A Discrete Analogue for Minkowski's Second Theorem on Successive Minima

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Abstract

The main result of this paper is an inequality relating the lattice point enumerator of a 3-dimensional, 0-symmetric convex body and its successive minima. This is an example of generalization of Minkowski's theorems on successive minima, where the volume is replaced by the discrete analogue, the lattice point enumerator. This problem is still open in higher dimensions, however, we introduce a stronger conjecture that shows a possibility of proof by induction on the dimension.

1 Introduction

A subset K of $\mathbb{R}^d$ is called a convex body if it is convex, compact, with nonempty interior (sometimes the technical condition $K \subset \overline{\text{int}(K)}$ is required, but we will not need it here). The set of all convex bodies of $\mathbb{R}^d$ will be denoted as $\mathcal{K}^d$, and the subset of the 0-symmetric elements will be denoted as $\mathcal{K}_0^d$. Furthermore, $\mathcal{L}^d$ will denote the set of all lattices in $\mathbb{R}^d$. We will denote the ith successive minimum of $K \in \mathcal{K}_0^d$ with respect to $\Lambda \in \mathcal{L}^d$ by $\lambda_i(K, \Lambda)$, i.e.,

$$\lambda_i = \inf\{\lambda > 0 \mid \dim(\lambda K \cap \Lambda) \geqslant i\},$$

where $\dim(A)$ for $A \subset \mathbb{R}^d$ is the dimension of the vector space spanned by all vectors of the set A.

Minkowski proved the following two inequalities relating the volume of $K \in \mathcal{K}_0^d$, i.e., its d-dimensional Lebesgue measure, with its successive minima, with respect to a lattice Λ:

$$\frac{\text{vol}(K)}{\det(\Lambda)} \leqslant \left(\frac{2}{\lambda_1(K, \Lambda)}\right)^d \tag{1}$$

and

$$\frac{1}{d!}\prod_{i=1}^{d}\frac{2}{\lambda_i(K,\Lambda)} \leqslant \frac{\text{vol}(K)}{\det(\Lambda)} \leqslant \prod_{i=1}^{d}\frac{2}{\lambda_i(K,\Lambda)} \tag{2}$$

which is an improvement of (1). The above are known as the first and second theorem on successive minima, respectively. About a century later, in 1993, Betke, Henk, and Wills [BHW] stated discrete analogues of these theorems, where the volume is replaced by the lattice point point enumerator, $G(K,\Lambda) := \#(K \cap \Lambda)$. They proved the analogue for the first theorem, which predicts the following inequality:

Theorem 1.1. *Let $K \in \mathcal{K}_0^d$ and $\Lambda \in \mathcal{L}^d$. Then*

$$G(K,\Lambda) \leqslant \left[\frac{2}{\lambda_1(K,\Lambda)} + 1\right]^d. \tag{3}$$

As for the second theorem, they stated a conjecture, which they verified in the planar case.

Conjecture 1.2. *Let $K \in \mathcal{K}_0^d$ and $\Lambda \in \mathcal{L}^d$. Then*

$$G(K,\Lambda) \leqslant \prod_{i=1}^{d}\left[\frac{2}{\lambda_i(K,\Lambda)} + 1\right]. \tag{4}$$

In Section 3.2 we shall see a proof for the 3-dimensional case of the above conjecture. There is a notion of induction in the proof; we need statements about intersections of a given convex body by hyperplanes passing through lattice points. The resulting convex bodies, whose dimension is $d-1$, are not centrally symmetric, in general. Therefore, it is necessary to extend the definition of the successive minima, as well as the results referred to them, to the class of all convex bodies, not necessarily 0-symmetric, namely

$$\lambda_i(K,\Lambda) := \lambda_i(\tfrac{1}{2}\mathfrak{D}K, \Lambda),$$

where $\mathfrak{D}K$ stands for the difference body of K, i.e.,

$$\mathfrak{D}K = K - K = \{x - y | x, y \in K\}.$$

Under this definition, Minkowski's theorems still hold; this is a simple consequence of the Brunn-Minkowski theorem ([GL], pp. 12 and 32), which predicts that

$$\text{vol}(K) \leqslant \text{vol}(\tfrac{1}{2}\mathfrak{D}K).$$

In Section 3.1, we will extend Theorem 1.1 to the non 0-symmetric case, as follows:

Theorem 1.3. *Let $K \in \mathcal{K}^d$, $\Lambda \in \mathcal{L}^d$. Then*

$$G(K, \Lambda) \leqslant \left[\frac{2}{\lambda_1(K, \Lambda)} + 1 \right]^d.$$

We conjecture that inequality (4) still holds when K is not 0-symmetric, and we will prove it in Section 3.2 for $d \leqslant 3$, which is the main result of this paper:

Theorem 1.4. *Let $K \in \mathcal{K}^d$, $\Lambda \in \mathcal{L}^d$, where $d \leqslant 3$. Then*

$$G(K, \Lambda) \leqslant \prod_{i=1}^{d} \left[\frac{2}{\lambda_i(K, \Lambda)} + 1 \right]$$

We will also prove some weaker estimates, exactly as in [M].

Theorem 1.5. *Let $K \in \mathcal{K}^d$, $\Lambda \in \mathcal{L}^d$. Then*

$$G(K, \Lambda) \leqslant \frac{4}{e} (\sqrt{3})^{d-1} \prod_{i=1}^{d} q_i(K, \Lambda).$$

If $K \in \mathcal{K}_0^d$, then

$$G(K, \Lambda) \leqslant \frac{4}{e} \left(\sqrt[3]{\frac{40}{9}} \right)^{d-1} \prod_{i=1}^{d} q_i(K, \Lambda).$$

2 Some auxiliary lemmata

We will first need the following standard convention; remember that the successive minima of $K \in \mathcal{K}^d$ with respect to a lattice Λ are those of $\frac{1}{2}\mathfrak{D}K$. By definition of the successive minima $\lambda_i(K, \Lambda)$, there are d linearly independent lattice vectors $a^i = a^i(K, \Lambda)$ such that

$$a^i(K, \Lambda) \in \tfrac{\lambda_i(K,\Lambda)}{2} \mathfrak{D}K \cap \Lambda.$$

Then we construct a basis of Λ denoted by $e^i = e^i(K, \Lambda)$, $1 \leqslant i \leqslant d$, such that

$$\lim(a^1, \ldots, a^i) = \lim(e^1, \ldots, e^i)$$

for all i, $1 \leqslant i \leqslant d$. Furthermore, we define the following subgroups of Λ:

$$\Lambda^i(K) := \mathbb{Z}e^1 \oplus \cdots \oplus \mathbb{Z}e^i$$

3

We will usually abbreviate the notation to Λ^i. It should be noted that there is an abuse of notation here; it is evident that the choice of the a^i's and the e^i's, as well as the Λ^i's, is not always unique. However, by this notation we shall always mean a choice of vectors or subgroups with the above properties. The main property that will be used later is

$$\text{int}(\tfrac{\lambda_i}{2}\mathfrak{D}K) \cap \Lambda \subset \Lambda^{i-1} \tag{5}$$

Lemma 2.1. *Let $K \in \mathcal{K}^d$, $\Lambda \in \mathcal{L}^d$. For each real n_i, satisfying $n_i > 2/\lambda_i$, we have*

$$\mathfrak{D}K \cap n_i(\Lambda \setminus \Lambda^{i-1}) = \varnothing$$

In particular,

$$\text{int}(\mathfrak{D}K) \cap \frac{2}{\lambda_i}(\Lambda \setminus \Lambda^{i-1}) = \varnothing$$

Proof. Assume otherwise; then the intersection

$$\tfrac{1}{n_i}\mathfrak{D}K \cap (\Lambda \setminus \Lambda^{i-1})$$

would be nonempty. The left part of this intersection is a subset of

$$\text{int}(\tfrac{\lambda_i}{2}\mathfrak{D}K),$$

since $n_i > 2/\lambda_i$. Therefore, the intersection

$$\text{int}(\tfrac{\lambda_i}{2}\mathfrak{D}K) \cap (\Lambda \setminus \Lambda^{i-1})$$

is nonempty, contradicting (5) above, as was to be shown. $\qquad\square$

The following is an adaptation of Lemma 2.1 in [H], for the case of all convex bodies, not necessarily 0-symmetric. Even though the proof is identical, we provide it here for convenience.

Lemma 2.2. *Let $K \in \mathcal{K}^d$ and $\Lambda, \widetilde{\Lambda} \in \mathcal{L}^d$, with $\widetilde{\Lambda} \subset \Lambda$. Then*

$$G(K, \Lambda) \leqslant \frac{\det(\widetilde{\Lambda})}{\det(\Lambda)} G(\mathfrak{D}K, \widetilde{\Lambda}). \tag{6}$$

Proof. Let $m = G(\mathfrak{D}K, \widetilde{\Lambda})$ and suppose there exist at least $m+1$ different lattice points $v^1, \ldots, v^{m+1} \in K \cap \Lambda$ such that $v^i \equiv v^1 \mod \widetilde{\Lambda}$, $1 \leqslant i \leqslant m+1$. Then we have

$$v^i - v^1 \in \mathfrak{D}K \cap \widetilde{\Lambda}, \quad 1 \leqslant i \leqslant m+1,$$

which contradicts the assumption $m = G(\mathfrak{D}K, \widetilde{\Lambda})$. Thus we have shown that every residue class of Λ with respect to $\widetilde{\Lambda}$ does not contain more than m points of $K \cap \Lambda$. Since there are precisely $\det(\widetilde{\Lambda})/\det(\Lambda)$ different residue classes, we obtain the desired bound. $\qquad\square$

4

The following two lemmata will be used for the proof of Conjecture 1.2 in the 3-dimensional case. Notice that they are statements in d dimensions.

Lemma 2.3. *Let $K \subset \mathbb{R}^d$ be a convex body, $\Lambda \in \mathcal{L}^d$, such that $K \cap \Lambda = \varnothing$. Then there is some $v \in \Lambda$ such that for any real $t > 1$,*

$$K \cap (v + t\Lambda) = \varnothing.$$

Proof. Take $v \in \Lambda$ such that $\#(\text{conv}(v, K) \cap \Lambda)$ is minimal. If this number is greater than 1, then there is some $w \in \Lambda$, $w \neq v$, such that $w \in \text{conv}(v, K)$. Hence, $\text{conv}(w, K) \subset \text{conv}(v, K)$, and $v \notin \text{conv}(w, K)$, contradicting the minimality of $\#(\text{conv}(v, K) \cap \Lambda)$. Thus, $\text{conv}(v, K) \cap \Lambda = \{v\}$. We claim that $K \cap (v + t\Lambda) = \varnothing$, for all $t > 1$. Suppose not; then there is some $u \in \Lambda$ such that $v + tu \in K$, for some $t > 1$. By convexity, and the fact that $t > 1$, we get $v + u \in \text{conv}(v, K)$, which implies $u = 0$, so $v \in K$, a contradiction, since $K \cap \Lambda = \varnothing$. This concludes the proof. $\square$

The next lemma generalizes the above:

Lemma 2.4. *Let $K \subset \mathbb{R}^d$ be a convex body, and $\Lambda \in \mathcal{L}^d$. Let $S \subset \Lambda$ be finite, and r be a positive integer, such that*

(1) $(K - S) \cap r\Lambda = \varnothing$.

(2) $\mathfrak{D}S \cap r(\Lambda \setminus \{0\}) = \varnothing$.

Now, let $t > r$ be an integer. There is a set $S' \subset \Lambda$, obtained by translating each $v \in S$ by some vector $r \cdot w(v)$, where $w(v) \in \Lambda$, such that

(1)′ $(K - S') \cap t\Lambda = \varnothing$.

(2)′ $\mathfrak{D}S' \cap t(\Lambda \setminus \{0\}) = \varnothing$.

Proof. The proof proceeds by induction on $\#(S)$. If $\#(S) = 1$; i.e., $S = \{v\}$, we use Lemma 2.3 for $K - v$ and the lattice $r\Lambda$. Since $t > r$, there is some $w(v) \in \Lambda$, such that $(K - v) \cap (r \cdot w(v) + t\Lambda) = \varnothing$. Put $S' = \{v + r \cdot w(v)\}$, and we see that (1)′ is satisfied. It should be noted that when $\#(S) = 1$, conditions (2) and (2)′ hold vacuously.

Now, assume that $\#(S) > 1$. Take $v \in S + r\Lambda$, such that $\#(\text{conv}(v, K) \cap (S + r\Lambda))$ is minimal. Again, as in the proof of Lemma 2.3, we must have

5

$\operatorname{conv}(v, K) \cap (S + r\Lambda) = \{v\}$. Apply induction for $\widetilde{K} = \operatorname{conv}(v, K)$ and $\widetilde{S} = S \backslash (S \cap (v + r\Lambda))$; we have $\#(\widetilde{S}) = \#(S) - 1$. Let's see why (1) and (2) are satisfied for $\widetilde{K}, \widetilde{S}$ (same r, Λ); (2) is obviously satisfied, as $\widetilde{S} \subset S$. If (1) were not satisfied, then there would be some $w \in \widetilde{S}$ and $u \in \Lambda$ such that $w + ru \in \operatorname{conv}(v, K)$. By the minimality assumption, $w + ru = v$. But $v \notin \widetilde{S} + r\Lambda$, a contradiction. Thus, (1) and (2) hold for $\widetilde{K}, \widetilde{S}$, and by induction there is some $\overline{S} \subset \Lambda$, obtained from $\widetilde{S}$ by translating each $u \in \widetilde{S}$ by $r \cdot w(u)$, $w(u) \in \Lambda$, such that

$$(\widetilde{K} - \overline{S}) \cap t\Lambda = \varnothing, \tag{7}$$

and

$$\mathfrak{D}\overline{S} \cap t(\Lambda \setminus \{0\}) = \varnothing. \tag{8}$$

Now, set $S' = \overline{S} \cup \{v\}$. $(2)'$ is satisfied for S'; if $x, y \in \overline{S}$, then $x - y \notin t(\Lambda \setminus \{0\})$ from (8). If $x \in \overline{S}$ and $y = v$, then again from above, $v - x \notin t\Lambda$, since $v \in \widetilde{K}$. If $x = y = v$, we have nothing to prove, so

$$\mathfrak{D}S' \cap t(\Lambda \setminus \{0\}) = \varnothing.$$

$(1)'$ is also satisfied for K, S'; suppose not. Then, there is some $w \in S'$, $u \in \Lambda$ such that $w + tu \in K$. If $w \in \overline{S}$, then $w + tu \in \widetilde{K}$, which contradicts $(\widetilde{K} - \overline{S}) \cap t\Lambda = \varnothing$. If $w = v$, then $v + tu \in K$, and by convexity, $v + ru \in K$, hence $u = 0$, by minimality assumption, and $v \in K$, a contradiction. This concludes the proof. $\qquad\square$

3 Inequalities for $G(K, \Lambda)$

Throughout the rest of the paper we will use the notation

$$q_i(K, \Lambda) = \left\lceil \frac{2}{\lambda_i(K, \Lambda)} + 1 \right\rceil.$$

When no confusion arises, we will simply write q_i instead of $q_i(K, \Lambda)$. Also, when Λ is the standard lattice $\mathbb{Z}^d$, we write $G(K)$ instead of $G(K, \mathbb{Z}^d)$.

3.1 The general case

The method of the following proof is similar to the proof of Theorem 1.5 in [H], from a slightly different viewpoint.

Theorem 3.1. *Let $K \in \mathcal{K}^d$, $\Lambda \in \mathcal{L}^d$. Let also $n_1, \ldots, n_d$ be a sequence of integers satisfying*

- n_{i+1} *divides* n_i, $1 \leqslant i \leqslant d - 1$.

- $q_i(K, \Lambda) \leqslant n_i$, $1 \leqslant i \leqslant d$.

Then,

$$G(K, \Lambda) \leqslant \prod_{i=1}^{d} n_i.$$

Proof. Let $e^i = e^i(K, \Lambda)$ and define

$$\widetilde{\Lambda} = \mathbb{Z} n_1 e^1 \oplus \cdots \oplus \mathbb{Z} n_d e^d.$$

By Lemma 2.2,

$$G(K, \Lambda) \leqslant \frac{\det(\widetilde{\Lambda})}{\det(\Lambda)} G(\mathfrak{D} K, \widetilde{\Lambda}) = G(\mathfrak{D} K, \widetilde{\Lambda}) \prod_{i=1}^{d} n_i.$$

It suffices to prove that $G(\mathfrak{D} K, \widetilde{\Lambda}) = 1$, or equivalently

$$\mathfrak{D} K \cap (\widetilde{\Lambda} \setminus \{0\}) = \varnothing.$$

This follows from Lemma 2.1 and the fact that

$$\widetilde{\Lambda} \setminus \{0\} \subset \bigcup_{i=1}^{d} n_i(\Lambda \setminus \Lambda^{i-1})$$

(recall that $n_i \geqslant q_i(K, \Lambda) > 2/\lambda_i(K, \Lambda)$). Indeed, let $g \in \widetilde{\Lambda} \setminus \{0\}$ be arbitrary, and let k be minimal such that

$$g \in \mathbb{Z} n_1 e^1 \oplus \cdots \oplus \mathbb{Z} n_k e^k.$$

Since n_k divides all $n_1, \ldots, n_{k-1}$ by assumption, we have $g \in n_k \Lambda$. By minimality of k, we also have $g \notin \Lambda^{k-1}$, hence $g \in n_k(\Lambda \setminus \Lambda^{k-1})$ as desired. $\quad\square$

As a simple consequence we can extend Theorem 1.1 to the non-symmetric case, and thus obtain Theorem 1.3, as follows:

Proof of Theorem 1.3. The numbers $n_1 = \cdots = n_d = q_1(K, \Lambda)$ satisfy the hypotheses of Theorem 3.1. $\quad\square$

We remind the following definition, given in [M]:

Definition. Let C_d denote the least positive constant, such that for any sequence of d integers, $0 < x_1 \leqslant x_2 \leqslant \cdots \leqslant x_d$, there exists a sequence of integers $y_1, y_2, \ldots, y_d$ satisfying:

a. $x_i \leqslant y_i$, for all i, $1 \leqslant i \leqslant d$

b. y_i divides y_{i+1}, for all i, $1 \leqslant i \leqslant d - 1$

c. $\dfrac{y_1 y_2 \cdots y_d}{x_1 x_2 \cdots x_d} \leqslant C_d$.

We can always choose $n_1, \ldots, n_d$ with the properties given in Theorem 3.1, such that

$$\prod_{i=1}^{d} n_i \leqslant C_d \prod_{i=1}^{d} q_i(K, \Lambda).$$

The estimates for C_d in [M], yield Theorem 1.5.

Notice that when K is not 0-symmetric, we cannot disregard all $q_i(K, \Lambda)$ that are less than or equal to 2, as it was done in [M]; the reason is that we may have $q_1(K, \Lambda) = \cdots = q_d(K, \Lambda) = 2$, and $K \cap \Lambda$ have full affine dimension (while if K was 0-symmetric, this would mean that $K \cap \Lambda = \{0\}$). Consider, for example, the d-dimensional cube $[0, 1]^d$. So, in the non 0-symmetric case we cannot reduce the base to $(\sqrt[3]{40/9})^{d-1}$, using this argument.

3.2 The case $d = 3$

We will introduce an inductive method in order to prove Conjecture 1.2; this method works up to dimension 3, thus obtaining Theorem 1.4, but it is inadequate for higher dimensions. As we will see in the next section, stronger versions of Lemmata 2.3 and 2.4 might be needed for this method to work in all dimensions.

Let $K \in \mathcal{K}_0^d$, $\Lambda \in \mathcal{L}^d$. Fix a basis $e^i = e^i(K, \Lambda)$ of Λ, that satisfies the properties given in Section 2. We will write each vector x of $\mathbb{R}^d$ with coordinates with respect to this basis:

$$\begin{aligned} x &= (x_1, \ldots, x_d) \\ &= x_1 e^1 + \cdots + x_d e^d. \end{aligned}$$

Define

$$K[t] := \{x \in K | x_d = t\};$$

i.e., the subset of K whose elements have fixed height, or the intersection of K with a hyperplane parallel to the vector subspace spanned by $e^1, \ldots, e^{d-1}$. We can write $G(K, \Lambda)$ in terms of lattice point enumerators of convex bodies whose dimension is $d-1$; this is the point where induction could be used. Namely,

$$G(K, \Lambda) = \sum_{t \in \mathbb{Z}} G(K[t] - te^d, \Lambda^{d-1}).$$

The bodies $K[t] - te^d$ are projections of the intersections $K[t]$ on the vector subspace spanned by $e^1, \ldots, e^{d-1}$ along the lattice vector e^d. As before, Λ^{d-1} is the $\mathbb{Z}$-span of $e^1, \ldots, e^{d-1}$. Apart from $K[0]$ which is 0-symmetric, the other projections are not necessarily 0-symmetric. This is the main reason for extending Theorem 1.1 and Conjecture 1.2 to the non-symmetric case.

Next, observe that
$$\tfrac{1}{2} \mathfrak{D}(K[t] - te^d) \subset \tfrac{1}{2} \mathfrak{D} K,$$

therefore, for $1 \leqslant i \leqslant d - 1$

$$\lambda_i(K[t] - te^d, \Lambda^{d-1}) \geqslant \lambda_i(K, \Lambda),$$

which implies
$$q_i(K[t] - te^d, \Lambda^{d-1}) \leqslant q_i(K, \Lambda),$$

for $1 \leqslant i \leqslant d - 1$. Assuming that Conjecture 1.2 holds for $d - 1$, we have

$$G(K[t] - te^d, \Lambda^{d-1}) \leqslant \prod_{i=1}^{d-1} q_i(K, \Lambda),$$

for all $t \in \mathbb{Z}$. Only the factor $q_d(K, \Lambda)$ is missing; we could normally expect that the number of the nonempty "slices", $K[t]$, is less than $q_d(K, \Lambda)$. But it is not always the case that this number is less than $q_d(K, \Lambda)$.

From now on, we will write q_i instead of $q_i(K, \Lambda)$. The next step is to group all intersections whose heights are congruent modulo q_d. Doing so, the above sum becomes

$$G(K, \Lambda) = \sum_{r=0}^{q_d - 1} \sum_{t \equiv r \,(\mathrm{mod}\, q_d)} G(K[t] - te^d, \Lambda^{d-1}).$$

It suffices to prove that for each fixed r, we have

$$\sum_{t \equiv r \,(\mathrm{mod}\, q_d)} G(K[t] - te^d, \Lambda^{d-1}) \leqslant \prod_{i=1}^{d-1} q_i.$$

Of course, we could have more than one convex body in the above sum, however, the above collection of convex bodies $K[t] - te^d$, $t \equiv r \,(\mathrm{mod}\, q_d)$ satisfies some restrictive conditions, namely:

(1) $\mathfrak{D}(K[t] - te^d) \cap q_i(\Lambda^{d-1} \setminus \Lambda^{i-1}) = \varnothing$ for all $t \equiv r(\mathrm{mod}\, q_d)$ and $1 \leqslant i \leqslant d-1$.

(2) $((K[t] - te^d) - (K[t'] - t'e^d)) \cap q_d\Lambda^{d-1} = \varnothing$ for all $t, t' \equiv r(\mathrm{mod}\, q_d)$, $t \neq t'$.

The two statements above are consequences of Lemma 2.1. Indeed, for **(1)** we observe that

$$\mathfrak{D}(K[t] - te^d) \cap q_i(\Lambda^{d-1} \setminus \Lambda^{i-1}) \subset \mathfrak{D}K \cap q_i(\Lambda \setminus \Lambda^{i-1}),$$

and the latter is empty since $q_i > 2/\lambda_i$. As for **(2)**, if

$$((K[t] - te^d) - (K[t'] - t'e^d)) \cap q_d\Lambda^{d-1} \neq \varnothing,$$

then there would exist some $v \in \Lambda^{d-1}$ such that $q_d v + (t - t')e^d \in K[t] - K[t'] \subset \mathfrak{D}K$. However, since $q_d | t - t'$, and $t \neq t'$, the intersection

$$\mathfrak{D}K \cap q_d(\Lambda \setminus \Lambda^{d-1})$$

is nonempty, contradicting Lemma 2.1.

It is natural to state the following conjecture:

Conjecture 3.2. *Let $K_1, \ldots, K_n \subset \mathbb{R}^d$ be convex bodies and $\Lambda \in \mathcal{L}^d$. Also, let $e^1, \ldots, e^d$ be a basis of Λ, and denote by Λ^i the $\mathbb{Z}$-span of $0, e^1, \ldots, e^i$, and let $q_1 \geqslant q_2 \geqslant \cdots \geqslant q_{d+1}$ be positive integers satisfying*

(C1) $\mathfrak{D}K_j \cap q_i(\Lambda \setminus \Lambda^{i-1}) = \varnothing$ *for all $1 \leqslant j \leqslant n$ and $1 \leqslant i \leqslant d$.*

(C2) $(K_j - K_l) \cap q_{d+1}\Lambda = \varnothing$ *for all $1 \leqslant j, l \leqslant n$, $j \neq l$.*

Then

$$\sum_{j=1}^{n} G(K_j, \Lambda) \leqslant \prod_{i=1}^{d} q_i.$$

From the above analysis, it is clear that the above conjecture implies Conjecture 1.2 for one dimension higher. We will verify this conjecture for $d = 1, 2$, thus proving Conjecture 1.2 in all dimensions up to three, which is the main result of this paper (Theorem 1.4). A statement in support of this conjecture is that condition **(C2)** is too restrictive for the convex bodies K_j, given the fact that q_{d+1} is smaller than the rest of the q_i's. This statement simply says that no two translates of K_j and K_l, $j \neq l$, by vectors of $q_{d+1}\Lambda$ intersect. In the next section, we present a more convincing reduction of Conjecture 3.2.

Proof of Conjecture 3.2, $d = 1$. Without loss of generality, we assume that $\Lambda = \mathbb{Z}$. Let $K_j = [a_j, b_j]$, $1 \leqslant j \leqslant n$. Conditions **(C1)** and **(C2)** read

(C1) $b_j - a_j < q_1$ for all $1 \leqslant j \leqslant n$.

(C2) $(K_j - K_l) \cap q_2 \mathbb{Z} = \varnothing$ for all $1 \leqslant j, l \leqslant n$, $j \neq l$.

If $b_1 - a_1 \geqslant q_2$, then the union of K_1 with all its translates by multiples of q_2 cover all of $\mathbb{R}$, so by condition **(C2)** we must have $n = 1$, therefore

$$\sum_{j=1}^{n} G(K_j, \Lambda) = G(K_1) \leqslant q_1$$

by **(C1)**. If $b_1 - a_1 < q_2$, there is a translate of each K_j by some multiple of q_2, $2 \leqslant j \leqslant n$, that lies in $(b_1, a_1 + q_2)$, again by **(C2)**. Since they do not intersect each other by **(C2)**, we have

$$\sum_{j=1}^{n} G(K_j) \leqslant G([a_1, a_1 + q_2)) = q_2 \leqslant q_1.$$

$\square$

Proof of Conjecture 3.2, $d = 2$. Let

$$D = \dim\left(\left(\bigcup_{j=1}^{n} \mathfrak{D}K_j\right) \cap q_3 \Lambda\right).$$

We distinguish cases for D:

$\underline{D \leqslant 1}$: There exists a primitive lattice vector, say v, such that

$$\left(\bigcup_{j=1}^{n} \mathfrak{D}K_j\right) \cap q_3 \Lambda \subset \mathbb{Z}(q_3 v)$$

therefore

$$\left(\left(\bigcup_{j=1}^{n} \mathfrak{D}K_j\right) \cap q_3\left(\Lambda \setminus \mathbb{Z}v\right)\right) = \varnothing.$$

Find $w \in \Lambda$ such that v, w is a basis for Λ. Then

$$\sum_{j=1}^{n} G(K_j, \Lambda) = \sum_{r=0}^{q_3-1} \sum_{j=1}^{n} \sum_{t \equiv r \,(\mathrm{mod}\ q_3)} G(K_j[t] - tw, \mathbb{Z}v).$$

11

We will prove that the above sum is less than or equal to q_1q_3 (which is less than or equal to q_1q_2); it suffices to prove that

$$\sum_{j=1}^{n} \sum_{t \equiv r \pmod{q_3}} G(K_j[t] - tw, \mathbb{Z}v) \leqslant q_1,$$

for a fixed r, where the notation $K_j[t]$ refers to the basis v, w. Naturally, we identify $\mathbb{R}v$ with $\mathbb{R}$, so the collection of all sets $K_{j,t} := K_j[t] - tw$ (of which only a finite number are nonempty) is a collection of compact intervals on $\mathbb{R}$. We have

$$\mathfrak{D}K_{j,t} \cap q_1(\mathbb{Z}v \setminus \{0\}) \subset \mathfrak{D}K_j \cap q_1(\Lambda \setminus \{0\}),$$

which is empty by assumption for all j, so condition **(C1)** of Conjecture 3.2 is satisfied, for the family of convex bodies $K_{j,t}$, the lattice $\mathbb{Z}v$ and the positive integers $q_1 \geqslant q_2 > 0$. Furthermore, when $t \neq t'$, if the intersection

$$(K_{j,t} - K_{j,t'}) \cap q_3(\mathbb{Z}v)$$

is nonempty, then there exists $u \in \mathbb{Z}v$ such that

$$q_3 u + (t - t')w \in K_j[t] - K_j[t'] \subset \mathfrak{D}K_j,$$

implying

$$\mathfrak{D}K_j \cap q_3(\Lambda \setminus \mathbb{Z}v) \neq \varnothing,$$

which provides a contradiction, since $D \leqslant 1$. If $i \neq j$, and if the intersection

$$(K_{i,t} - K_{j,t'}) \cap q_3(\mathbb{Z}v)$$

is nonempty, then there is $u \in \mathbb{Z}v$ such that

$$q_3 u + (t - t')w \in K_i[t] - K_j[t'] \subset K_i - K_j,$$

implying

$$(K_i - K_j) \cap q_3\Lambda \neq \varnothing,$$

which provides another contradiction. Thus, condition **(C2)** is satisfied, and since the 1-dimensional case is true, we have

$$\sum_{j=1}^{n} \sum_{t \equiv r \pmod{q_3}} G(K_j[t] - tw, \mathbb{Z}v) \leqslant q_1,$$

as desired.

$\underline{D = 2}$: This means that there are two primitive, linearly independent vectors of Λ in $\bigcup \mathfrak{D}K_j$, say v, w. We may assume that $q_3 v \in \mathfrak{D}K_i$ and $q_3 w \in \mathfrak{D}K_j$, for some indices i, j. We must show that $i = j$ (if $n = 1$, this is vacuously true, so we assume $n \geqslant 2$). We have

$$K_i \cap (K_i - q_3 v) \neq \varnothing,$$

so we pick an element x from this intersection. Hence, $x, x + q_3 v \in K_i$ and,

$$K_j \cap (K_j + q_3 w) \neq \varnothing,$$

from which we pick an element y, hence $y, y - q_3 w \in K_j$. Let $\widetilde{\Lambda} = \mathbb{Z}v \oplus \mathbb{Z}w$, and consider the fundamental parallelogram of $q_3 \widetilde{\Lambda}$ with vertices $x, x + q_3 v, x + q_3 w, x + q_3(v + w)$, say $\mathcal{P}$. Since $\mathcal{P}$ is a fundamental parallelogram, there is a translate of y by $q_3 \widetilde{\Lambda}$ (and hence by $q_3 \Lambda$ as well) in $\mathcal{P}$. Without loss of generality, we may assume that $y \in \mathcal{P}$ (if we translate any K_i by an element of $q_3 \Lambda$, conditions **(C1)** and **(C2)** still hold). Assume that $y = x + \alpha q_3 v + \beta q_3 w$, where $0 \leqslant \alpha, \beta < 1$. Note that the element

$$y - \beta q_3 w = x + \alpha q_3 v$$

belongs to both $\mathrm{conv}(x, x + q_3 v)$ and $\mathrm{conv}(y, y - q_3 w)$, i.e., the intersection $K_i \cap K_j$ is nonempty. This contradicts condition **(C2)** if $i \neq j$, so we must have $i = j$.

Without loss of generality, assume that $i = 1$, that is, $v, w \in \mathfrak{D}K_1$. Choose v, w so that the index $[\Lambda : \widetilde{\Lambda}]$ is minimal. Assume that $[\Lambda : \widetilde{\Lambda}] > 1$. Then there is a point $q_3 u \in q_3 \Lambda$, such that $q_3 u = \mu q_3 v + \nu q_3 w$, with $0 < \mu, \nu < 1$. It is not hard to see that any point in $\mathbb{R}^2$ is congruent modulo $q_3 \widetilde{\Lambda}$ to some point in the parallelogram $\mathrm{conv}(\pm q_3 v, \pm q_3 w)$. So, we may assume that $q_3 u \in \mathrm{conv}(\pm q_3 v, \pm q_3 w)$, and by convexity we also have $u \in \mathfrak{D}K_1$. Since $0 < \mu, \nu < 1$, the lattice generated by v, u has strictly smaller index in Λ than $\widetilde{\Lambda}$, contradicting the minimality assumption, therefore we must have $\Lambda = \widetilde{\Lambda}$. By Lemma 3.3 below, there is some $x \in K_1$ such that the boundary of the fundamental parallelogram of $q_3 \Lambda$ with vertices $x, x + q_3 v, x + q_3 w, x + q_3(v+w)$ (call it $\mathcal{P}$ again) is a subset of $K_1 + q_3 \Lambda$. By condition **(C2)**, all K_j, $j \neq 1$ avoid $K_1 + q_3 \Lambda$, and hence the boundary of $\mathcal{P}$. Since one translate of K_j by $q_3 \Lambda$ intersects $\mathcal{P}$, as it is a fundamental parallelogram of $q_3 \Lambda$, *this translate must lie inside of* $\mathcal{P}$, by convexity since the boundary of $\mathcal{P}$ splits the plane $\mathbb{R}^2$ into two disjoint regions. Thus, all K_j for $j > 1$ satisfy the additional property

$$\mathfrak{D}K_j \cap q_3(\Lambda \setminus \{0\}) = \varnothing.$$

Now, let

$$S = \Big(\bigcup_{j>1} K_j \Big) \cap \Lambda.$$

From the previous identity we get

$$\mathfrak{D}S \cap q_3(\Lambda \setminus \{0\}) = \varnothing,$$

and condition **(C2)** implies

$$(K_1 - S) \cap q_3 \Lambda = \varnothing,$$

13

Therefore, K_1 and S satisfy the conditions of Lemma 2.4, for $r = q_3$, and $d = 2$. So, there is a finite set $S' \subset \Lambda$, obtained from S by translating each element of S with an element of $q_3\Lambda$, satisfying

$$\mathfrak{D}S' \cap q_2(\Lambda \setminus \{0\}) = \varnothing$$

and

$$(K_1 - S') \cap q_2\Lambda = \varnothing,$$

since $q_2 \geqslant q_3$. Then,

$$\sum_{j=1}^{n} G(K_j, \Lambda) = G(K_1, \Lambda) + \#(S') =$$

$$= \sum_{r=0}^{q_2-1} \sum_{t \equiv r \, (\mathrm{mod} \, q_2)} G(K_1[t] - te^2, \mathbb{Z}e^1) + \sum_{r=0}^{q_2-1} \sum_{t \equiv r \, (\mathrm{mod} \, q_2)} \#(S'[t] - te^2).$$

Here, the notation $K[t]$ refers to the original basis e^1, e^2. It suffices to prove that for fixed r,

$$\sum_{t \equiv r \, (\mathrm{mod} \, q_2)} G(K_{1,t}, \mathbb{Z}e^1) + \sum_{t \equiv r \, (\mathrm{mod} \, q_2)} \#(S'[t] - te^2) \leqslant q_1.$$

We identify $\mathbb{R}e^1$ with $\mathbb{R}$. Hence, we have a finite collection of nonempty compact intervals, $K_{1,t}$, and some lattice points which come from $S'[t] - te^2$. Assume that $S'[t] - te^2 = \{m_1e^1, \ldots, m_ke^1\}$, where $m_1, m_2, \ldots, m_k$ are distinct integers. Again, we have

$$\mathfrak{D}K_{1,t} \cap q_1(\mathbb{Z}e^1 \setminus \{0\}) \subset \mathfrak{D}K_j \cap q_1(\Lambda \setminus \{0\}) = \varnothing,$$

so condition **(C1)** is satisfied for the intervals $K_{1,t}$ and $m_1e^1, \ldots, m_ke^1$ (it is trivial for a point). If the intersection

$$(K_{1,t} - K_{1,t'}) \cap q_2(\mathbb{Z}e^1)$$

is nonempty for some $t \neq t'$, then there is some $u \in \mathbb{Z}e^1$, such that

$$q_2u + (t - t')e^2 \in K_1[t] - K_1[t'] \subset \mathfrak{D}K_1,$$

which implies (since $q_2|t - t'$)

$$\mathfrak{D}K_1 \cap q_2(\Lambda \setminus \Lambda^1) \neq \varnothing,$$

contradicting condition **(C1)**. Furthermore,

$$(K_{1,t} - \{m_ie^1\}) \cap q_2(\mathbb{Z}e^1) \subset (K_1 - S') \cap q_2\Lambda = \varnothing,$$

and for $i \neq j$,

$$\{m_ie^1\} - \{m_je^1\} \cap q_2(\mathbb{Z}e^1) \subset \mathfrak{D}S' \cap q_2\Lambda = \varnothing,$$

14

so condition **(C2)** holds as well for the intervals $K_{1,t}$ and the points $m_1 e^1, m_2 e^1,$ $\ldots, m_k e^1$, with respect to the lattice $\mathbb{Z}e^1$ and the integers $q_1 \geqslant q_2$, hence

$$\sum_{t \equiv r \,(\mathrm{mod}\ q_2)} G(K_{1,t}, \mathbb{Z}e^1) + \sum_{t \equiv r \,(\mathrm{mod}\ q_2)} \#(S'[t] - te^1) \leqslant q_1,$$

as desired, completing the proof. $\qquad\square$

This implies that Conjecture 1.2 is true for $d \leqslant 3$. We observe that in order to prove Conjecture 3.2 for $d = 2$, we used the result for $d = 1$. This is exactly the purpose of stating a stronger conjecture than inequality (4); we might be able to use induction on the dimension, something that did not seem possible in this inequality. However, when $d > 2$, we need something more than just induction. For $d = 2$, Lemma 2.4 was used, because when $D = 2$, all but one of the K_j must be confined in a fundamental parallelogram. This is not true in higher dimensions in general; perhaps we need a stronger version of Lemma 2.4.

We conclude this section with the following lemma, that was used for the proof of Conjecture 3.2, case $d = 2$:

Lemma 3.3. *Let $K \in \mathcal{K}^2$, and $v^1, v^2 \in \mathbb{R}^2$ two linearly independent vectors such that the intersections $K \cap (K + v^1)$ and $K \cap (K + v^2)$ are nonempty. Then there exists a point $x \in K$ such that the boundary of the parallelogram with vertices $x, x + v^1, x + v^2, x + v^1 + v^2$ is contained in $K + \Lambda$, where Λ is the lattice generated by v^1, v^2.*

Proof. From the hypothesis, there is a line parallel to v^1 contained in $K + \mathbb{Z}v^1$, and similarly, a line parallel to v^2 contained in $K + \mathbb{Z}v^2$. Let y be the point of intersection; then the lines parallel to v^1, v^2, passing through y are contained in $K + \Lambda$. The same happens with any lattice translate of y. Pick one such translate that belongs to K, say x. Considering the translates $x + v^1$, $x + v^2$, $x + v^1 + v^2$, we deduce that the union of lines parallel to v^1, v^2 and passing through x, $x + v^1$, $x + v^2$, $x + v^1 + v^2$ is a subset of $K + \Lambda$. It is clear that this union of lines contains the boundary of the fundamental parallelogram with vertices $x, x + v^1, x + v^2, x + v^1 + v^2$, as desired. $\qquad\square$

4 Reductions of Conjecture 1.2

Two reductions of Conjecture 1.2 will be given; the first one is a reduction of Conjecture 3.2, while the second one is a certain monotonicity property for the discrete measure that is satisfied by the Lebesgue measure.

4.1 A simultaneous translation problem

The main technique in the proof of the two-dimensional case of Conjecture 3.2 was using a projection onto a certain hyperplane and then applying induction, i.e. the result in the one-dimensional case. Can we do this in the general case? In particular, what happens when we consider the projections $K_{j,t} = K_j[t] - te^d$ for $1 \leqslant j \leqslant n$, $t \equiv r \pmod{q_d}$, for a fixed r? Do they satisfy conditions **(C1)**, **(C2)** of the conjecture, for the lattice Λ^{d-1}, the basis $e^1, \ldots, e^{d-1}$ and the integers $q_1 \geqslant \cdots \geqslant q_d$? Not in general. They do, however, in the special case when q_{d+1} divides q_d. If so, we can replace **(C2)** with the weaker condition

$$(K_j - K_l) \cap q_d \Lambda = \varnothing,$$

simply because $q_d \Lambda$ is a sublattice of $q_{d+1} \Lambda$. Indeed,

$$\mathfrak{D} K_{j,t} \cap q_i (\Lambda^{d-1} \setminus \Lambda^{i-1}) \subset \mathfrak{D} K_j \cap q_i (\Lambda \setminus \Lambda^{i-1}) = \varnothing.$$

For $t \neq t'$, $t \equiv t' \pmod{q_d}$, we have

$$
\begin{aligned}
(K_{j,t} - K_{j,t'}) \cap q_d \Lambda^{d-1} &= (K_j[t] - K_j[t']) \cap (q_d \Lambda^{d-1} + (t - t')e^d) \\
&\subset \mathfrak{D} K_j \cap q_d (\Lambda \setminus \Lambda^{d-1}) = \varnothing,
\end{aligned}
$$

and for $j \neq l$, $t \equiv t' \pmod{q_d}$, we have

$$
\begin{aligned}
(K_{j,t} - K_{l,t'}) \cap q_d \Lambda^{d-1} &= (K_j[t] - K_l[t']) \cap (q_d \Lambda^{d-1} + (t - t')e^d) \\
&\subset (K_j - K_l) \cap q_d \Lambda = \varnothing.
\end{aligned}
$$

Hence, as long as q_{d+1} divides q_d, we can apply the induction step, using the projection technique. Given the result of Conjecture 3.2 for $d = 2$, we establish the following:

Theorem 4.1. *Let $K_1, \ldots, K_n \subset \mathbb{R}^d$ be convex bodies and $\Lambda \in \mathcal{L}^d$. Also, let $e^1, \ldots, e^d$ be a basis of Λ, and denote by Λ^i the $\mathbb{Z}$-span of $0, e^1, \ldots, e^i$, and let $q_1 \geqslant q_2 \geqslant \cdots \geqslant q_{d+1}$ be positive integers satisfying*

(C1) $\mathfrak{D} K_j \cap q_i (\Lambda \setminus \Lambda^{i-1}) = \varnothing$ *for all $1 \leqslant j \leqslant n$ and $1 \leqslant i \leqslant d$.*

(C2) $(K_j - K_l) \cap q_{d+1} \Lambda = \varnothing$ *for all $1 \leqslant j, l \leqslant n$, $j \neq l$.*

(C3) $q_{d+1} | q_d | \cdots | q_3$.

Then

$$\sum_{j=1}^{n} G(K_j, \Lambda) \leqslant \prod_{i=1}^{d} q_i.$$

16

Our next objective is to get rid of the successive divisibility property, **(C3)**. What happens when q_{d+1} does not divide q_d? We cannot use the same technique anymore, as the projected convex bodies will not always satisfy condition **(C2)**. Can we somehow replace q_{d+1} by q_d in condition **(C2)**? We might need to translate the given convex bodies, but we should translate them by a lattice vector, so that the lattice point enumerator remains invariant. We pose the following:

Problem. *Let $K_1, K_2, \ldots, K_n$ be convex bodies in $\mathbb{R}^d$, Λ a lattice, and r be a positive integer, such that the following property holds:*

$$(K_i - K_j) \cap r\Lambda = \varnothing,$$

for $i \neq j$, $1 \leqslant i, j \leqslant n$. Given a positive integer $t \geqslant r$, is it true that we can translate each K_i by a lattice vector, thus obtaining the convex bodies $K_1', \ldots, K_n'$, so that the following property holds for $i \neq j$, $1 \leqslant i, j \leqslant n$

$$(K_i' - K_j') \cap t\Lambda = \varnothing?$$

It is obvious from the analysis at the beginning of the subsection that if this problem is answered in the affirmative, then it implies Conjecture 3.2, and consequently inequality (4) for all dimensions. It should be noted that Lemma 2.4 is a special case of this problem and the case $n = 2$ is covered as a simple consequence of Lemma 2.3. Lastly, the one-dimensional case is trivial, or the case where r divides t. In this case, we do not have to translate the convex bodies at all.

Finally, we state the following corollary to Theorem 4.1, which is a slight improvement of Theorem 3.1:

Corollary 4.2. *Let $K \in \mathcal{K}^d$, $\Lambda \in \mathcal{L}^d$, $q_i = q_i(K, \Lambda)$. Let $n_1, n_2, \ldots, n_d$ be a decreasing sequence of positive integers such that*

- $q_i \leqslant n_i$, *for* $1 \leqslant i \leqslant d$.

- $n_d | n_{d-1} | \cdots | n_3$.

Then

$$G(K, \Lambda) \leqslant \prod_{i=1}^{d} n_i.$$

Proof. Let $e^i = e^i(K, \Lambda)$, $\Lambda^i = \Lambda^i(K)$. From the analysis at the beginning of Section 3.2, it is clear that the slices $K[t] - te^d$, for $t \equiv r \pmod{n_d}$, and numbers $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_d$ satisfy conditions **(C1)**, **(C2)**, and **(C3)** of Theorem 4.1, whence the desired inequality. $\qquad\square$

In particular, Conjecture 1.2 is verified when $q_d | q_{d-1} | \cdots | q_3$. This shows that the verification of Conjecture 3.2 for $d = 2$ implies that we need not include the first two terms in this successive divisibility property. And it is clear, that if Conjecture 3.2 is proven for, say $d = s$, then Conjecture 1.2 is verified when $q_d | q_{d-1} | \cdots | q_{s+1}$.

4.2 The discrete monotonicity property

In every proof of Minkowski's second theorem, a monotonicity property for the Lebesgue measure is proven in one form or another. For example, Bambah [BWZ] proves that

$$\mathrm{vol}(tK/L) \geqslant t^{d-i} \, \mathrm{vol}(K/L),$$

where $t \geqslant 1$, $K \in \mathcal{K}^d$, L is a discrete subgroup of $\mathbb{R}^d$ whose rank is equal to i, and $\mathrm{vol}(K/L)$ is the Lebesgue measure of K taken modulo L; i.e., identifying two points of K that are congruent modulo K. The above is equivalent to the assertion that

$$\frac{\mathrm{vol}(K/rL)}{r^i}$$

is decreasing in $r > 0$. This so-called continuous monotonicity property, and holds for all convex bodies K and discrete subgroups L of $\mathbb{R}^d$, unconditionally.

We now state the discrete monotonicity property; we first replace the d-dimensional Lebesgue measure by a discrete measure corresponding to a lattice Λ, so that the measure of a given set A is simply the cardinality of $A \cap \Lambda$. Instead of discrete subgroups of $\mathbb{R}^d$ we consider subgroups of Λ.

Definition. Let $K \in \mathcal{K}^d$, $\Lambda \in \mathcal{L}^d$. We say that K satisfies the **discrete monotonicity property** with respect to Λ, if for any subgroup of Λ, say $\widetilde{\Lambda}$, the following sequence is decreasing in $r > 0$, $r \in \mathbb{Z}$:

$$\frac{D_\Lambda(K, r\widetilde{\Lambda})}{r^i},$$

where i is the rank of $\widetilde{\Lambda}$.

Here $D_\Lambda(K, r\widetilde\Lambda)$ denotes the cardinality of the set $K \cap \Lambda$ taken modulo $r\widetilde\Lambda$. In this setting, we require that r be an integer, because we need $r\widetilde\Lambda$ to be a subset of Λ. It is clear that $D_\Lambda(K, r\widetilde\Lambda)$ is the corresponding quantity of $\mathrm{vol}(K/r\Lambda)$ above. Next we prove the following helpful lemma:

Lemma 4.3. *Let $K \in \mathcal{K}^d$, $\Lambda \in \mathcal{L}^d$, $a^1, \dots, a^d$ d linearly independent vectors of Λ and*
$$L^i := \mathbb{Z}a^1 \oplus \cdots \oplus \mathbb{Z}a^i.$$
Assume that $\mathfrak{D}K \cap (L^d \setminus L^i) = \varnothing$. Then
$$D_\Lambda(K, L^d) = D_\Lambda(K, L^{d-1}) = \cdots = D_\Lambda(K, L^i).$$

Proof. The hypothesis simply implies that if two points $x, y \in K \cap \Lambda$ are congruent modulo L^d, then they must be congruent modulo L^i, and consequently congruent modulo L^j, for $i \leqslant j \leqslant d$. The lemma then follows from the definition of $D_\Lambda(K, L^i)$. $\qquad\square$

Theorem 4.4. *Assume that $K \in \mathcal{K}^d$ satisfies the discrete monotonicity property with respect to $\Lambda \in \mathcal{L}^d$. Then*
$$G(K, \Lambda) \leqslant \prod_{i=1}^{d} q_i(K, \Lambda).$$

Proof. Let $\Lambda^i = \Lambda^i(K)$, for $0 \leqslant i \leqslant d$, and $q_i = q_i(K, \Lambda)$. By Lemma 2.1, we have $\mathfrak{D}K \cap q_i(\Lambda \setminus \Lambda^{i-1})$ for all i, and by the virtue of Lemma 4.3 we have the following series of equalities/inequalities:

$$
\begin{aligned}
q_d^d \;\geqslant\;& D_\Lambda(K, q_d\Lambda) = D_\Lambda(K, q_d\Lambda^{d-1}) \\[4pt]
\geqslant\;& \left(\frac{q_d}{q_{d-1}}\right)^{d-1} D_\Lambda(K, q_{d-1}\Lambda^{d-1}) = \left(\frac{q_d}{q_{d-1}}\right)^{d-1} D_\Lambda(K, q_{d-1}\Lambda^{d-2}) \\[4pt]
&\;\;\vdots \\[4pt]
\geqslant\;& \left(\frac{q_d}{q_{d-1}}\right)^{d-1}\left(\frac{q_{d-1}}{q_{d-2}}\right)^{d-2} \cdots \frac{q_2}{q_1} D_\Lambda(K, q_1\Lambda^1) \\[4pt]
=\;& \left(\frac{q_d}{q_{d-1}}\right)^{d-1}\left(\frac{q_{d-1}}{q_{d-2}}\right)^{d-2} \cdots \frac{q_2}{q_1} D_\Lambda(K, q_1\Lambda^0) \\[4pt]
=\;& \left(\frac{q_d}{q_{d-1}}\right)^{d-1}\left(\frac{q_{d-1}}{q_{d-2}}\right)^{d-2} \cdots \frac{q_2}{q_1} G(K, \Lambda)
\end{aligned}
$$

whence
$$G(K, \Lambda) \leqslant \prod_{i=1}^{d} q_i.$$

$\qquad\square$

The continuous monotonicity property is proven using the homogeneity of the Lebesgue measure. This property is not valid for the discrete measure, so we expect that it might be very difficult to prove the discrete monotonicity property for all convex bodies and lattices.

References

[BHW]   Ulrich Betke, Martin Henk, and Jörg Wills. "Successive-minima-type inequalities." *Discrete Comput. Geom.*, **9**(2):165–175, 1993.

[BWZ]   R. P. Bambah, A. C. Woods, and H. Zassenhaus. "Three proofs of Minkowski's second inequality in the geometry of numbers." *J. Aust. Math. Soc.*, **5**:453–462, 1965.

[GL]   P. M. Gruber and C. G. Lekkerkerker. *Geometry of numbers*, volume 37 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, second edition, 1987.

[H]   Martin Henk. "Successive minima and lattice points." *Rend. Circ. Mat. Palermo (2) Suppl.*, **70**(part I):377–384, 2002.

[M]   Romanos-Diogenes Malikiosis. "An optimization problem related to Minkowski's successive minima." *Discrete Comput. Geom.*, **43**:784–797, 2010.