	K	S	P

The following relations are valid:

$$\begin{aligned}
\tau - a &= |AD| = |AZ| = |CL| = |CO| = |BS| = |BK| \\
\tau - b &= |BD| = |BH| = |CJ| = |CN| = |AK| = |AP| \\
\tau - c &= |CZ| = |CH| = |AR| = |AO| = |BM| = |BJ| \\
|HJ| &= |c - b|, \quad |OZ| = |a - c|, \quad |KD| = |b - a|.
\end{aligned}$$

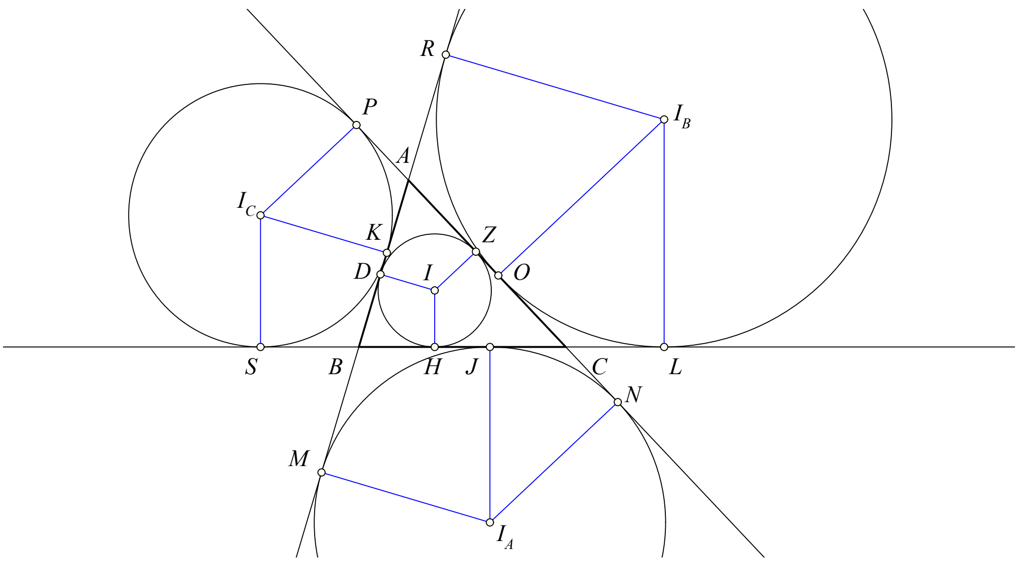


Figure 3: Segments on the sides

Proof. That it holds $\tau - a = |AD| = |AZ|$, we saw in the theorem 3. For the other equalities on the same line write

$$|CJ| = |BC| - |BJ| = \tau - a.$$

Similarly follow also the equalities in the second and third line. Equality $|HJ| = |c - b|$ follows from the previous

$$|HJ| = |BC| - |BH| - |CJ| = a - (\tau - b) - (\tau - b) = b - c.$$

Similarly follow also the two last equalities. □

Exercise 4. *Using figure 3 and the previous relations show that*

1. $|SJ| = |KM| = b, \ \ |JL| = |ON| = c, \ \ |OP| = |RK| = a.$
2. $|DM| = |ZN| = a, \ \ |HL| = |DR| = b, \ \ |ZP| = |HS| = c.$
3. $|PN| = a + c, \ \ |SL| = b + c, \ \ |MR| = a + b.$

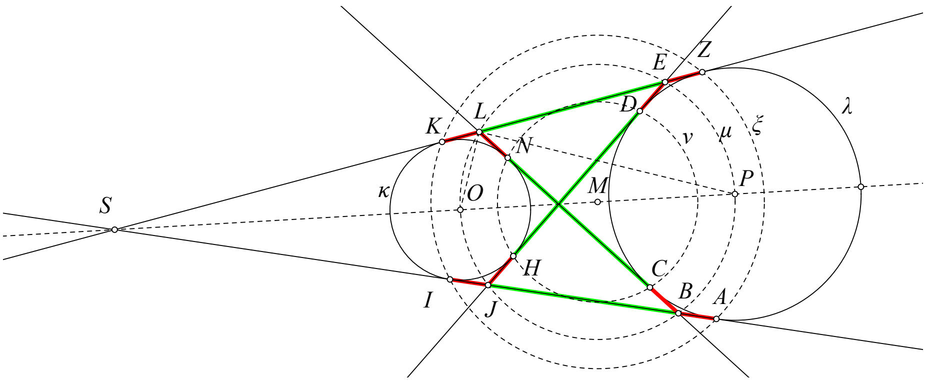


Figure 4: Common tangents of two circles

Exercise 5. *Given are two not congruent and external to each other circles $\kappa(O)$ and $L(P)$. Show that the relations suggested by figure (see figure 4).*

1. $|KL| = |LN| = |DE| = |EZ|$ *etc.*
2. $|LE| = |NC|$ *etc.*
3. *The circles $\nu = (CDNH)$, $\mu = (BELJ)$, $J = (AZKI)$ are concentric.*
4. *The circle μ passes through the centers O and P.*

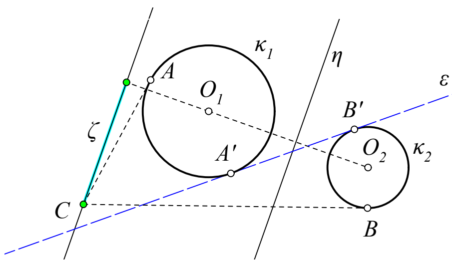


Figure 5: The common tangent ε of two circles

Exercise 6. *Show that point C is contained in the internal common tangent of two external to each other circles $\{\kappa_1(O_1), \kappa_2(O_2)\}$ and external to the segment $A'B'$ of the contact points, if and only if, for the other (external) tangents $\{CA, CB\}$ the following relation is valid: $||CA| - |CB|| = |A'B'|$ (see figure 5).*

Hint: Obviously the relation is valid if C is on ε and outside $A'B'$. For the converse consider the movement of C on a line ζ parallel to the radical axis η of the two circles. Show that the function $f(x) = ||CA| - |CB|| = k/(|CA| + |CB|)$, with k constant, is a decreasing function of the distance x of C from the line of centers O_1O_2.

3 Relations between the radii

Most of the following formulas are simple consequences of the relations discussed in section 2. All of them have simple proofs and are formulated as exercises. In some of them it is of help to use the formula for the area Δ of the triangle in dependence from the *circumradius* R :

$$\Delta = \frac{a \cdot b \cdot c}{4R}, \tag{1}$$

which results from the area formula $\Delta = ab\sin(\gamma)/2 = abc/(4R)$, using the well known sine rule $c/\sin(\gamma) = 2R$.

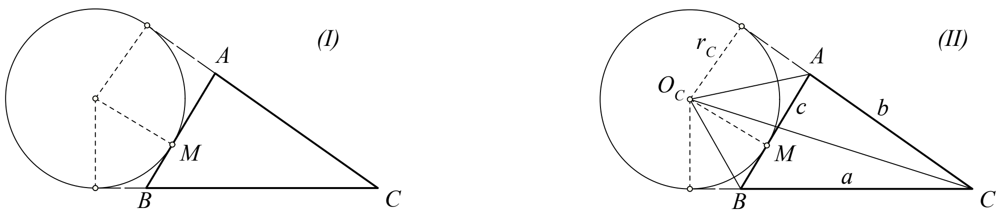


Figure 6: M bisects the perimeter $\qquad (ABC) = \frac{1}{2}(a+b-c)\cdot r_C$

Exercise 7. *Find a point M on the side AB of triangle ABC (See Figure 6-I), such that*

$$|MA| + |AC| = |MB| + |BC|.$$

Exercise 8. *Show that for the radii $\{r, r_A, r_B, r_C\}$ of the inscribed and escribed circles, the altitudes $\{h_A, h_B, h_C\}$ and the area $\Delta = (ABC)$ of the triangle ABC holds*

$$\begin{aligned}
\Delta &= \tau \cdot r && (2)\\
&= (\tau - a)r_A \;=\; (\tau - b)r_B \;=\; (\tau - c)r_C, && (3)\\
\frac{1}{r} &= \frac{1}{r_A} + \frac{1}{r_B} + \frac{1}{r_C} = && (4)\\
&= \frac{1}{h_A} + \frac{1}{h_B} + \frac{1}{h_C}. && (5)
\end{aligned}$$

Hint: For the first three equalities see figure 6-II. For the rest see that $\frac{1}{h_A} = \frac{a}{2\Delta}$, which implies that $\frac{1}{h_A} + \frac{1}{h_B} + \frac{1}{h_C} = \frac{\tau}{\Delta} = \frac{1}{r}$.

Exercise 9. *Show the following formulas:*

$$\begin{aligned}
4R + r &= r_A + r_B + r_C, && (6)\\
R^2 &= \frac{a^2b^2c^2}{2(b^2c^2 + c^2a^2 + a^2b^2) - (a^4 + b^4 + c^4)}. && (7)
\end{aligned}$$

Hint: The last equality follows from $abc = 4\Delta R$ and the identity

$$2(a^2b^2 + b^2c^2 + c^2a^2) - a^4 - b^4 - c^4 = (a+b+c)(b+c-a)(c+a-b)(a+b-c). \tag{8}$$

Exercise 10. *Show that, if the incenter of the triangle ABC coincides with its centroid or its orthocenter, then the triangle is equilateral.*

4 Radii of the tritangent circles related to side-lengths

Theorem 5. *The radius r of the inscribed and r_A of the escribed circle of the triangle ABC are given respectively by the formulas*

$$r^2 = \frac{(\tau-a)(\tau-b)(\tau-c)}{\tau}, \qquad r_A^2 = \frac{\tau(\tau-b)(\tau-c)}{\tau-a}. \tag{9}$$

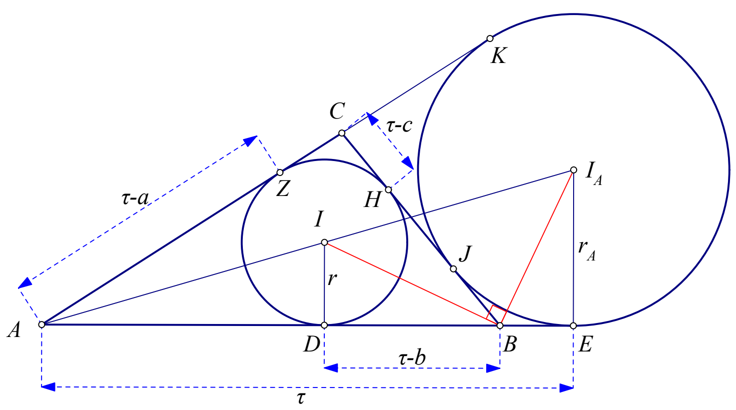


Figure 7: r, r_A as functions of the sides

Proof. In figure 7 are seen two circles: the inscribed $I(r)$ and the escribed $I_A(r_a)$. The proof uses the relations of section 3 and the similarity between two pairs of triangles. The first pair of triangles is (ADI, AEI_A). The second pair is (DIB, EBI_A). Both pairs consist of right triangles and we have:

$$\frac{|DI|}{|EI_A|} = \frac{|AD|}{|AE|} \quad \Leftrightarrow \quad \frac{r}{r_A} = \frac{\tau-a}{\tau}, \tag{10}$$

$$\frac{|DI|}{|DB|} = \frac{|EB|}{|EI_A|} \quad \Leftrightarrow \quad \frac{r}{\tau-b} = \frac{\tau-c}{r_A}. \tag{11}$$

Solving the second relative to r_A and substituting into the first expression, we find the formula for r^2. Squaring the first formula and substituting with the found expression for r^2, we prove the second formula as well. □

5 Relations between angles and side-parts

Next formulas relating the side parts $\{\tau-a,\ \tau-b,\ \tau-c\}$ and τ, defined by the tritangent circles, with the angles follow by inspecting figure 7 and are formulated as exercises. In these $\{\alpha, \beta, \gamma\}$ denote the angles of the triangle and R its circumradius.

Exercise 11. *Show that for every triangle hold the formulas*

$$\cot(\alpha) = \frac{b^2+c^2-a^2}{4\Delta} \quad \Rightarrow \quad \cot(\alpha)+\cot(\beta)+\cot(\gamma) = \frac{a^2+b^2+c^2}{4\Delta}. \tag{12}$$

Remark 1. For every triangle ABC there is a special angle ω, called *"Brocard angle"* of the triangle and satisfying $\cot(\omega) = \cot(\alpha)+\cot(\beta)+\cot(\gamma)$. It follows that

$$a^2+b^2+c^2 = 4\Delta\cot(\omega).$$

Exercise 12. *Show that in every triangle the following formulas are valid*

$$\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{(\tau-b)(\tau-c)}{bc}}, \quad \sin\left(\frac{\beta}{2}\right) = \sqrt{\frac{(\tau-c)(\tau-a)}{ca}}, \quad \sin\left(\frac{\gamma}{2}\right) = \sqrt{\frac{(\tau-a)(\tau-b)}{ab}}. \tag{13}$$

Hint: Start with the formula for the cosine and show first that

$$2\sin\left(\frac{\alpha}{2}\right)^2 = \frac{(a-b+c)(a+b-c)}{2bc}. \tag{14}$$

Exercise 13. *Show that for every triangle the following formulas are valid*

$$\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{\tau(\tau-a)}{bc}}, \quad \cos\left(\frac{\beta}{2}\right) = \sqrt{\frac{\tau(\tau-b)}{ca}}, \quad \cos\left(\frac{\gamma}{2}\right) = \sqrt{\frac{\tau(\tau-c)}{ab}}. \tag{15}$$

$$\tan\left(\frac{\alpha}{2}\right) = \sqrt{\frac{(\tau-b)(\tau-c)}{\tau(\tau-a)}}, \quad \tan\left(\frac{\beta}{2}\right) = \sqrt{\frac{(\tau-c)(\tau-a)}{\tau(\tau-b)}}, \quad \tan\left(\frac{\gamma}{2}\right) = \sqrt{\frac{(\tau-a)(\tau-b)}{\tau(\tau-c)}}. \tag{16}$$

Exercise 14. *Show that for every triangle holds next formula and the similars resulting by cyclic permutations of the letters*

$$\cos(\alpha/2)\cos(\beta/2)\sin(\gamma/2) = \frac{r_A}{4R}. \tag{17}$$

Exercise 15. *Show that for every triangle the following formulas are valid*

$$\frac{a-b}{c} = \frac{\sin\left(\frac{1}{2}(\alpha-\beta)\right)}{\cos\left(\frac{1}{2}\gamma\right)}, \qquad \frac{a+b}{c} = \frac{\cos\left(\frac{1}{2}(\alpha-\beta)\right)}{\sin\left(\frac{1}{2}\gamma\right)}. \tag{18}$$

Exercise 16. *Given is a triangle ABC with centroid M and an arbitrary point X. Show that for the sum of squares holds*

$$|XA|^2 + |XB|^2 + |XC|^2 = |MA|^2 + |MB|^2 + |MC|^2 + 3|XM|^2. \tag{19}$$

Also show the converse, that is, if the point M satisfies the above equation, for every point X of the plane, then it coincides with the centroid of the triangle.

Exercise 17. *Show that, for a triangle ABC with circumcircle $c(O,R)$ and centroid M, the folowing formula is valid*

$$|OM|^2 = R^2 - \frac{1}{9}(a^2+b^2+c^2). \tag{20}$$

6 Heron's triangle area formula

This formula relates the area Δ of the triangle ABC to its side-lengths $\{a,b,c\}$ and the half-perimeter $\tau = (a+b+c)/2$:

$$\Delta^2 = \tau\cdot(\tau-a)\cdot(\tau-b)\cdot(\tau-c). \tag{21}$$

The proof of Heron's formula follows by substituting in the formula for the area $\Delta = r\cdot\tau$ the radius r through the formula of theorem 5 (see figure 8-(I)).

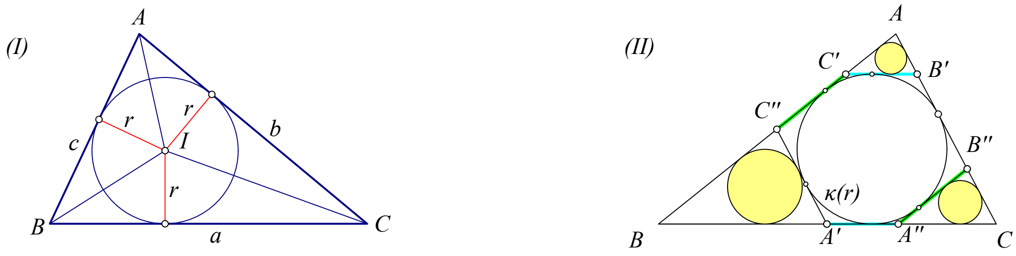


Figure 8: Heron's proof

Symmetric inscribed hexagon

Exercise 18. *Let $\{B'C', C''A', A''B''\}$ be tangents of the inscribed circle $\kappa(r)$ of the triangle ABC, respectively parallel to the sides $\{BC, CA, AB\}$. Show that the hexagon $A'A''B''B'C'C''$ is symmetric and has its opposite sides equal and parallel (see figure 8-(II)). Show also that the sum of the inradii of the small circles $r_A + r_B + r_C = r$. Finally show that the circumcircles of the small triangles are tangent to the circumcircle of triangle ABC.*

Exercise 19. *Show that the area Δ of the triangle ABC is expressed with the help of its altitudes $\{h_A, h_B, h_C\}$ through the formula:*

$$\frac{1}{\Delta^2} = \left(\frac{1}{h_A} + \frac{1}{h_B} + \frac{1}{h_C}\right) \cdot \left(-\frac{1}{h_A} + \frac{1}{h_B} + \frac{1}{h_C}\right) \cdot \left(\frac{1}{h_A} - \frac{1}{h_B} + \frac{1}{h_C}\right) \cdot \left(\frac{1}{h_A} + \frac{1}{h_B} - \frac{1}{h_C}\right). \tag{22}$$

Remark 2. In the articles of Baker [Bak85a], [Bak85b] are contained 110 formulas for the area Δ of the triangle among which the following interesting, in that they relate the area Δ to the areas of other triangles intimately related to ABC :

$$\Delta = L \cdot \frac{1}{2\cos(\alpha)\cos(\beta)\cos(\gamma)}, \tag{23}$$

$$= M \cdot \frac{(a+b)(b+c)(c+a)}{2abc}, \tag{24}$$

$$= N \cdot \frac{2R}{r}. \tag{25}$$

In these L is the area of the *"orthic"* triangle, with vertices the feet of the altitudes, M is the area of the triangle having for vertices the traces of the internal bisectors, and N is the area of the triangle having for vertices the points of tangency of the incircle.

7 A symmetric equilateral hexagon

Intimately related to the triangle ABC is the hexagon created by projecting on the sides of the triangle its excenters and extending them to their intersections $\{A_3, B_3, C_3\}$. This creates the hexagon $h = I_A B_3 I_C A_3 I_B C_3$ (see figure 9) of which next theorem sums up the basic properties.

Theorem 6. *The sides of h are equal to the circumdiameter $2R$ of ABC. The hexagon is symmetric w.r.t. to the circumcenter O of ABC and its exterior angles equal those of the triangle.*

Proof. The claim about the outer angles is obvious. The claim on the equality of sides follows from the triangles $\{A_3 I_B I_C, B_3 I_C I_A, C_3 I_A I_B\}$ easily seen to be isosceli. The symmetry follows from the equality of sides and the parallelity of opposite sides, since these are orthogonal, each to a corresponding side of ABC. That the center is O follows by considering a diagonal, I_CC_3 say. The symmetry center is on the parallel to I_CB_3, which passes through the middle of A_1A_2, which is identical with the middle of BC. Finally, the claim about $2R$ follows by calculating the side A_3I_B in terms of I_BI_C, which in turn is expressible in terms of $\{I_BB'', I_CC''\}$. □

Corollary 1. *The lengths of the sides of the triangle $I_AI_BI_C$ are*

$$I_AI_B = 4R\cos(\gamma/2), \quad I_BI_C = 4R\cos(\alpha/2), \quad I_CI_A = 4R\cos(\beta/2). \tag{26}$$

Proof. Follows by applying the cosine formula to triangle $A_3I_BI_C$ and his alikes. □

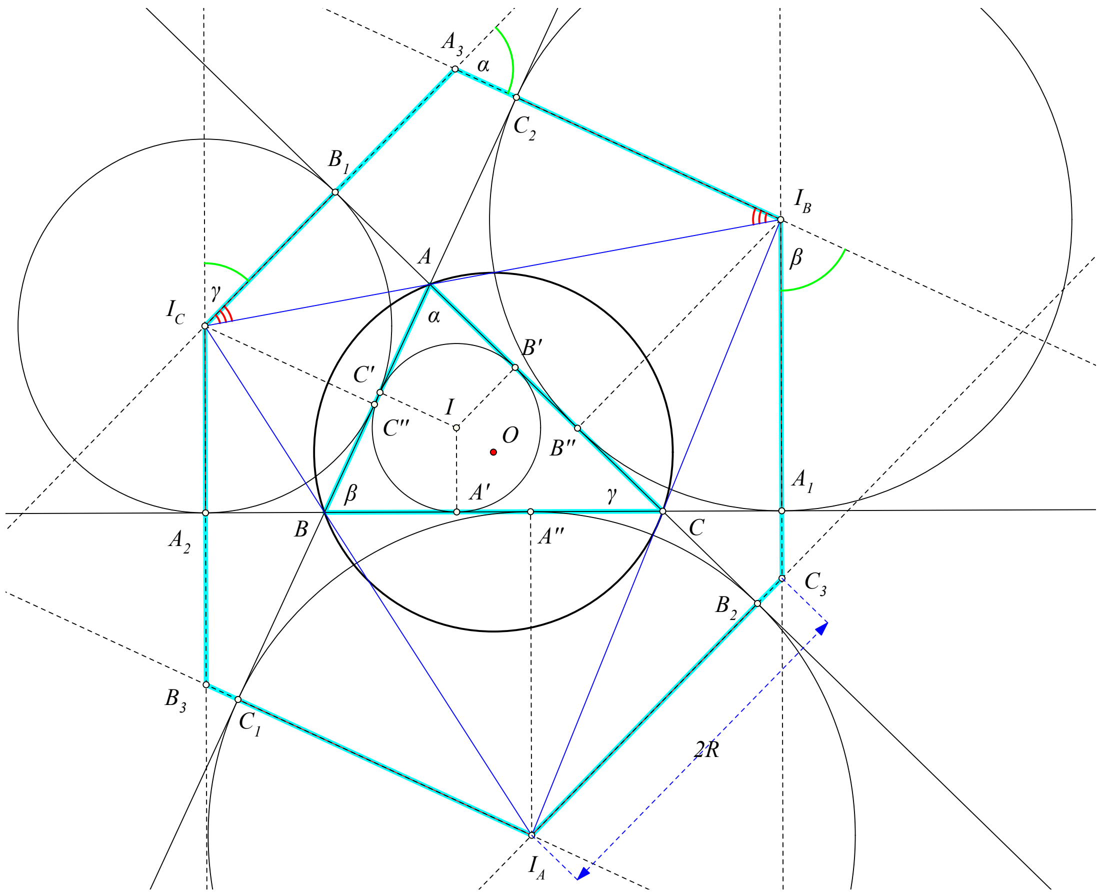


Figure 9: Equilateral symmetric hexagon

Corollary 2. *The lengths of the diagonals of the hexagon h are*

$$(I_AA_3)^2 = 4R(R + 2r_A), \quad (I_BB_3)^2 = 4R(R + 2r_B), \quad (I_CC_3)^2 = 4R(R + 2r_C). \tag{27}$$

Proof. Follows by applying the parallelogram rule $x^2 + y^2 = 2(a^2 + b^2)$ relating the diagonals $\{x, y\}$ to the sides $\{a, b\}$. Indeed, setting

$$x = (I_AA_3)^2,\ y = (I_BB_3)^3,\ z = (I_CC_3)^2, \quad \text{and} \quad u = (I_BI_C)^2,\ v = (I_CI_A)^2,\ w = (I_AI_B)^2$$

and applying this rule we obtain the simple linear system w.r.t. to $\{x, y, z\}$:

$$\begin{aligned} z + y &= 2(4R^2 + u), \\ x + z &= 2(4R^2 + v), \\ y + x &= 2(4R^2 + w). \end{aligned}$$

This has the solution

$$x = v + w - u + 4R^2, \quad y = w + u - v + 4R^2, \quad z = u + v - w + 4R^2,$$

which, in view of corollary 1, leads to the claimed formulas. To see this, after applying corollary 1, use the trigonometric identity

$$\cos^2(\alpha/2) + \cos^2(\beta/2) - \cos^2(\gamma/2) = 2\cos(\alpha/2)\cos(\beta/2)\sin(\gamma/2), \tag{28}$$

and the formulas of exercise 14. □

Remark 3. The formulas of corllary 2 are equivalent to *"Euler's theorem"* connecting the radii $\{R, r_A\}$ with the distance $|OI_A|$ of the centers of the corresponding circles. In fact, since it holds $|OI_A| = |I_A A_3|/2$, and the likes for the other centers, these formulas become equivalent to

$$|OI_A|^2 = R(R + 2r_A), \qquad |OI_B|^2 = R(R + 2r_B), \qquad |OI_C|^2 = R(R + 2r_C), \tag{29}$$

which are *Euler's formulas* for the circumcenter and the *excenters* of the triangle. The corresponding formula for the incenter is handled below.

Remark 4. The correspondence of the hexagon to the triangle $ABC \leftrightarrow h$ is reversible. The triangle can be constructed if we know the symmetric equilateral hexagon h. For this it suffices to construct the triangle of the diagonals $I_A I_B I_C$ and take the *orthic* ABC of the latter. The selection of the other diagonals and the corresponding triangle $A_3B_3C_3$ leads, because of the symmetry w.r. to O to a congruent to ABC triangle.

Exercise 20. *Construct a triangle ABC knowing its circumradius R and*

1. *an angle α and one of the sides $\{I_A I_B, \; I_A I_C.\}$*
2. *a sum of two sides $b + c$ and the corresponding angle α.*
3. *two sums of sides $\{b + c, c + a\}$.*
4. *an exradius r_A and the sum $a + c$.*
5. *two exradii $\{r_A, \; r_B\}$.*

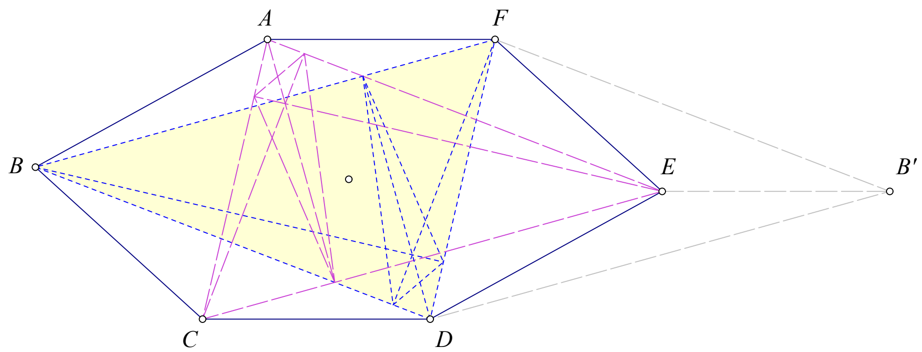


Figure 10: Double area $(ABCDEF) = 2(BDF)$

Exercise 21. *In the convex symmetric hexagon $ABCDEF$ show that the triangle of the diagonals BDF has half the area of the hexagon (see figure 10).*

Hint: Show that the hexagon has area equal to the parallelogram $BDB'F$.

Exercise 22. *Show that the triangles $\{A'B'C', I_AI_BI_C\}$ are homothetic (see figure 11). Show also that their homothety center L is described in trilinear coordinates through*

$$L = \left(\frac{1}{\tau - a} : \frac{1}{\tau - b} : \frac{1}{\tau - c} \right).$$

Hint: The homothety results from the parallelity of corresponding sides of the triangles. The length ratio of two parallel sides, $\{AC/I_AI_C\}$ say, is easily computed to be:

$$A'C' = 2BC' \sin(\beta/2), \; I_AI_C = 4R\cos(\beta/2) \quad \Rightarrow \quad \frac{A'C'}{I_AI_C} = \frac{\tau - b}{2R} \tan(\beta/2).$$

Using this and the formulas established so far, we find that

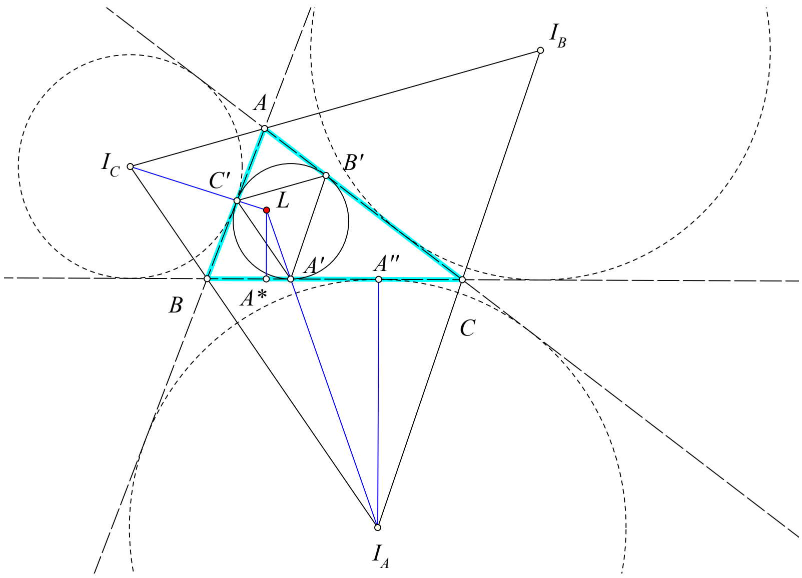


Figure 11: Similar triangles

$$LA^* = \frac{\Delta r_A}{2\tau R - \Delta} = \frac{r r_A}{2R - r} = \left(\frac{r^2 \tau}{2R - r} \right) \frac{1}{\tau - a},$$

in which the factor before $1/(\tau - a)$ is independent of the particular side of the triangle.

Remark 5. Point L coincides with the *"triangle center"* X_{57} in Kimberling's list ([Kim18]). It is the *"isogonal conjugate"* of the so-called *"Mittenpunkt"* $X_9 = (\tau - a : \tau - b : \tau - c)$ of the triangle ABC.

8 Euler's theorem for the incenter/excenter

Every triangle has a circumscribed circle L and an inscribed κ. If we hide the triangle we see two circles (κ in the interior of L) and the question rises, whether there are other triangles which have these two circles respectively as inscribed and circumscribed. More generally, for two circles $\{\kappa, L\}$, the first of which is contained in the second, one may ask, whether there is a triangle circumscribed to the first and inscribed to the second.

The next Euler's (1707-1783) theorem (one of many [Ric08], [Nah06]) leads to the expression of the power of the incenter I relative to the circumcircle of the triangle as a function of the radii of the circumcircle and the incircle (see figure 12-(I)). The formula which results gives also a quantitative criterion which answers the previous question, of the existence of a triangle circumscribed/inscribed to two given circles.

Theorem 7. *(Euler's theorem) In every triangle the radius R of its circumcircle, the radius r of its incircle and the distance OI of the centers of these circles are related through the formula*

$$|OI|^2 = R(R-2r). \tag{30}$$

Proof. The formula reminds of the power $p(I) = |OI|^2 - R^2$ of the incenter I (center of the inscribed circle) relative to the circumcircle, especially if we write it as $|OI|^2 - R^2 = -2rR$. It suffices therefore to show that the power of the incenter I relative to the circumcircle is $-2rR$. The basic observations which lead to the proof are two.

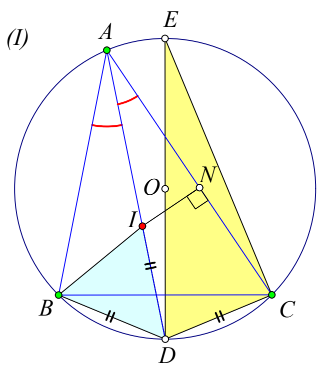
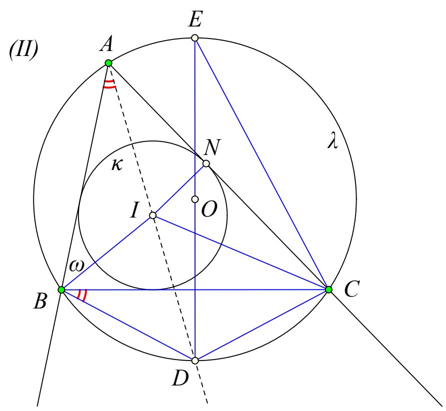


Figure 12: Euler's theorem



Triangles with the same incircle and circumcircle

First, that the extension of the bisector AI passes through the middle D of the arc BC and the line segments $\{DB, DI, DC\}$ are equal (See Figure 12-I). Segments DB and DC are of course equal, since D is the middle of the arc BC. Triangle BDI is however isosceles, because its angle at B is the sum $\frac{\alpha+\beta}{2}$ and the same happens with its angle at I, as external of the triangle BIA. The power of I then is $|AI||ID| = |AI||DC|$.

The second observation is that the right triangles AIN and EDC are similar. Here, N is the projection of I on AC, therefore its length is $|IN| = r$. Point E is the diametrically opposite of D. Obviously the triangles are similar because they have their acute angles at A and E equal. Then their sides are proportional:

$$\frac{|AI|}{|IN|} = \frac{|ED|}{|DC|} \Rightarrow |AI||DC| = |IN||ED|,$$

which, with what we said, translates to the claimed:

$$-p(I) = |AI||ID| = |AI||DC| = |IN||ED| = r(2R). \qquad \square$$

Theorem 8. *If for two circles $\kappa(I, r)$ and $L(O, R)$ holds the relation $|OI|^2 = R(R-2r)$, then for every point A of L there exists a triangle ABC inscribed in L and cirumscribed to κ.*

Proof. To begin with, κ is inside circle L. This is seen by considering a point X of κ and calculating the difference

$$|XO|^2 < (|XI| + |IO|)^2 < |XI|^2 + |IO|^2 = r^2 + R^2 - 2Rr = (R-r)^2,$$

where in the previous to last equality we used the assumed relation. We draw therefore from an arbitrary point A of L the tangents to κ, which intersect again the circle L at B and C (See Figure 12-II). It suffices to show that line BC is tangent to κ. To see this, we draw the bisector of the angle $\widehat{BAC}$, which intersects the circle L at the middle D of arc BC. Suppose again that E is the diametrically opposite of D and N is the projection of I on AC. The right triangles AIN and EDC have their acute angles at A and E equal, therefore they are similar. Consequently

$$\frac{|AI|}{|IN|} = \frac{|ED|}{|DC|} \quad \Rightarrow \quad |AI||DC| = |IN||ED| = r(2R).$$

However, by hypothesis, the power of I relative to L is also $-p(I) = |AI||ID| = R^2 - |OI|^2 = 2Rr$. Therefore

$$|AI||ID| = |AI||DC| \quad \Rightarrow \quad |ID| = |DC|.$$

Thus, there are defined two isosceli triangles, BDI and DIC. Let $\omega = \widehat{IBA}$. Taking into account that angles $\widehat{DBC}$ and $\widehat{DAB}$ are equal to $\alpha/2$, where $\alpha = \widehat{BAC}$, we have

$$\widehat{BID} = \omega + \frac{\alpha}{2} = \widehat{IBD} = \widehat{IBC} + \frac{\alpha}{2} \quad \Rightarrow \widehat{IBC} = \omega.$$

This means that BI is a bisector of the angle $\widehat{ABC}$ hence BC is also tangent to κ. □

Next exercise settles the analogous relation between the circumcenter O and the excenter I_A of a triangle.

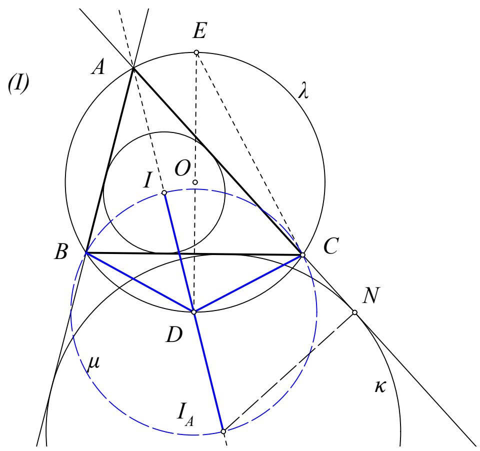
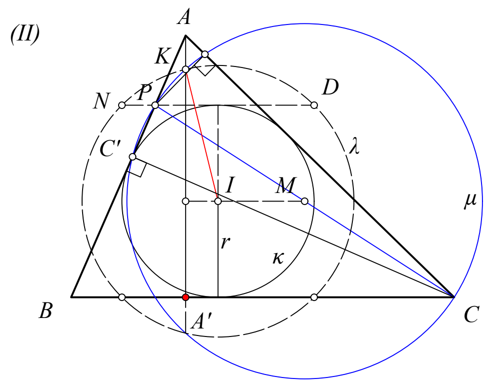


Figure 13: Euler's theorem for escribed



Circle $L(I, \sqrt{2} \cdot r)$

Exercise 23. *Show that the Euler's theorem 7 as well as 8 holds similarly also for the escribed circle of triagle ABC. Specifically, if $I_A(r_A)$ is the escribed circle contained in the angle $\widehat{A}$, then*

$$|OI_A|^2 = R(R + 2r_A). \tag{31}$$

Conversely, if for two circles $\kappa(I,r)$ and $L(O,R)$ holds $|OI|^2 = R(R+2r)$, then for every point A of L there is a triangle ABC inscribed in L and escribed to κ.

Hint: With the notation of figure 13-(I), the proof of the theorem 7 is transferred verbatim to this case. An alternative proof was given in section 7. The proof of the converse claim is similar. The key in the transfer of these proofs is the fact that the circle μ with diameter II_A passes through points $\{B,C\}$ and has for center the point D.

Exercise 24. *Assume that the tangent ND of the inscribed circle $\kappa(I,r)$, which is parallel to the base BC of the triangle ABC, intersects side AB at the point P. Show that the circle μ, with diameter PC, intersects the altitude AA' at a point K, whose distance from the incenter I is equal to $\sqrt{2}\cdot r$.*

Hint: Show that A lies on the radical axis of circles $L(I,\sqrt{2}\cdot r)$ and μ (see figure 13-(II)), by calculating its two powers $\{\delta_1,\delta_2\}$ relative to the circles $\{L,\mu\}$. If $\{\tau, r, v = |AA'|\}$ denote respectively the half perimeter, the radius of the inscribed circle and the altitude from A, these powers are

$$\delta_1 = (\tau - a)^2 - r^2 \quad = \quad \delta_2 = \frac{(v-2r)bc\cos(\alpha)}{v} \quad = \frac{1}{2}(b^2+c^2-a^2).$$

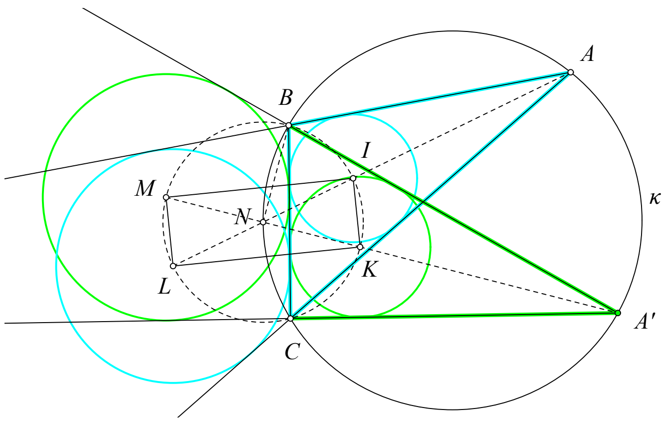


Figure 14: Rectangle of tritangent centers

Exercise 25. *For the triangles $\{ABC, A'BC\}$ with common base BC we consider the four centers the tritangent of the inscribed and escribed cirlces in angle $\widehat{A}$ (see figure 14). Show that the corresponding quadrilateral IKLM is a rectangle centerred at the middle N of the arc BC of the circumcircle κ.*

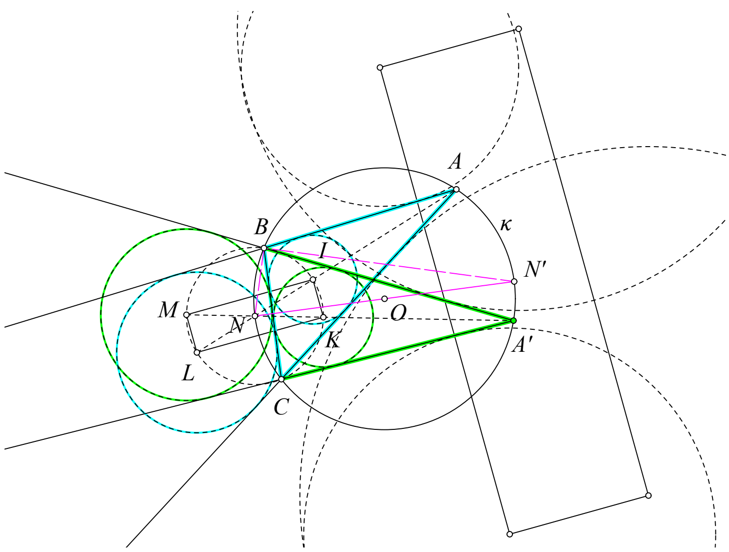


Figure 15: Two similar rectangles

Exercise 26. *Continuing the preceding exercise, consider all four tritangent circles of the two triangles $\{ABC\ ,\ A'BC\}$. Show that their centers are the vertices of two similar rectangles (see figure 15). For this consider the diameter NN' of κ orthogonal to BC and show that the similarity with center B, rotation angle $\widehat{NBN'}$ and ratio $t = BN'/BN$ transforms the rectangle with center N to that with center N'.*

Exercise 27. *Using the notation of this section and figure 15-I, show the relations:*

$$|CD| = \frac{a}{2\cos\left(\frac{\alpha}{2}\right)} = 2R\sin\left(\frac{\alpha}{2}\right), \tag{32}$$

$$|IB| = |II_A|\sin\left(\frac{\gamma}{2}\right) = 4R\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\gamma}{2}\right), \tag{33}$$

$$r = 4R\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right), \tag{34}$$

$$\tau - a = |AI|\cos\left(\frac{\alpha}{2}\right) = 4R\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right)\cos\left(\frac{\alpha}{2}\right), \tag{35}$$

$$\cos\left(\frac{\alpha}{2}\right) = \frac{\tau}{|AI_A|}, \qquad \widehat{ADE} = \frac{|\beta-\gamma|}{2} \tag{36}$$

$$|AD| = 2R\cos\left(\frac{|\beta-\gamma|}{2}\right), \qquad |AE| = 2R\sin\left(\frac{|\beta-\gamma|}{2}\right), \tag{37}$$

$$\tau^2 - r^2 - 4Rr = \frac{1}{2}(a^2+b^2+c^2), \qquad \tau^2 + r^2 + 4rR = ab + bc + ca. \tag{38}$$

Hint: The relations in the first five lines result immediately from the figure and the rule of sine for triangles. For the relations of the sixth line start from $\tan\left(\frac{\alpha}{2}\right) = \frac{r}{\tau - a}$ and the relations of theorem 5. The last equation follows from the previous one and the identity $2(ab+bc+ca) = (a+b+c)^2 - (a^2+b^2+c^2)$.

9 Some properties of the bisectors

Exercise 28. *Show that the line AD, passing through a vertex of triangle ABC and intersecting the opposite side at D and the circumcircle at E is a bisector of the angle $\widehat{BAC}$, if and only if it holds (see figure 16-(I)):*

$$|ED||EA| = |EB|^2 = |EC|^2. \tag{39}$$

Hint: Point E is the middle of the arc BC and the triangles ABE and BDE are similar.

Exercise 29. *Show that for the isosceles triangle ABC and the point D of its base BC and with the notation $\{a = |AB|, d = |AD|, x = |BD|, y = |DC|\}$, the relation $a^2 = d^2 + x \cdot y$ holds true.*

Hint: Almost identical to exercise 28 (see figure 16-(I)).

Exercise 30. *Construct a triangle from its elements $\alpha = |BAC|$, $a = |BC|$ and $\delta_A = |AD|$, where AD is the bisector of angle A.*

Hint: From the first two given elements the circumcircle of the wanted triangle ABC can be constructed and the position of B, C on it can be determined. Using figure 16-(I), the conclusion of the previous exercise, and setting $x = |EA|$, we have

$$x(x - \delta_A) = |EB|^2,$$

from which $|EA|$ is determined. With center E and radius $|EA|$ we draw a circle which intersects the previously constructed circumcircle at A.

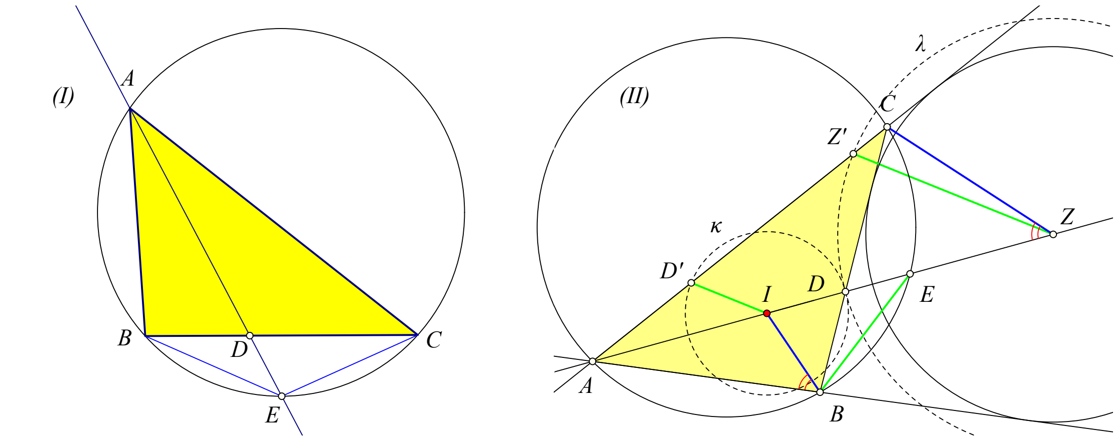


Figure 16: Relation with bisector Relations of segments on the bisector

Exercise 31. *Given is a triangle ABC with incenter I, and in the bisector from A the excenter Z, the intersection E with the circumcircle and the intersection point D with BC. Show that the circles $\{\kappa, \lambda\}$ with centers, respectively, $\{I, Z\}$ and radii $\{|ID|, |ZD|\}$ define points $\{D', Z'\}$ on AC (see figure 16-(II)), such that the triangles $\{ABE, AD'I, ADC, AZ'Z\}$ are similar. Also similar are the triangles $\{AIB, ACZ\}$, as well as the triangles $\{AIC, ABZ\}$ and the following relations hold:*

1. $|AE| \cdot |ID| = |ZI| \cdot |AI|$.
2. $|AE| \cdot |AD| = |AI| \cdot |AZ| = |AB| \cdot |AC|$.
3. $|IA| \cdot |ZD| = |ZA| \cdot |ID|$.

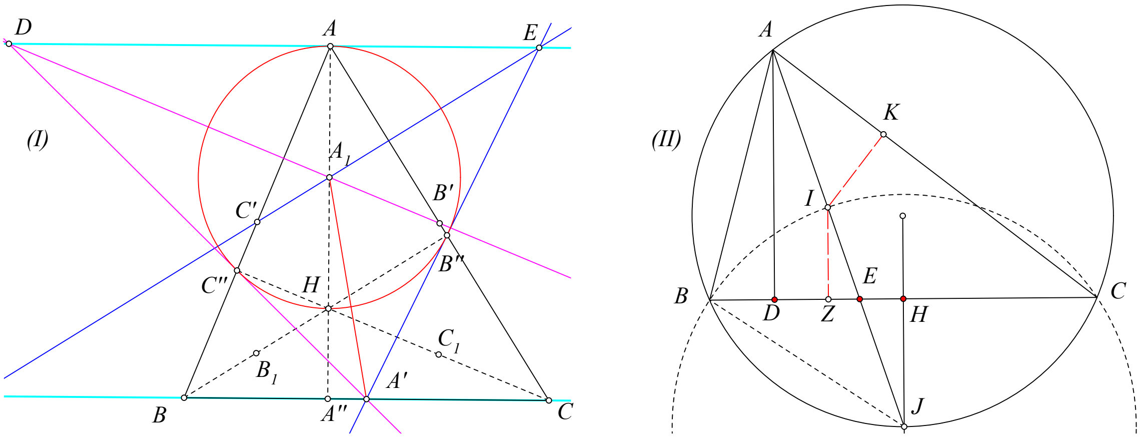


Figure 17: $\{A, D, E\}$ collinear Traces of altitudes, bisectors and medians

Exercise 32. *In the triangle ABC, the middles of the sides are respectively A', B', C', the traces of the altitudes are A'', B'', C'', the orthocenter is H and A_1, B_1, C_1 are the middles of HA, HB, HC. Show that the points of intersection of the lines $D = (A'C'', A_1B')$ and $E = (A_1C', A'B'')$ are contained in the parallel to BC from A. Moreover line EA_1 is a bisector of the angle $\widehat{AEA'}$ and line DB' is a bisector of the angle $\widehat{ADA'}$. Show that the circle with diameter AH is the inscribed circle of triangle DEA' (see figure 17-(I)).*

Exercise 33. *Let $\{D, E, H\}$ be respectively the traces of the altitude, bisector and median on the side BC of the triangle ABC. Let also Z be the projection of the incenter I on the side BC. Show that $|HZ|^2 = |HD| \cdot |HE|$ (see figure 17-(II)). Using $|ZH| = \frac{|b-c|}{2}$, construct the triangle, whose given are the lengths $\{v_A = |AD|,\ \ \mu_A = |AH|,\ \ |b-c|\}$.*

Hint: The relation follows from the equality of the first and last term of the equations

$$\frac{|DZ|}{|ZH|} = \frac{|AI|}{|IJ|} = \frac{|AI|}{|BJ|} = \frac{|IK|}{|HJ|} = \frac{|IZ|}{|HJ|} = \frac{|ZE|}{|EH|} \quad \Rightarrow \quad |EH| \cdot |HD - HZ| = |ZH| \cdot |ZE|.$$

10 Euler's line and circle of the triangle

Theorem 9. *The centroid G of the triangle ABC is contained in the line segment with end points the circumcenter O and the orthocenter H and divides it into ratio 1:2 ($|GH| = 2|GO|$). The line $\varepsilon = OH$ is called "Euler line" of the triangle (see figure 18).*

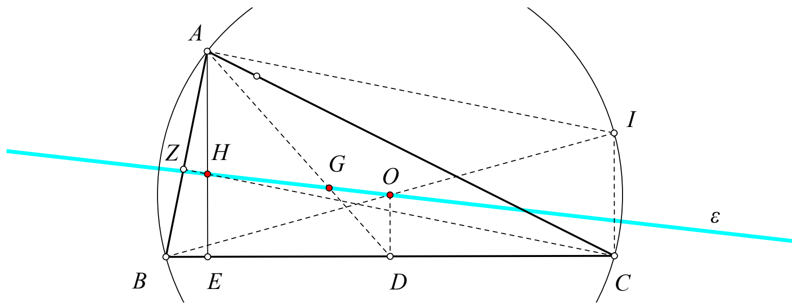


Figure 18: The Euler line HO

Proof. Consider the diametrically opposite I of vertex B relative to the circumcircle. A and C are also on the circumcircle, therefore they see the diameter BI under a right angle. It follows easily that $AHCI$ is a parallelogram, hence $|AH| = |IC|$. However, if D is the middle of BC, OD joins side middles of the triangle BCI, therefore $|IC| = 2|OD|$ and consequently AH has twice the length and is parallel to OD. Suppose G is the intersection point of the median AD with OH. Triangles AHG and DOG are similar, having corresponding angles equal. Consequently, their sides will be proportional and, because $|AH| = 2|OD|$, the same will happen also with the other corresponding sides, in other words:

$$|AG| = 2|GD| \quad \text{and} \quad |HG| = 2|GO|,$$

the first characterizing the centroid and the second proving the claim. □

"Euler's circle" of the triangle, is the one which passes through the three middles of the sides. Its particularity lies in the fact that it also passes through six more noteworthy points of the triangle. That's why it is often called *"circle of nine points"* of the triangle (see figure 19).

Theorem 10. *The circle κ, which passes through the middles M, N, J of the sides of triangle ABC, has the following properties:*

1. *It passes also through the traces $\{D, E, Z\}$ of the altitudes of the triangle.*
2. *It passes also through the middles $\{P, S, I\}$ of the line segments which join the vertices with the orthocenter H of the triangle.*
3. *Its center P is the middle of the segment which joins the orthocenter H with the circumcenter O of the triangle.*
4. *Its radius $|PI|$ is half that of the radius $R = |OA|$ of the circumscribed circle L of the triangle.*
5. *Point H is a similarity center of κ and the circumcircle of the triangle, with similarity ratio $1:2$.*

Proof. The proof relies on the existence of three rectangles which have, by two, a common diagonal. The rectangles are $STNJ$, $SMNI$ and $IJMT$. First, let us see that these rectangles exist. I show that $STNJ$ is such a rectangle. The proof for rectangles $SMNI$ and $IJMT$ is similar.

In $STNJ$ then, SJ joins the middles of sides of the triangle BHA. Therefore it is parallel and the half of HA. Similarly TN joins the middles of sides of triangle AHC. Therefore it is parallel and the half of HA. Consequently SJ and TN, being parallel and equal, they define a parallelogram $STNJ$. That this is actually a rectangle, follows from the fact that

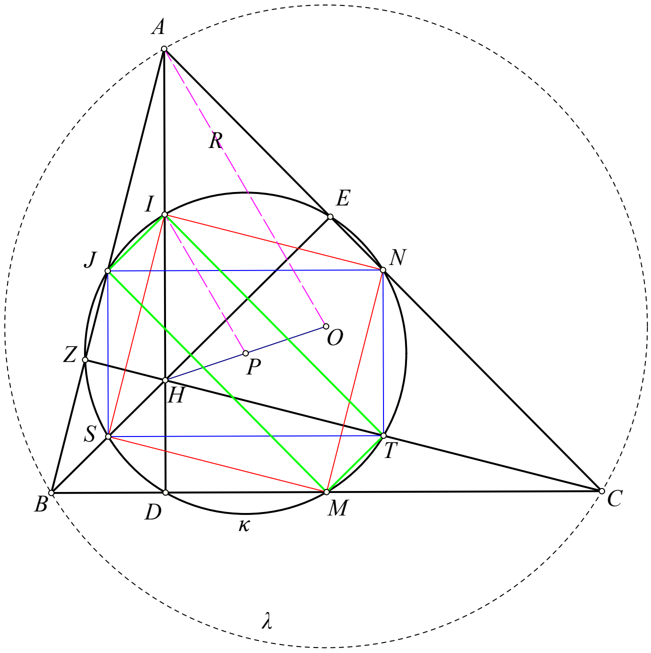


Figure 19: Euler's circle κ of the triangle ABC

ST joins middles of sides of triangle HBC, therefore it is parallel and the half of BC. Since AD and BC are mutually orthogonal, the same will happen also with their parallels SJ and ST.

The three rectangles have by two a common diagonal, which is the diameter of their circumscribed circle. This implies that the three circumscribed circles of these rectangles coincide. This completes the proof of the first two claims of the proposition.

For the proof of the remaining two claims, it suffices to observe that in the triangle HOA the segment PI joins the middles of the sides, therefore it is parallel and the half of OA. However OA is a radius of the circumscribed circle L and PI is a radius of circle κ. Latter follows from the fact $|AH| = 2|OM|$, therefore $IHMO$ is a parallelogram and its diagonals are bisected at P. However IM, as seen previously, is a diameter of the circle κ. The last claim is a consequence of the two previous ones. □

11 Some properties of Euler's circle

Here we discuss some properties of Euler's circle formulated as exercises.

Exercise 34. *Show that the triangle ABC coincides with the orthic triangle of triangle $I_A I_B I_C$, with vertices the excenters of ABC. Conclude that the circumcircle of the triangle ABC coincides with the Euler circle of $I_A I_B I_C$ (see figure 20).*

Exercise 35. *The orthocenter H of the triangle ABC is projected on the bisectors of angle $\widehat{A}$, internal and external, at the points S and P. Show that the line SP passes through the middle of BC and the center of its Euler circle.*

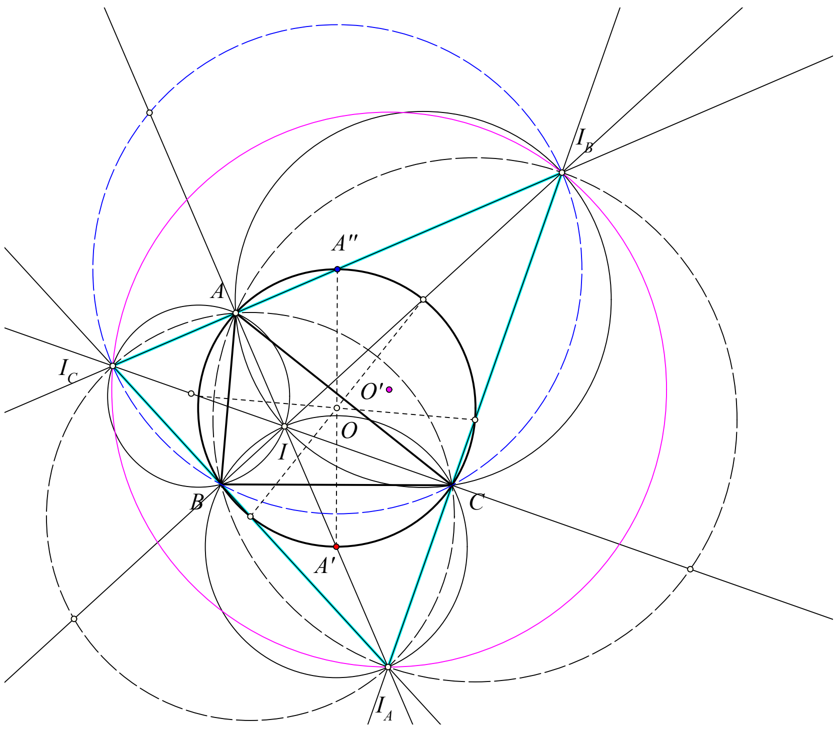


Figure 20: Circumcircle of ABC as Euler circle of $I_AI_BI_C$

Exercise 36. *Show that for every triangle ABC with orthocenter H, the Euler circles of the triangles ABC, ABH, BCH and CAH coincide (see figure 21-(I)). Conclude that these four triangles have circumscribed circles of equal radii.*

Exercise 37. *The diagonals of the quadrilateral $ABCD$ define four triangles ABC, CDA, BCD, DAB (see figure 21-(II)). Show that the Euler circles of these triangles pass through a common point O.*

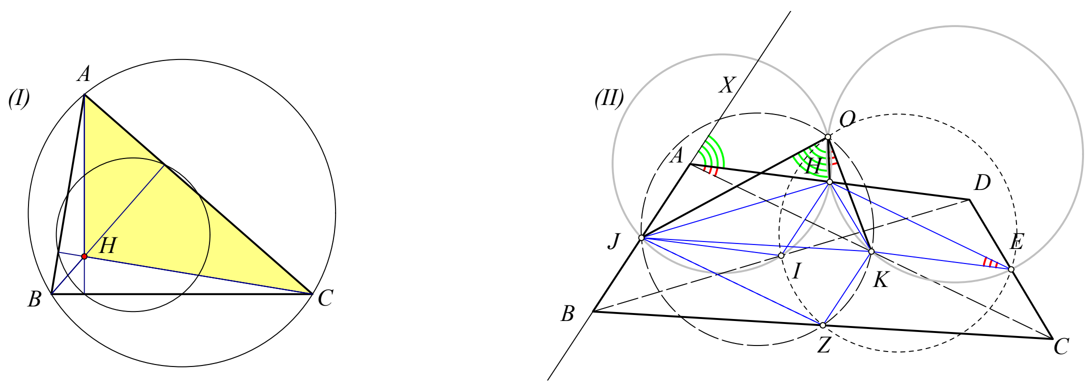


Figure 21: Common Euler circle Intersection of four Euler circles

Hint: Consider the parallelogram $EZHJ$ of the middles of the sides of the quadrilateral and the middles $\{I, K\}$ of the diagonals. Let also O be the intersection of two of these Euler circles e.g. of $\{ABD, ACD\}$. Show that the other Euler circles pass through the same point, proving f.e. that $\widehat{JZK}$ and $\widehat{JOK}$ are supplementary angles.

Exercise 38. *The intersection point O of the diagonals of the quadrilateral $ABCD$ defines four triangles ABO, BCO, CDO, DAO. Show that the circumcenters $\{E, Z, H, J\}$ of these triangles are vertices of a parallelogram. Show the same also for the centers $\{I, K, L, M\}$ of the Euler circles of these triangles (see figure 22-(I)).*

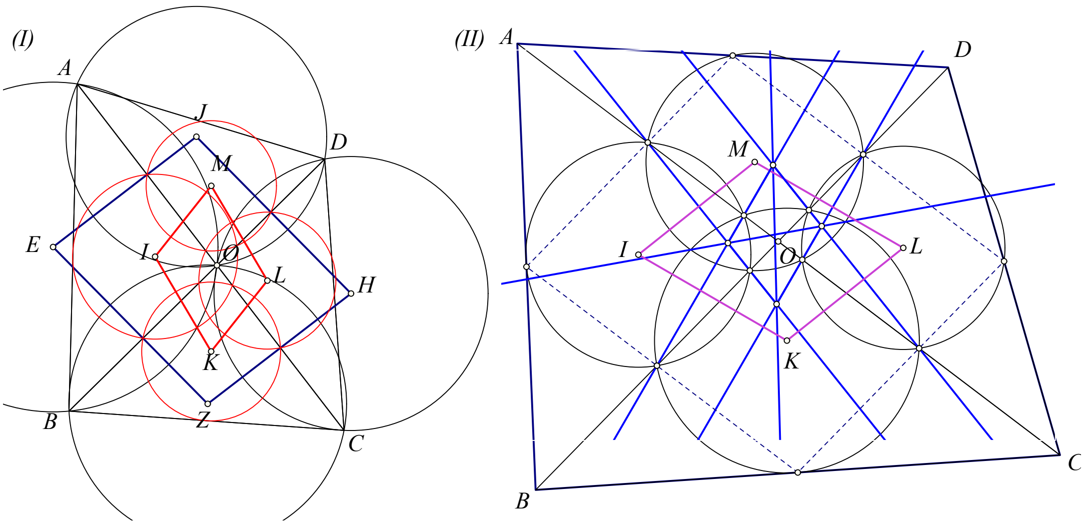


Figure 22: Four circumscribed circles Four Euler circles

Exercise 39. *For the quadrilateral and the four triangles, defined in the previous exercise, show that the radical axes of the Euler circles of these triangles define the sides and the diagonals of a parallelogram, which is similar to IKLM (see figure 22-(II)).*

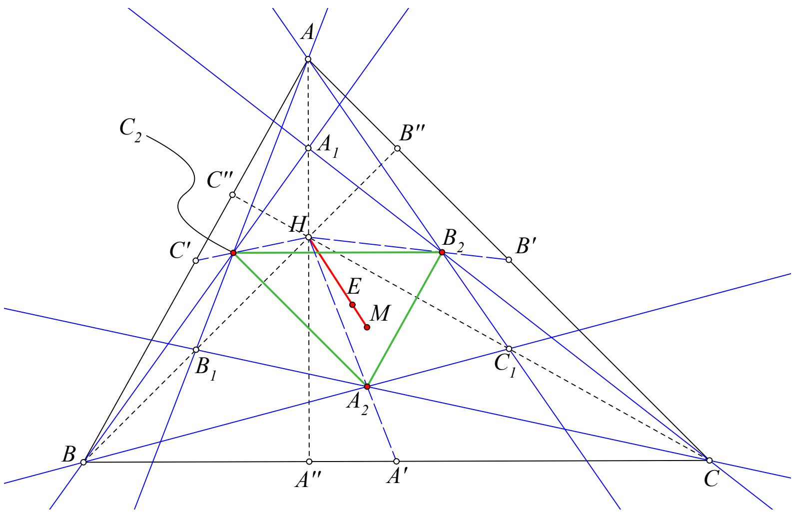


Figure 23: Property of the center of the Euler circle

Exercise 40. *For a triangle ABC the middles of its sides are respectively $\{A', B', C'\}$, the traces of its altitudes are $\{A'', B'', C''\}$, the orthocenter, the center of mass and the center of its Euler circle are respectively $\{H, M, E\}$ (see figure 23). Also $\{A_1, B_1, C_1\}$ are respectively the middles of*

$\{HA, HB, HC\}$. Show that the triangle with vertices the centroids $\{A_2, B_2, C_2\}$ of the respective triangles $\{HBC, HCA, HAB\}$ is homothetic to ABC with center of homothecy the point E and ratio of homothecy 1:3. Also the orthocenter of $A_2B_2C_2$ coincides with M.

Exercise 41. *Given a triangle ABC and a point D not contained in the side-lines of the triangle, show that the Euler circles $\{\kappa_A, \kappa_B, \kappa_C\}$ respectively of triangles $\{DBC, DCA, DAB\}$ intersect at a point L contained in the Euler circle of the triangle ABC.*

Hint: Consider the second intersection point L of circles κ_B, κ_C (the first is the middle E of the segment DA) (see figure 24-(I)). From the inscriptible quadrilateral $LHJE$, the angle

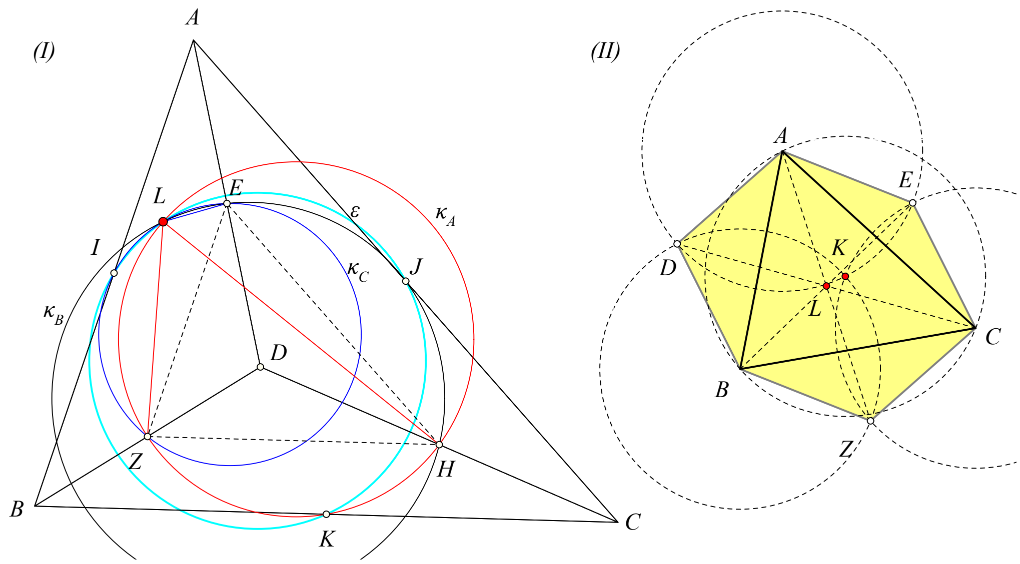


Figure 24: Concurring circles of Euler Symmetric circumscribed hexagon

$\widehat{ELH}$ is supplementary to $\widehat{EJH}$, which is equal to $\widehat{EDH}$, because $EJHD$ is a parallelogram. Similarly, $\widehat{ZLE}$ is supplementary to $\widehat{ZDE}$ and $\widehat{ZLH}$ is equal to $\widehat{ZDH}$. The circle κ_A will also pass through L. Next show that point L lies on the Euler circle of ABC by proving that point L sees KI under an angle of measure $180° - \beta$.

Exercise 42. *Show that for every acute-angled triangle ABC there exists a convex symmetric hexagon $ADBZCE$ with equal sides, whose center coincides with the center L of its Euler circle (see figure 24-(II)). Show that, conversely in each convex symmetric hexagon with equal sides, the triangle which results by taking non successive vertices of it has as center of its Euler circle the center of symmetry of the hexagon.*

Exercise 43. *Construct a triangle ABC for which are given the position of the vertex A, the position of the projection D of A on the opposite side BC and the position of the center P of its Euler circle.*

12 Feuerbach's theorem

In 1822 Feuerbach (1800-1834), who was a high school teacher, published a small book, which, among other noteworthy theorems on the triangle ([Joh60, p.190]), contained also the theorem we prove below. In this proof ([Aud02, p. 110], [CG67, p.117]) the key role is played by the circle κ with diameter KL, where $\{K, L\}$ are the projections on BC of the incenter I and of the excenter O contained in the angle $\widehat{A}$ of triangle ABC (see figure 25-(I)). From theorem 4 we know that this circle has its center at the middle A' of BC. The proof of

Feuerbach's theorem results from properties of the inversion relative to that circle. Next theorem formulates these properties beginning with the acute triangle with $b > c$, using arguments which hold in all cases.

Theorem 11. *With the previous definitions and notations, hold the properties:*

1. *The circle κ is orthogonal to the inscribed λ as well as to the escribed μ.*
2. *The intersection point J of the bisector AI with BC is the inverse relative to κ of the trace A'' of the altitude.*
3. *The angles with BC, of the tangent ε to λ from J and the tangent to the Euler circle at the middle A' of BC are equal to $|\beta - \gamma|$.*
4. *The inversion relative to κ maps the Euler circle ν to the line ε.*

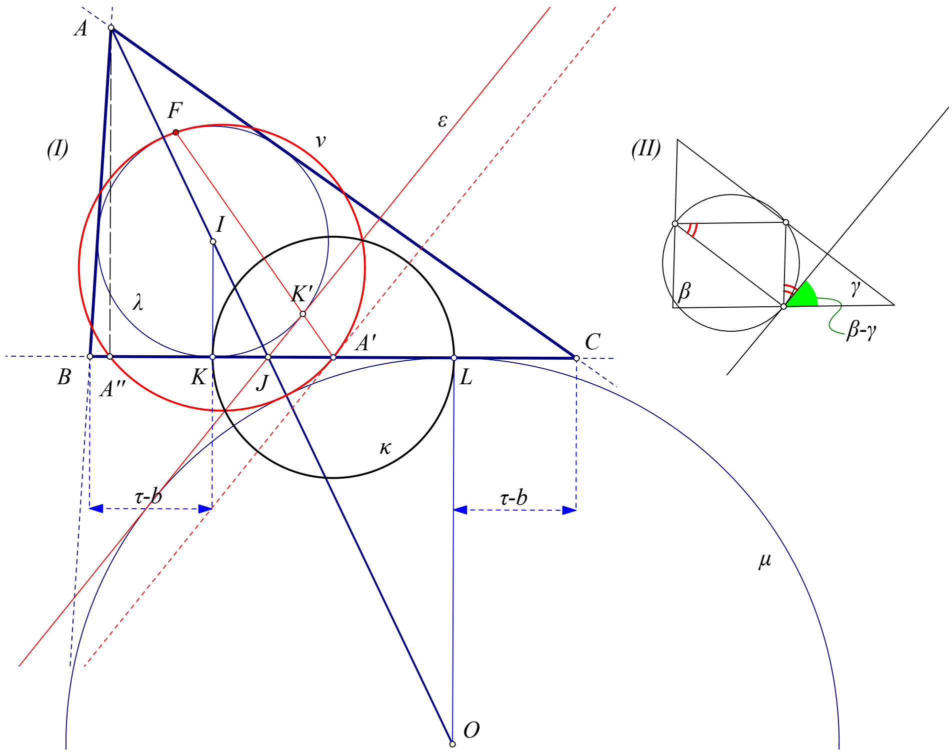


Figure 25: The circle κ and its inversion

Proof. *Nr-1* is obvious, since the radii of the circles at K and L, respectively, are orthogonal to the corresponding ones of κ.

Nr-2 follows from calculations we performed in previous sections. According to proposition 4, the radius of κ is equal to $r = |b - c|/2$. The length $|A'A''|$ is calculated by considering the power of B relative to the circle with diameter AC, and is found to be equal to $\frac{|b^2-c^2|}{2a}$. The length $|A'J|$ is calculated through the ratio in which the bisector divides BC, and is found to be equal to $\frac{a|c-b|}{b+c}$. The claim follows from the fact $r^2 = |A'J||A'A''|$.

Nr-3 follows, on one hand from the figure 25-(II), which shows that the measure of the angle at A is equal to $|\beta - \gamma|$, and on the other from the fact, that the angle at J will have measure $|180° - 2\widehat{BJA}|$, which is also easily seen to be equal to $|\beta - \gamma|$.

Nr-4 follows from the properties of inversion and the previous claims. Indeed, since the Euler circle ν passes through the center of inversion and through A'', J, its image will be a line passing through J and forming at J an angle equal to that formed by the circle ν with BC, therefore coincident with ε. This conclusion follows from *nr-3*. □

Theorem 12. *(Feuerbach) In every triangle the Euler circle is tangent to the inscribed, as well as, to its three escribed circles.*

Proof. With the notation of the previous theorem, we consider the inversion relative to circle κ (see figure 25-(I)). In it the inverse of the inscribed λ is itself and, according to the previous proposition, the inverse of the Euler circle ν is the line ε, i.e. the second tangent to λ from J. Since ε is tangent to *lambda*, its inverse, which is the circle ν, will be also tangent to λ at a point F, which is the inverse of the contact point K' of ε with λ. Note that, because of the symmetry of λ relative to the bisector AI, point K' is the symmetric of K relative to AI.

The same argument is applied also to the escribed circle μ with center O. This circle, as well, is orthogonal to κ and has ε as a tangent. Therefore the inverse of this circle is itself and the inverse of the line ε, which is the Euler circle, will be tangent to it.

What we said proves then that the Euler circle is tangent simultaneously to the inscribed and escribed contained in the angle A. Similarly we prove that the Euler circle is tangent to the escribed circles contained in the other angles. □

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