University of California

Los Angeles

Discrete and Other Analogues of Minkowski's Theorems on Successive Minima

A dissertation submitted in partial satisfaction

of the requirements for the degree

Doctor of Philosophy in Mathematics

by

Romanos Diogenes Malikiosis

2010

The dissertation of Romanos Diogenes Malikiosis is approved.

Richard Saouma

William Duke

Terence Tao

Don Blasius, Committee Chair

University of California, Los Angeles

2010

Table of Contents

Acknowledgments

This thesis, like any other, has only one name on it but could never have been completed without help from many people. I need to thank my whole family for their support, especially my wife Katerina, who always stood beside me and supported all my choices. I would also like to thank Martin Henk and Jörg Wills for taking the time to respond to my e-mail queries, as well as introducing me to new problems and already established facts in the field of the geometry of numbers. Last but not least, I wish to express my deep gratitude for my adviser Don Blasius. Our endless discussions on number theory during the last three years provided a very sound background for my research.

Vita

1980	Born, Alexandroupolis, Greece.
2002	B.A. (Mathematics), Aristoteleion University of Thessaloniki.
2005	M.Sc. (Mathematics), Aristoteleion University of Thessaloniki.
2008	M.A. Mathematics, UCLA, Los Angeles, California.
2006–2009	Teaching Assistant, Mathematics Department, UCLA.
2010	Research Assistant, Mathematics Department, UCLA.

Presentations

Malikiosis, Romanos-Diogenes. "Minkowski-type inequalities for the lattice point enumerator." Presented at the Convex and Discrete Geometry seminar at the Otto-von-Guericke University, Magdeburg, Germany, May 2009.

—. "A discrete problem in the geometry of numbers." Presented at the seminar of Theoretical Computer Science and Discrete Mathematics at the Aristoteleion University of Thessaloniki, Greece, June 2009.

Abstract of the Dissertation

Discrete and Other Analogues of Minkowski's Theorems on Successive Minima

by

Romanos Diogenes Malikiosis

Doctor of Philosophy in Mathematics

University of California, Los Angeles, 2010

Professor Don Blasius, Chair

We will discuss about certain generalizations of Minkowski's theorems on successive minima. The discrete generalizations are statements where the volume is replaced by the lattice point enumerator. Furthermore, we will present a notable generalization by Davenport, and provide some independent proofs of already known results related to this generalization.

CHAPTER 1

Introduction

Minkowski's two theorems on successive minima were the beginning of the subfield of mathematics called "Geometry of Numbers". Currently, the term "Geometry of Numbers" is considered outdated, as it is generally regarded as a collection of problems relating convex bodies with lattices in a finite dimensional real vector space.

Minkowski's first theorem was proven in 1896 [Min96], and was used by Minkowski himself in order to obtain a lower bound on the absolute value of the discriminant of a number field. This implies that no nontrivial extension of $\mathbb{Q}$ is unramified at all primes, and is a key result towards the proof of the Kronecker-Weber theorem, which states that any abelian extension of $\mathbb{Q}$ is contained in a cyclotomic extension of $\mathbb{Q}$. Furthermore, the Hermite-Minkowski theorem, asserting that there are finitely many number fields of bounded discrriminant, is also a consequence of Minkowski's first theorem on successive minima. This implies further that there are finitely many extensions of $\mathbb{Q}$ of bounded degree, unramified outside a finite set of primes. Minkowski's first theorem was also used by Faltings in proving Mordell's conjecture regarding rational points on algebraic curves.

Some definitions and notations are in order:

Definition. A subset K of $\subset \mathbb{R}^d$ is called a convex body if it is convex, compact and has nonempty interior. The set of all convex bodies in $\mathbb{R}^d$ will be denoted as $\mathcal{K}^d$, and the subset of all its 0-symmetric elements will be denoted as $\mathcal{K}_0^d$ (a subset S of $\mathbb{R}^d$ will be called 0-symmetric when it is symmetric with respect to the origin, i.e., $x \in S$ if and only if $-x \in S$).

Definition. A subset Λ of $\mathbb{R}^d$ is called a lattice if it forms the $\mathbb{Z}$-span of a basis for $\mathbb{R}^d$ (alternatively, we could require that Λ is a discrete subgroup of $\mathbb{R}^d$, having full rank). If $e^1, e^2, \ldots, e^d$ is such a basis, then the absolute value of the determinant of the matrix having as columns these vectors, is defined to be the determinant of Λ, and is simply denoted as $d(\Lambda)$. This is also equal to the volume (i.e., d-dimensional Lebesgue measure) of a fundamental parallelepiped of Λ. It should be noted that $d(\Lambda)$ is independent of the choice of basis for Λ.

Minkowski's first theorem first appeared in this form, where $K \in \mathcal{K}_0^d$ and $\Lambda \in \mathcal{L}^d$:

If $\operatorname{vol}(K) \geqslant 2^d \cdot d(\Lambda)$, *then* K *contains a nonzero lattice point of* Λ.

A proof is as follows, let P be a fundamental parallelepiped of Λ. Then

$$\operatorname{vol}(\tfrac{1}{2}K) = \sum_{v \in \Lambda} \operatorname{vol}(\tfrac{1}{2}K \cap (v + P))$$

or equivalently

$$\operatorname{vol}(\tfrac{1}{2}K) = \sum_{v \in \Lambda} \operatorname{vol}((\tfrac{1}{2}K + v) \cap P). \tag{1.1}$$

By hypothesis,

$$\operatorname{vol}(P) = d(\Lambda) \leqslant 2^{-d} \operatorname{vol}(K) = \operatorname{vol}(\tfrac{1}{2}K),$$

so if the family of convex bodies $\frac{1}{2}K + v$, $v \in \Lambda$ is pairwise disjoint, we would have $\operatorname{vol}(\frac{1}{2}K) \leqslant d(\Lambda)$ by (1.1), thus $\operatorname{vol}(\frac{1}{2}K) = \operatorname{vol}(P)$. The complement of the union $\bigcup_{v \in \Lambda}(\frac{1}{2}K + v)$ is open and invariant under translation by a lattice vector of Λ. The intersection of this complement with P has measure zero since $\mathrm{vol}(\frac{1}{2}K) = \mathrm{vol}(P)$, or by (1.1)

$$\sum_{v\in\Lambda} \mathrm{vol}((\tfrac{1}{2}K + v) \cap P) = \mathrm{vol}(P).$$

Therefore, this complement has measure zero, and since it is open, it must be empty, implying that $\frac{1}{2}K$ and its translates by Λ cover the entire space. However, this also implies that the distance between the closed sets $\frac{1}{2}K$ and $\bigcup_{v\in\Lambda, v\neq 0}(\frac{1}{2}K + v)$ is zero, and since both sets are closed they must intersect one another, yielding a contradiction. Thus, the family $\frac{1}{2}K + v$, $v \in \Lambda$ is not pairwise disjoint, so there exist $v, w \in \Lambda$, $v \neq w$, such that $\frac{1}{2}K + v$ and $\frac{1}{2}K + w$ have nonempty intersection. Say that $x, y \in \frac{1}{2}K$ satisfy $x + v = y + w$. Then $x - y = w - v \in \Lambda \setminus \{0\}$, and by convexity and 0-symmetry of K, we have $x - y \in K$. So, finally, the intersection of K by Λ contains a nontrivial point.

Minkowski introduced the notion of the successive minima of $K \in \mathcal{K}_0^d$ with respect to a lattice Λ:

Definition. The ith successive minimum of $K \in \mathcal{K}_0^d$ with respect to $\Lambda \in \mathcal{L}^d$, denoted by $\lambda_i = \lambda_i(K, \Lambda)$, is the least positive real number λ such that the dilate λK contains at least i linearly independent lattice points of Λ.

In particular, $\lambda_1 K$ is the smallest dilate of K that contains a nontrivial lattice point of Λ. Thus, Minkowski's first theorem can be reformulated as follows:

Theorem 1.1. *Let $K \in \mathcal{K}_0^d$ and $\Lambda \in \mathcal{L}^d$. Then*

$$\mathrm{vol}(K) \leqslant \left(\frac{2}{\lambda_1(K, \Lambda)}\right)^d d(\Lambda). \tag{1.2}$$

Proof. Let λ be an arbitrary positive real, satisfying $\lambda < \lambda_1(K, \Lambda)$. If

$$\mathrm{vol}(\lambda K) \geqslant 2^d d(\Lambda),$$

then λK contains a nontrivial point of Λ, contradicting the definition of $\lambda_1(K,\Lambda)$. Therefore,

$$\mathrm{vol}(\lambda K) < 2^d d(\Lambda)$$

for all $\lambda < \lambda_1$, yielding the desired inequality. $\square$

Another obvious property of the successive minima is the following set of inequalities

$$0 < \lambda_1(K,\Lambda) \leqslant \lambda_1(K,\Lambda) \leqslant \cdots \leqslant \lambda_d(K,\Lambda) < +\infty.$$

Minkowski's second theorem provides a stronger inequality than (1.2).

Theorem 1.2. *Let $K \in \mathcal{K}_0^d$ and $\Lambda \in \mathcal{L}^d$. Then*

$$\frac{1}{d!}\prod_{i=1}^{d}\frac{2}{\lambda_i(K,\Lambda)} \leqslant \frac{\mathrm{vol}(K)}{d(\Lambda)} \leqslant \prod_{i=1}^{d}\frac{2}{\lambda_i(K,\Lambda)}. \tag{1.3}$$

Apart from those mentioned in the beginning of this chapter, Theorem 1.1 has many deep applications, especially in number theory. In particular, Theorem 1.1 is very useful in the theory of quadratic forms, as well as Diophantine approximation. It usually serves as an existence theorem; it provides the existence of solutions of certain Diophantine equations satisfying certain properties, that define a centrally symmetric convex body.

Theorem 1.2 does not share this variety of applications, but nevertheless there have seen many attempts to strengthen inequality (1.3) or generalize it to other settings. In particular, there are versions of Theorem 1.2 in the discrete setting (where the volume is replaced by the lattice point enumerator) and the adelic setting, where instead of finite dimensional real vector spaces we deal with convex bodies in adelic fibres. One notable application of a discrete analogue of Theorem 1.2 was used by Gaudron [Gau09] in the adelic setting to prove an adelic analogue of Siegel's lemma.

In Chapter 2 we present the attempts for stating and proving discrete analogues of Minkowski's two theorems, as well as the author's contributions.

Chapter 3 will include a notable generalization by Davenport, and Appendix A will provide a proof of Theorem 1.2 motivated by the ideas of Chapter 2.

CHAPTER 2

Discrete analogues

2.1 A conjecture by Betke, Henk, and Wills

In 1993, Betke, Henk, and Wills [BHW93] attempted to establish similar results for the lattice point enumerator, instead of the volume of a convex body.

Definition. Let $K \in \mathcal{K}^d$ and $\Lambda \in \mathcal{L}^d$. The lattice point enumerator of K with respect to Λ is simply the cardinality of the intersection $K \cap \Lambda$, and is denoted as $G(K, \Lambda)$. When Λ is the standard lattice $\mathbb{Z}^d$, we will simply write $G(K)$.

$G(\lambda K, \Lambda)$ approximates $\mathrm{vol}(\lambda K)/d(\Lambda)$ as λ tends to infinity, but for small λ, there is no good relation between these two quantities. Hence it is interesting to see whether similar bounds exist for $G(K, \Lambda)$. For Minkowski's first theorem, Betke, Henk, and Wills were successful; they proved that such a bound exists for $G(K, \Lambda)$.

Theorem 2.1. *Let $K \in \mathcal{K}_0^d$ and $\Lambda \in \mathcal{L}^d$. Then*

$$G(K, \Lambda) \leqslant \left\lfloor \frac{2}{\lambda_1(K, \Lambda)} + 1 \right\rfloor^d . \tag{2.1}$$

Proof. Let

$$q_1 = \left\lfloor \frac{2}{\lambda_1(K, \Lambda)} + 1 \right\rfloor .$$

It suffices to show that all lattice points of K are pairwise incongruent modulo q_1. Assuming otherwise, there should exist two such points, say $x, y \in K \cap \Lambda$,

$x \neq y$, congruent modulo q_1. Then the point $\frac{1}{q_1}(x-y)$ would be a lattice point of Λ. Furthermore, by 0-symmetry and convexity of K, and the fact that $2/q_1 < \lambda_1(K,\Lambda)$ we have

$$\frac{1}{q_1}(x-y) = \frac{1}{2}\left(\frac{2}{q_1}x\right) + \frac{1}{2}\left(-\frac{2}{q_1}y\right) \in \frac{2}{q_1}K \subset \operatorname{int}(\lambda_1(K,\Lambda)K),$$

which contradicts the definition of $\lambda_1(K,\Lambda)$. So, our initial assertion is true, and we obtain 2.1. □

For Minkowski's second theorem, they proposed a conjecture for the discrete case, which they verified up to the planar case.

Conjecture 2.2. *Let $K \in \mathcal{K}_0^d$ and $\Lambda \in \mathcal{L}^d$. Then*

$$G(K,\Lambda) \leqslant \prod_{i=1}^{d} \left\lfloor \frac{2}{\lambda_i(K,\Lambda)} + 1 \right\rfloor. \tag{2.2}$$

It should be noted that the above statements are stronger than the corresponding theorems of Minkowski, due to a simple argument involving the Riemann integral. Indeed, using just the definition we would have

$$\frac{\operatorname{vol}(K)}{d(\Lambda)} = \lim_{r\to 0} r^d G(K, r\Lambda) \leqslant \lim_{r\to 0} \prod_{i=0}^{d} r\left\lfloor \frac{2}{\lambda_i(K,r\Lambda)} + 1 \right\rfloor = \prod_{i=1}^{d} \frac{2}{\lambda_i(K,\Lambda)}.$$

In the same paper, Betke, Henk, and Wills, proved a weaker inequality for $G(K,\Lambda)$, namely

$$G(K,\Lambda) \leqslant \prod_{i=1}^{d} \left(\frac{2i}{\lambda_i(K,\Lambda)} + 1 \right), \tag{2.3}$$

so roughly

$$G(K,\Lambda) = O(d!) \prod_{i=1}^{d} \left\lfloor \frac{2}{\lambda_i(K,\Lambda)} + 1 \right\rfloor,$$

which is inequality (2.2) with an additional factor, which is roughly equal to $d!$. Later, in 2002, Henk [Hen02] managed to improve inequality (2.3) by replacing the factorial by an exponential factor of magnitude 2^{d-1}, and in 2009, the author decreased the base to $\sqrt[3]{40/9} \approx 1.64414$ [Mal09b], as well as proving the 3-dimensional case [Mal09a]. The method used for this case uses induction on the dimension, and it seems that it can be generalized in order to obtain the full result. Some obstructions arise, that necessitate the proof of stronger lemmata. We will provide some reductions of Conjecture 2.2 at the end of this chapter.

2.2 Some definitions, notations, and lemmata

In the course of developing the inductive method mentioned at the end of the previous section, it was necessary to bound lattice point enumerators of convex bodies that are not 0-symmetric. Therefore, it was necessary to extend the definition of successive minima for these bodies as well. The most natural way to extend is the following:

Definition. Let $K \in \mathcal{K}^d$, $\Lambda \in \mathcal{L}^d$. The successive minima of K are defined to be the same as those of the 0-symmetric convex body $\frac{1}{2}\mathfrak{D}K$, that is

$$\lambda_i(K, \Lambda) := \lambda_i(\tfrac{1}{2}\mathfrak{D}K, \Lambda),$$

where $\mathfrak{D}K := K - K = \{x - y | x, y \in K\}$.

Notation. Let $K \in \mathcal{K}^d$, $\Lambda \in \mathcal{L}^d$. By definition of the successive minima $\lambda_i(K, \Lambda)$, there are d linearly independent lattice vectors a^i, $1 \leqslant i \leqslant d$, such that

$$a^i(K, \Lambda) \in \tfrac{\lambda_i(K,\Lambda)}{2}\mathfrak{D}K \cap \Lambda.$$

We will denote a choice of such vectors by $a^i(K, \Lambda)$ for all i. We then construct a basis of Λ, denoted by e^i, $1 \leqslant i \leqslant d$, such that

$$\operatorname{lin}(a^1, \ldots, a^i) = \operatorname{lin}(e^1, \ldots, e^i)$$

for all i, $1 \leqslant i \leqslant d$. We will denote a choice of such vectors by $e^i(K, \Lambda)$. Furthermore, we define the following subgroups of Λ:

$$\Lambda^i := \mathbb{Z}e^1 \oplus \cdots \oplus \mathbb{Z}e^i,$$

which we denote by $\Lambda^i(K)$. Finally, we define

$$q_i(K, \Lambda) := \left[\frac{2}{\lambda_i(K, \Lambda)} + 1 \right].$$

We will suppress mention of K and Λ when no contradiction arises, and simply write a^i, e^i, Λ^i, q_i respectively. In particular, when Λ is the standard lattice $\mathbb{Z}^d$, we will always suppress mention of the lattice, and write $a^i(K), e^i(K), q_i(K)$. Furthermore, $\mathrm{conv}(A)$ will denote the convex hull of a set $A \subset \mathbb{R}^d$. When A is a union of a single point v and another set K, we will write $\mathrm{conv}(v, K)$.

It should be noted that there is an abuse of notation here; it is evident that the choice of the a^i's and the e^i's, as well as the Λ^i's, is not always unique. However, by this notation we shall always mean a choice of vectors or subgroups with the above properties. The main property that will be used later is

$$\mathrm{int}(\tfrac{\lambda_i}{2}\mathfrak{D}K) \cap \Lambda \subset \Lambda^{i-1} \tag{2.4}$$

Lemma 2.3. *Let $K \in \mathcal{K}^d$, $\Lambda \in \mathcal{L}^d$. For each real n_i, satisfying $n_i > 2/\lambda_i$, we have*

$$\mathfrak{D}K \cap n_i(\Lambda \setminus \Lambda^{i-1}) = \varnothing$$

In particular,

$$\mathrm{int}(\mathfrak{D}K) \cap \frac{2}{\lambda_i}(\Lambda \setminus \Lambda^{i-1}) = \varnothing$$

Proof. Assume otherwise; then the intersection

$$\tfrac{1}{n_i}\mathfrak{D}K \cap (\Lambda \setminus \Lambda^{i-1})$$

would be nonempty. The left part of this intersection is a subset of

$$\mathrm{int}(\tfrac{\lambda_i}{2}\mathfrak{D}K),$$

since $n_i > 2/\lambda_i$. Therefore, the intersection

$$\mathrm{int}(\tfrac{\lambda_i}{2}\mathfrak{D}K) \cap (\Lambda \setminus \Lambda^{i-1})$$

is nonempty, contradicting (2.4) above, as was to be shown. □

The following is an adaptation of Lemma 2.1 in [Hen02], for the case of all convex bodies, not necessarily 0-symmetric. Even though the proof is identical, we provide it here for convenience.

Lemma 2.4. *Let $K \in \mathcal{K}^d$ and $\Lambda, \widetilde{\Lambda} \in \mathcal{L}^d$, with $\widetilde{\Lambda} \subset \Lambda$. Then*

$$G(K, \Lambda) \leqslant \frac{d(\widetilde{\Lambda})}{d(\Lambda)} G(\mathfrak{D}K, \widetilde{\Lambda}). \tag{2.5}$$

Proof. Let $m = G(\mathfrak{D}K, \widetilde{\Lambda})$ and suppose there exist at least $m+1$ different lattice points $v^1, \ldots, v^{m+1} \in K \cap \Lambda$ such that $v^i \equiv v^1 \mod \widetilde{\Lambda}$, $1 \leqslant i \leqslant m+1$. Then we have

$$v^i - v^1 \in \mathfrak{D}K \cap \widetilde{\Lambda}, \quad 1 \leqslant i \leqslant m+1,$$

which contradicts the assumption $m = G(\mathfrak{D}K, \widetilde{\Lambda})$. Thus we have shown that every residue class of Λ with respect to $\widetilde{\Lambda}$ does not contain more than m points of $K \cap \Lambda$. Since there are precisely $d(\widetilde{\Lambda})/d(\Lambda)$ different residue classes, we obtain the desired bound. □

The following two lemmata will be used for the proof of inequality (2.2) in the 3-dimensional case. Notice that they are statements in d dimensions.

Lemma 2.5. *Let $K \subset \mathbb{R}^d$ be a convex body, $\Lambda \in \mathcal{L}^d$, such that $K \cap \Lambda = \varnothing$. Then there is some $v \in \Lambda$ such that for any real $t > 1$,*

$$K \cap (v + t\Lambda) = \varnothing.$$

Proof. Take $v \in \Lambda$ such that $\#(\mathrm{conv}(v, K) \cap \Lambda)$ is minimal. If this number is greater than 1, then there is some $w \in \Lambda$, $w \neq v$, such that $w \in \mathrm{conv}(v, K)$. Hence, $\mathrm{conv}(w, K) \subset \mathrm{conv}(v, K)$, and $v \notin \mathrm{conv}(w, K)$, contradicting the minimality of $\#(\mathrm{conv}(v, K) \cap \Lambda)$. Thus, $\mathrm{conv}(v, K) \cap \Lambda = \{v\}$. We claim that $K \cap (v + t\Lambda) = \varnothing$, for all $t > 1$. Suppose not; then there is some $u \in \Lambda$ such that $v + tu \in K$, for some $t > 1$. By convexity, and the fact that $t > 1$, we get $v + u \in \mathrm{conv}(v, K)$, which implies $u = 0$, so $v \in K$, a contradiction, since $K \cap \Lambda = \varnothing$. This concludes the proof. □

The next lemma generalizes the above:

Lemma 2.6. *Let $K \subset \mathbb{R}^d$ be a convex body, and $\Lambda \in \mathcal{L}^d$. Let $S \subset \Lambda$ be finite, and r be a positive integer, such that*

(1) $(K - S) \cap r\Lambda = \varnothing$.

(2) $\mathfrak{D}S \cap r(\Lambda \setminus \{0\}) = \varnothing$.

Now, let $t > r$ be an integer. There is a set $S' \subset \Lambda$, obtained by translating each $v \in S$ by some vector $r \cdot w(v)$, where $w(v) \in \Lambda$, such that

(1)' $(K - S') \cap t\Lambda = \varnothing$.

(2)' $\mathfrak{D}S' \cap t(\Lambda \setminus \{0\}) = \varnothing$.

Proof. The proof proceeds by induction on $\#(S)$. If $\#(S) = 1$; i.e., $S = \{v\}$, we use lemma 2.5 for $K - v$ and the lattice $r\Lambda$. Since $t > r$, there is some $w(v) \in \Lambda$, such that $(K - v) \cap (r \cdot w(v) + t\Lambda) = \varnothing$. Put $S' = \{v + r \cdot w(v)\}$, and we see that (1)′ is satisfied. It should be noted that when $\#(S) = 1$, conditions (2) and (2)′ hold vacuously.

Now, assume that $\#(S) > 1$. Take $v \in S + r\Lambda$, such that $\#(\mathrm{conv}(v, K) \cap (S + r\Lambda))$ is minimal. Again, as in the proof of Lemma 2.5, we must have $\mathrm{conv}(v, K) \cap (S + r\Lambda) = \{v\}$. Apply induction for $\widetilde{K} = \mathrm{conv}(v, K)$ and $\widetilde{S} = S \setminus (S \cap (v + r\Lambda))$; we have $\#(\widetilde{S}) = \#(S) - 1$. Let's see why (1) and (2) are satisfied for $\widetilde{K}, \widetilde{S}$ (same r, Λ); (2) is obviously satisfied, as $\widetilde{S} \subset S$. If (1) were not satisfied, then there would be some $w \in \widetilde{S}$ and $u \in \Lambda$ such that $w + ru \in \mathrm{conv}(v, K)$. By the minimality assumption, $w + ru = v$. But $v \notin \widetilde{S} + r\Lambda$, a contradiction. Thus, (1) and (2) hold for $\widetilde{K}$, $\widetilde{S}$, and by induction there is some $\overline{S} \subset \Lambda$, obtained from $\widetilde{S}$ by translating each $u \in \widetilde{S}$ by $r \cdot w(u)$, $w(u) \in \Lambda$, such that

$$(\widetilde{K} - \overline{S}) \cap t\Lambda = \varnothing, \tag{2.6}$$

and

$$\mathfrak{D}\overline{S} \cap t(\Lambda \setminus \{0\}) = \varnothing. \tag{2.7}$$

Now, set $S' = \overline{S} \cup \{v\}$. $(2)'$ is satisfied for S'; if $x, y \in \overline{S}$, then $x - y \notin t(\Lambda \setminus \{0\})$ from (2.7). If $x \in \overline{S}$ and $y = v$, then again from above, $v - x \notin t\Lambda$, since $v \in \widetilde{K}$. If $x = y = v$, we have nothing to prove, so

$$\mathfrak{D}S' \cap t(\Lambda \setminus \{0\}) = \varnothing.$$

$(1)'$ is also satisfied for K, S'; suppose not. Then, there is some $w \in S'$, $u \in \Lambda$ such that $w + tu \in K$. If $w \in \overline{S}$, then $w + tu \in \widetilde{K}$, which contradicts $(\widetilde{K} - \overline{S}) \cap t\Lambda = \varnothing$. If $w = v$, then $v + tu \in K$, and by convexity, $v + ru \in K$, hence $u = 0$, by minimality assumption, and $v \in K$, a contradiction. This concludes the proof. □

2.3 The general case

2.3.1 Henk's inequality

Henk's inequality is the following:

Theorem 2.7. *Let $K \in \mathcal{K}_0^d$ and $\Lambda \in \mathcal{L}^d$. Then*

$$G(K,\Lambda) \leqslant 2^{d-1} \prod_{i=1}^{d} q_i(K,\Lambda). \tag{2.8}$$

We will not present Henk's proof of Theorem 2.7 here, but rather modify it in order to obtain a stronger result, from which Theorem 2.7 follows. It should be noted that the proof resembles Rogers' proof [Rog49] for an upper bound on $\mathrm{vol}(K)$, involving the density of the densest lattice packing, in an attempt towards Davenport's problem (which we will see in chapter 3).

2.3.2 An optimization problem

Theorem 2.8. *Let $K \in \mathcal{K}^d$, $\Lambda \in \mathcal{L}^d$, $q_i = q_i(K,\Lambda)$. Let also $n_1, \ldots, n_d$ be a sequence of integers satisfying*

- *n_{i+1} divides n_i, $1 \leqslant i \leqslant d-1$.*
- *$q_i \leqslant n_i$, $1 \leqslant i \leqslant d$.*

Then,

$$G(K,\Lambda) \leqslant \prod_{i=1}^{d} n_i.$$

Proof. Let $e^i = e^i(K,\Lambda)$ and define

$$\widetilde{\Lambda} = \mathbb{Z}n_1 e^1 \oplus \cdots \oplus \mathbb{Z}n_d e^d.$$

By Lemma 2.4,

$$G(K,\Lambda) \leqslant \frac{d(\widetilde{\Lambda})}{d(\Lambda)} G(\mathfrak{D}K,\widetilde{\Lambda}) = G(\mathfrak{D}K,\widetilde{\Lambda}) \prod_{i=1}^{d} n_i.$$

It suffices to prove that $G(\mathfrak{D}K,\widetilde{\Lambda}) = 1$, or equivalently

$$\mathfrak{D}K \cap (\widetilde{\Lambda} \setminus \{0\}) = \varnothing.$$

This follows from Lemma 2.3 and the fact that

$$\widetilde{\Lambda} \setminus \{0\} \subset \bigcup_{i=1}^{d} n_i(\Lambda \setminus \Lambda^{i-1})$$

(recall that $n_i \geqslant q_i > 2/\lambda_i$). Indeed, let $g \in \widetilde{\Lambda} \setminus \{0\}$ be arbitrary, and let k be minimal such that

$$g \in \mathbb{Z}n_1e^1 \oplus \cdots \oplus \mathbb{Z}n_ke^k.$$

Since n_k divides all $n_1, \ldots, n_{k-1}$ by assumption, we have $g \in n_k\Lambda$. By minimality of k, we also have $g \notin \Lambda^{k-1}$, hence $g \in n_k(\Lambda \setminus \Lambda^{k-1})$ as desired. □

As a simple consequence we can extend Theorem 2.1 to the non-symmetric case.

Corollary 2.9. *Let $K \in \mathcal{K}^d$, $\Lambda \in \mathcal{L}^d$. Then*

$$G(K, \Lambda) \leqslant q_1(K, \Lambda)^d.$$

Proof. The numbers $n_1 = \cdots = n_d = q_1$ satisfy the hypotheses of Theorem 2.8. □

At this point, we may wonder what is the least possible value that the product of the n_i's in Theorem 2.8 can take relative to the product of the q_i's. We are naturally led to the following definition.

Definition. Let C_d denote the least positive constant, such that for any sequence of d integers, $0 < x_1 \leqslant x_2 \leqslant \cdots \leqslant x_d$, there exists a sequence of integers $y_1, y_2, \ldots, y_d$ satisfying:

a. $x_i \leqslant y_i$, for all i, $1 \leqslant i \leqslant d$

b. y_i divides y_{i+1}, for all i, $1 \leqslant i \leqslant d - 1$

c. $\dfrac{y_1 y_2 \cdots y_d}{x_1 x_2 \cdots x_d} \leqslant C_d$.

Using this definition, we have:

Corollary 2.10. *Let* $K \in \mathcal{K}^d$, $\Lambda \in \mathcal{L}^d$. *Then:*

$$G(K, \Lambda) \leqslant C_d \prod_{i=1}^{d} q_i(K, \Lambda).$$

Henk [Hen02] essentialy proved the following:

Proposition 2.11. $C_d \leqslant 2^{d-1}$.

Proof. It suffices to put $y_1 = x_1$, and inductively construct $x_i \leqslant y_i < 2x_i$, such that $y_i | y_{i+1}$. Such a construction is possible; assuming that we have constructed $y_1, \ldots, y_k$, satisfying the above requirements we now construct y_{k+1}. If $x_{k+1} \leqslant y_k$, we simply set $y_{k+1} = y_k$. Obviously, $x_{k+1} \leqslant y_{k+1} < 2x_k \leqslant 2x_{k+1}$. Otherwise, we consider the euclidean division of x_{k+1} by y_k, say $x_{k+1} = m \cdot y_k + r$, where $0 \leqslant r < y_k$. Then, setting $y_{k+1} = (m+1)y_k$ satisfies the desired requirements. $\square$

In order to obtain a better estimate on C_d, we drop the hypothesis on integrality of the x_i's and y_i's; in this setting, $y_i | y_{i+1}$ means $y_{i+1}/y_i \in \mathbb{Z}$. We call the corresponding constant c_d:

Definition. Let c_d denote the least positive constant, such that for any sequence of d positive real numbers, $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_d$, there exists a sequence of real numbers $y_1, y_2, \ldots, y_d$ satisfying:

a. $x_i \leqslant y_i$, for all i, $1 \leqslant i \leqslant d$

b. $y_{i+1}/y_i \in \mathbb{Z}$, for all i, $1 \leqslant i \leqslant d-1$

c. $\dfrac{y_1 y_2 \cdots y_d}{x_1 x_2 \cdots x_d} \leqslant c_d$.

We will later prove that $c_d \leqslant C_d$. The following lemma was proven by Rogers in [Rog49] and independently by Chabauty in [Cha49]. We provide a proof here, for convenience (see also [GL87], p.190):

Lemma 2.12. ***(Rogers)*** $c_d = 2^{(d-1)/2}$.

Proof. For each i, $1 \leqslant i \leqslant d$, we construct the sequence $y^i_1, \ldots, y^i_d$ that satisfies

$$y^i_i = x_i, \quad y^i_j = 2^{a_{ij}} x_i, \text{ for } j \neq i \text{ where } a_{ij} = -\left[\log_2 \frac{x_i}{x_j}\right].$$

In other words, a_{ij} is the unique integer satisfying

$$x_j \leqslant 2^{a_{ij}} x_i < 2x_j.$$

Therefore,

$$\log_2 \frac{y^i_j}{x_j} = \{\log_2 x_i - \log_2 x_j\},$$

for all j, so

$$\log_2 \frac{y^i_1 \cdots y^i_d}{x_1 \cdots x_d} = \sum_{j=1}^{d} \{\log_2 x_i - \log_2 x_j\}.$$

Summing over all i, we obtain

$$\sum_{i=1}^{d} \log_2 \frac{y^i_1 \cdots y^i_d}{x_1 \cdots x_d} = \sum_{i,j=1}^{d} \{\log_2 x_i - \log_2 x_j\}.$$

For any pair (i, j) with $i \neq j$, $\{\log_2 x_i - \log_2 x_j\} + \{\log_2 x_j - \log_2 x_i\} \leqslant 1$ (for $i = j$ it vanishes). Since there are $d(d-1)/2$ such pairs, we get

$$\sum_{i=1}^{d} \log_2 \frac{y^i_1 \cdots y^i_d}{x_1 \cdots x_d} \leqslant \frac{d(d-1)}{2}.$$

Hence, there is an index i such that

$$\log_2 \frac{y^i_1 \cdots y^i_d}{x_1 \cdots x_d} \leqslant \frac{d-1}{2}$$

and thus

$$\frac{y_1^i \cdots y_d^i}{x_1 \cdots x_d} \leqslant 2^{\frac{d-1}{2}}.$$

Since the increasing sequence $x_1, \ldots, x_d$ is arbitrary, we have $c_d \leqslant 2^{(d-1)/2}$. We will show, by an example, that $c_d = 2^{(d-1)/2}$; let $x_i = 2^{(i-1)/d}$. Let $y_1, \ldots, y_d$ be an increasing sequence satisfying $x_i \leqslant y_i$ and $y_i | y_{i+1}$ for all i. Dividing all y_i's by an appropriate number, we may assume that $x_i = y_i$ for some i. Since $x_d < 2$, we must have $y_j = x_i$ for all $j \leqslant i$ and of course $y_j \geqslant 2x_i$ for all $j > i$. Thus,

$$\frac{y_1 \cdots y_d}{x_1 \cdots x_d} \geqslant 2^{\frac{i-1}{d}} \cdot 2^{\frac{i-2}{d}} \cdots 1 \cdot 2^{\frac{d-1}{d}} \cdot 2^{\frac{d-2}{d}} \cdots 2^{\frac{i}{d}} = 2^{\frac{d-1}{2}}.$$

Since $y_1, \ldots, y_d$ is an arbitrary sequence with the above properties, we have established that $c_d = 2^{(d-1)/2}$. □

It is a more difficult task to compute C_d exactly, though the following Proposition provides an upper and lower bound.

Proposition 2.13. $2^{(d-1)/2} \leqslant C_d \leqslant (4/e) \cdot 3^{(d-1)/2}$, *and the lower bound is tight.*

Proof. The averaging process is slightly different than before; for each integer a with $x_1 \leqslant a < 2x_1$ we construct a sequence $y_1^a, \ldots, y_d^a$ satisfying $y_1^a = a$ and

$$y_i^a = 2^{b_{ai}} a, \text{ where } b_{ai} = -\left[\log_2 a - \log_2 x_i\right].$$

As before,

$$\log_2 \frac{y_1^a \cdots y_d^a}{x_1 \cdots x_d} = \sum_{i=1}^{d} \left\{\log_2 a - \log_2 x_i\right\}.$$

Summing over all a, we obtain

$$\sum_{a=x_1}^{2x_1-1} \log_2 \frac{y_1^a \cdots y_d^a}{x_1 \cdots x_d} = \sum_{a=x_1}^{2x_1-1} \sum_{i=1}^{d} \left\{\log_2 a - \log_2 x_i\right\}.$$

For $i = 1$, we obtain

$$\sum_{a=x_1}^{2x_1-1} \left\{\log_2 \frac{a}{x_1}\right\} < x_1 \int_1^2 \log_2 x dx.$$

Now let $i > 1$. The following equality holds

$$\{\log_2 a - \log_2 x_i\} = \left\{\log_2 \frac{a}{x_1}\right\} - \left\{\log_2 \frac{x_i}{x_1}\right\} + \varepsilon,$$

where $\varepsilon = 0$ or 1, depending on whether $\left\{\log_2 \frac{x_i}{x_1}\right\} \leqslant \left\{\log_2 \frac{a}{x_1}\right\}$ or $\left\{\log_2 \frac{x_i}{x_1}\right\} > \left\{\log_2 \frac{a}{x_1}\right\}$. If $\left\{\log_2 \frac{x_i}{x_1}\right\} = 0$, then we get the same result as in the case $i = 1$. Otherwise, let l be the unique integer satisfying

$$\log_2 \frac{x_1 + l - 1}{x_1} < \left\{\log_2 \frac{x_i}{x_1}\right\} \leqslant \log_2 \frac{x_1 + l}{x_1}.$$

Of course, $1 \leqslant l \leqslant x_1$. Thus we obtain

$$\begin{aligned}
\sum_{a=x_1}^{2x_1-1} \left\{\log_2 \frac{a}{x_i}\right\} &= \sum_{a=x_1}^{2x_1-1} \log_2 \frac{a}{x_1} - x_1 \left\{\log_2 \frac{x_i}{x_1}\right\} + l \\
&< \sum_{a=x_1}^{2x_1-1} \log_2 \frac{a}{x_1} - x_1 \log_2 \frac{x_1 + l - 1}{x_1} + l \\
&= \sum_{a=x_1}^{2x_1-1} \log_2 \frac{a}{x_1} - \log_2 \left(1 + \frac{l-1}{x_1}\right)^{x_1} + l \\
&< \sum_{a=x_1}^{2x_1-1} \log_2 \frac{a}{x_1} - \log_2 2^{l-1} + l \\
&= \sum_{a=x_1}^{2x_1-1} \log_2 \frac{a}{x_1} + 1 \\
&= \sum_{a=x_1+1}^{2x_1} \log_2 \frac{a}{x_1}.
\end{aligned}$$

The latter is an upper Riemann sum, multiplied by x_1, for the function $f(x) = \log_2 x$ for the partition

$$1 = \frac{x_1}{x_1} < \frac{x_1 + 1}{x_1} < \cdots < \frac{2x_1 - 1}{x_1} < \frac{2x_1}{x_1} = 2.$$

It is a simple task to prove that

$$\frac{1}{x_1} \sum_{a=x_1+1}^{2x_1} \log_2 \frac{a}{x_1}$$

is decreasing in x_1 and converges to $\int_1^2 \log_2 x dx$. Without loss of generality, we may assume that $x_1 \geqslant 2$; otherwise we disregard all terms equal to 1, because we can set $y_i = x_i = 1$, and we consider the first term of the sequence $x_1, \dots, x_d$ which is greater than 1. So, the maximal value of the Riemann sum is

$$\frac{1}{2}\left(\log_2 \frac{3}{2} + \log_2 \frac{4}{2}\right) = \log_2 \sqrt{3},$$

and hence

$$\sum_{a=x_1}^{2x_1-1} \left\{\log_2 \frac{a}{x_i}\right\} < x_1 \log_2 \sqrt{3}.$$

Thus,

$$\sum_{a=x_1}^{2x_1-1} \sum_{i=1}^{d} \{\log_2 a - \log_2 x_i\} < x_1 \left(\int_1^2 \log_2 x dx + (d-1)\log_2 \sqrt{3}\right),$$

therefore, there is a number a for which the following inequality holds:

$$\sum_{i=1}^{d} \{\log_2 a - \log_2 x_i\} < 2 - \frac{1}{\ln 2} + (d-1)\log_2 \sqrt{3},$$

so finally

$$\frac{y_1^a \cdots y_d^a}{x_1 \cdots x_d} < \frac{4}{e} \cdot 3^{(d-1)/2},$$

as desired.

As for the other inequality, we will base our arguments on the example at the end of the proof of Lemma 2.12, which shows $c_d \geqslant 2^{(d-1)/2}$. We will actually prove that for all $\delta > 0$, the following inequality holds:

$$C_d > (1-\delta)^{\frac{d(d-1)}{2}} \cdot 2^{\frac{d-1}{2}}.$$

Let $\delta > 0$ be arbitrary, and let M be a positive integer such that

$$M > \frac{1}{\delta \sqrt[d]{2}}.$$

Define $x_1 = M$ and $x_{i+1} = \left[x_i \sqrt[d]{2}\right]$ for $1 \leqslant i \leqslant d-1$. Let $y_1, \ldots, y_d$ be a sequence of positive integers satisfying $x_i \leqslant y_i$ and $y_i | y_{i+1}$ for all i, such that the product $y_1 y_2 \cdots y_d$ is minimal. Since $x_d < 2x_1$, we deduce that $y_d = y_1$ or $2y_1$. If $y_1 = y_d$, then by the minimality assumption, $y_i = x_d$, for all i. Otherwise, let i be the maximal index such that $y_i = x_i$ (i.e., $y_1 = y_2 = \cdots = y_i$, $2y_i = y_{i+1} = \cdots = y_d$). Then again, by minimality we have that $y_i = x_i$. So, the sequence $y_1, \ldots, y_d$ has the form

$$\underbrace{x_i, \ldots, x_i}_{i \text{ terms}}, \underbrace{2x_i, \ldots, 2x_i}_{d-i \text{ terms}},$$

for some index i. We will prove that $i = d$. Indeed, from the definition of the sequence $\{x_i\}_{i=1}^d$ we have that

$$\frac{x_i \cdot \sqrt[d]{2} - 1}{x_i} < \frac{x_{i+1}}{x_i} < \sqrt[d]{2},$$

which implies

$$\sqrt[d]{2} - \frac{1}{M} < \frac{x_{i+1}}{x_i} < \sqrt[d]{2},$$

and since $M > 1/(\delta \sqrt[d]{2})$, we get

$$(1-\delta)\sqrt[d]{2} < \frac{x_{i+1}}{x_i} < \sqrt[d]{2},$$

thus for $j > i$,

$$(1-\delta)^{j-i} \cdot 2^{\frac{j-i}{d}} < \frac{x_j}{x_i} < 2^{\frac{j-i}{d}}. \tag{2.9}$$

For $j = d$, the right-hand side becomes

$$\left(\frac{x_d}{x_i}\right)^d < 2^{d-i},$$

or

$$x_d^d < 2^{d-i} x_i^d = \underbrace{x_i \cdots x_i}_{i \text{ terms}} \cdot \underbrace{2x_i \cdots 2x_i}_{d-i \text{ terms}}.$$

So, we proved that $y_i = x_d$, for all i. Using the left-hand side inequalities of (4), for $j = d$, we obtain

$$\prod_{i=1}^{d-1} \frac{x_d}{x_i} > (1-\delta)^{\frac{d(d-1)}{2}} \cdot 2^{\frac{d-1}{2}},$$

hence

$$C_d > (1-\delta)^{\frac{d(d-1)}{2}} \cdot 2^{\frac{d-1}{2}}$$

for all $\delta > 0$, thus

$$C_d \geqslant 2^{\frac{d-1}{2}},$$

completing the proof. $\square$

We are now able to establish the following inequalities for $G(K, \Lambda)$, by virtue of Proposition 2.13, and the methods within the proof:

Theorem 2.14. *Let $K \in \mathcal{K}^d$, $\Lambda \in \mathcal{L}^d$. Then*

$$G(K, \Lambda) \leqslant \frac{4}{e}(\sqrt{3})^{d-1} \prod_{i=1}^{d} q_i(K, \Lambda).$$

If $K \in \mathcal{K}_0^d$, then

$$G(K, \Lambda) \leqslant \frac{4}{e}\left(\sqrt[3]{\frac{40}{9}}\right)^{d-1} \prod_{i=1}^{d} q_i(K, \Lambda).$$

Proof. The first inequality follows immediately from Corollary 2.10 and Proposition 2.13. For the 0-symmetric case, let k be the smallest index such that $\lambda_k > 1$. If $k = 1$, then $G(K) = 1$, and the conjecture is verified. If $k > 1$, then we have a reduction to fewer dimensions, namely $k-1$, because $K \cap \Lambda$ has at most $k-1$ linearly independent vectors, by the definition of the successive minima. So, if we intersect K and Λ with the linear hull of these vectors, we get a $(k-1)$-dimensional convex body K' and a $(k-1)$-dimensional lattice Λ' such that $\lambda_i(K', \Lambda') \leqslant 1$ for all i. Furthermore, $G(K, \Lambda) = G(K', \Lambda')$. This shows that the problem reduces to the setting where all successive minima are less than or equal to 1. In this case, all q_i are at least equal to 3.

Combining this observation with the proof of Proposition 2.13, allows us to take $x_1 \geqslant 3$ for the purposes of our geometric problem. Therefore, the maximal value for the upper Riemann sum

$$\frac{1}{x_1} \sum_{a=x_1+1}^{2x_1} \log_2 \frac{a}{x_1},$$

is obtained for $x_1 = 3$, which is

$$\frac{1}{3}\left(\log_2 \frac{4}{3} + \log_2 \frac{5}{3} + \log_2 \frac{6}{3}\right) = \log_2 \sqrt[3]{\frac{40}{9}}.$$

Thus, the corresponding constant, under the restriction $x_1 \geqslant 3$ is less than or equal to

$$\frac{4}{e}\left(\frac{40}{9}\right)^{\frac{d-1}{3}} \approx 1.47152 \cdot 1.64414^{d-1},$$

concluding the proof. □

2.4 A method by induction

It is clear from the proof of Proposition 2.13 that unless we develop a stronger geometric argument, we will have an additional exponential constant whose base is at least $\sqrt{2}$. It is natural to approach Conjecture 2.2, by counting lattice points on the intersections of K by hyperplanes passing through lattice points.

Let $K \in \mathcal{K}_0^d$, $\Lambda \in \mathcal{L}^d$. Fix a basis $e^i = e^i(K, \Lambda)$ of Λ, that satisfies the properties given in section 2.2. We will write each vector x of $\mathbb{R}^d$ with coordinates with respect to this basis:

$$\begin{aligned} x &= (x_1, \ldots, x_d) \\ &= x_1 e^1 + \cdots + x_d e^d. \end{aligned}$$

Define

$$K[t] := \{x \in K | x_d = t\};$$

i.e., the subset of K whose elements have fixed height, or the intersection of K by the hyperplane parallel to the vector subspace spanned by $e^1, \ldots, e^{d-1}$. We can write $G(K, \Lambda)$ in terms of lattice point enumerators of convex bodies whose dimension is $d-1$; this is the point where induction could be used. Namely,

$$G(K, \Lambda) = \sum_{t \in \mathbb{Z}} G(K[t] - te^d, \Lambda^{d-1}).$$

The bodies $K[t] - te^d$ are projections of the intersections $K[t]$ on the vector subspace spanned by $e^1, \ldots, e^{d-1}$ along the lattice vector e^d. As before, Λ^{d-1} is the $\mathbb{Z}$-span of $e^1, \ldots, e^{d-1}$. Apart from $K[0]$ which is 0-symmetric, the other projections are not necessarily 0-symmetric. This is the main reason for extending inequalities (2.1) and (2.2) to the non symmetric case.

Next, observe that

$$\tfrac{1}{2}\mathfrak{D}(K[t] - te^d) \subset \tfrac{1}{2}\mathfrak{D}K,$$

therefore, for $1 \leqslant i \leqslant d-1$

$$\lambda_i(K[t] - te^d, \Lambda^{d-1}) \geqslant \lambda_i(K, \Lambda),$$

which implies

$$q_i(K[t] - te^d, \Lambda^{d-1}) \leqslant q_i(K, \Lambda),$$

for $1 \leqslant i \leqslant d-1$. Assuming that inequality (2.2) holds for $d-1$, we have

$$G(K[t] - te^d, \Lambda^{d-1}) \leqslant \prod_{i=1}^{d-1} q_i(K, \Lambda),$$

for all $t \in \mathbb{Z}$. Only the factor q_d is missing; we could normally expect that the number of the nonempty “slices”, $K[t]$, is less than q_d. But it is not always the case that this number is less than $q_d(K, \Lambda)$.

The next step is to group all intersections whose heights are congruent modulo q_d. Doing so, the above sum becomes

$$G(K,\Lambda)=\sum_{r=0}^{q_d-1}\sum_{t\equiv r(\mathrm{mod} q_d)} G(K[t]-te^d,\Lambda^{d-1}).$$

It suffices to prove that for each fixed r, we have

$$\sum_{t\equiv r(\mathrm{mod} q_d)} G(K[t]-te^d,\Lambda^{d-1})\leqslant\prod_{i=1}^{d-1} q_i.$$

Of course, we could have more than one convex body in the above sum, however, the above collection of convex bodies $K[t]-te^d$, $t\equiv r(\mathrm{mod} q_d)$ satisfies some restricting conditions, namely:

(1) $\mathfrak{D}(K[t]-te^d)\cap q_i(\Lambda^{d-1}\setminus\Lambda^{i-1})=\varnothing$ for all $t\equiv r(\mathrm{mod} q_d)$ and $1\leqslant i\leqslant d-1$.

(2) $((K[t]-te^d)-(K[t']-t'e^d))\cap q_d\Lambda^{d-1}=\varnothing$ for all $t,t'\equiv r(\mathrm{mod} q_d)$, $t\neq t'$.

The two statements above are consequences of Lemma 2.3. Indeed, for (1) we observe that

$$\mathfrak{D}(K[t]-te^d)\cap q_i(\Lambda^{d-1}\setminus\Lambda^{i-1})\subset\mathfrak{D}K\cap q_i(\Lambda\setminus\Lambda^{i-1}),$$

and the latter is empty since $q_i>2/\lambda_i$. As for (2), if

$$((K[t]-te^d)-(K[t']-t'e^d))\cap q_d\Lambda^{d-1}\neq\varnothing,$$

then there would exist some $v\in\Lambda^{d-1}$ such that $q_dv+(t-t')e^d\in K[t]-K[t']\subset\mathfrak{D}K$. However, since $q_d|t-t'$, and $t\neq t'$, the intersection

$$\mathfrak{D}K\cap q_d(\Lambda\setminus\Lambda^{d-1})$$

is nonempty, contradicting Lemma 2.3.

It is natural to state the following conjecture:

Conjecture 2.15. *Let $K_1, \dots, K_n \subset \mathbb{R}^d$ be convex bodies and $\Lambda \in \mathcal{L}^d$. Also, let $e^1, \dots, e^d$ be a basis of Λ, and denote by Λ^i the $\mathbb{Z}$-span of $0, e^1, \dots, e^i$, and let $q_1 \geqslant q_2 \geqslant \cdots \geqslant q_{d+1}$ be positive integers satisfying*

(1) *$\mathfrak{D}K_j \cap q_i(\Lambda \setminus \Lambda^{i-1}) = \varnothing$ for all $1 \leqslant j \leqslant n$ and $1 \leqslant i \leqslant d$.*

(2) *$(K_j - K_l) \cap q_{d+1}\Lambda = \varnothing$ for all $1 \leqslant j, l \leqslant n$, $j \neq l$.*

Then

$$\sum_{j=1}^{n} G(K_j, \Lambda) \leqslant \prod_{i=1}^{d} q_i.$$

From the above analysis, it is clear that the above conjecture implies inequality (2.2) for one dimension higher. We will verify this conjecture for $d = 1, 2$, thus proving inequality (2.2) in all dimensions up to three. A statement in support of this conjecture is that condition (2) is too restricting for the convex bodies K_j, given the fact that q_{d+1} is smaller than the rest of the q_i's. This statement simply says that no two translates of K_j and K_l, $j \neq l$, by vectors of $q_{d+1}\Lambda$ intersect. In the next section, we present a more convincing reduction of Conjecture 2.15.

2.4.1 Proof of Conjecture 2.15, $d = 1$

Without loss of generality, we assume that $\Lambda = \mathbb{Z}$. Let $K_j = [a_j, b_j]$, $1 \leqslant j \leqslant n$. Conditions (1) and (2) read

(1) $b_j - a_j < q_1$ for all $1 \leqslant j \leqslant n$.

(2) $(K_j - K_l) \cap q_2\mathbb{Z} = \varnothing$ for all $1 \leqslant j, l \leqslant n$, $j \neq l$.

If $b_1 - a_1 \geqslant q_2$, then the union of K_1 with all its translates by multiples of q_2 cover all of $\mathbb{R}$, so by condition (2) we must have $n = 1$, therefore

$$\sum_{j=1}^{n} G(K_j, \Lambda) = G(K_1) \leqslant q_1$$

by (1). If $b_1 - a_1 < q_2$, there is a translate of each K_j by some multiple of q_2, $2 \leqslant j \leqslant n$, that lies in $(b_1, a_1 + q_2)$, again by (2). Since they do not intersect each other by (2), we have

$$\sum_{j=1}^{n} G(K_j) \leqslant G([a_1, a_1 + q_2)) = q_2 \leqslant q_1.$$

2.4.2 Proof of Conjecture 2.15, $d = 2$

Let

$$D = \dim\left(\left(\bigcup_{j=1}^{n} \mathfrak{D}K_j\right) \cap q_3\Lambda\right).$$

We distinguish cases for D:

$\underline{D \leqslant 1}$: There exists a primitive lattice vector, say v, such that

$$\left(\bigcup_{j=1}^{n} \mathfrak{D}K_j\right) \cap q_3\Lambda \subset \mathbb{Z}(q_3 v)$$

therefore

$$\big((\bigcup_{j=1}^{n} \mathfrak{D}K_j) \cap q_3\big(\Lambda \setminus \mathbb{Z}v\big)\big) = \varnothing.$$

Find $w \in \Lambda$ such that v, w is a basis for Λ. Then

$$\sum_{j=1}^{n} G(K_j, \Lambda) = \sum_{r=0}^{q_3-1} \sum_{j=1}^{n} \sum_{t \equiv r (\mathrm{mod} q_3)} G(K_j[t] - tw, \mathbb{Z}v).$$

We will prove that the above sum is less than or equal to $q_1 q_3$ (which is less than or equal to $q_1 q_2$); it suffices to prove that

$$\sum_{j=1}^{n} \sum_{t \equiv r (\mathrm{mod} q_3)} G(K_j[t] - tw, \mathbb{Z}v) \leqslant q_1,$$

for a fixed r, where the notation $K_j[t]$ refers to the basis v, w. Naturally, we identify $\mathbb{R}v$ with $\mathbb{R}$, so the collection of all sets $K_{j,t} := K_j[t] - tw$ (of which only a finite number are nonempty) is a collection of compact intervals on $\mathbb{R}$. We have

$$\mathfrak{D}K_{j,t} \cap q_1(\mathbb{Z}v \setminus \{0\}) \subset \mathfrak{D}K_j \cap q_1(\Lambda \setminus \{0\}),$$

which is empty by assumption for all j, so condition (1) of Conjecture 2.15 is satisfied, for the family of convex bodies $K_{j,t}$, the lattice $\mathbb{Z}v$ and the positive integers $q_1 \geqslant q_2 > 0$. Furthermore, when $t \neq t'$, if the intersection

$$(K_{j,t} - K_{j,t'}) \cap q_3(\mathbb{Z}v)$$

is nonempty, then there exists $u \in \mathbb{Z}v$ such that

$$q_3 u + (t-t')w \in K_j[t] - K_j[t'] \subset \mathfrak{D}K_j,$$

implying

$$\mathfrak{D}K_j \cap q_3(\Lambda \setminus \mathbb{Z}v) \neq \varnothing,$$

which provides a contradiction, since $D \leqslant 1$. If $i \neq j$, and if the intersection

$$(K_{i,t} - K_{j,t'}) \cap q_3(\mathbb{Z}v)$$

is nonempty, then there is $u \in \mathbb{Z}v$ such that

$$q_3 u + (t-t')w \in K_i[t] - K_j[t'] \subset K_i - K_j,$$

implying

$$(K_i - K_j) \cap q_3\Lambda \neq \varnothing,$$

which provides another contradiction. Thus, condition (2) is satisfied, and since the 1-dimensional case is true, we have

$$\sum_{j=1}^{n} \sum_{t \equiv r (\mathrm{mod} q_3)} G(K_j[t] - tw, \mathbb{Z}v) \leqslant q_1,$$

as desired.

<u>$D = 2$</u>: This means that there are two primitive, linearly independent vectors of Λ in $\bigcup \mathfrak{D}K_j$, say v, w. We may assume that $q_3 v \in \mathfrak{D}K_i$ and $q_3 w \in \mathfrak{D}K_j$, for

some indices i, j. We must show that $i = j$ (if $n = 1$, this is vacuously true, so we assume $n \geqslant 2$). We have

$$K_i \cap (K_i - q_3 v) \neq \varnothing,$$

so we pick an element x from this intersection. Hence, $x, x + q_3 v \in K_i$ and,

$$K_j \cap (K_j + q_3 w) \neq \varnothing,$$

from which we pick an element y, hence $y, y - q_3 w \in K_j$. Let $\widetilde{\Lambda} = \mathbb{Z}v \oplus \mathbb{Z}w$, and consider the fundamental parallelogram of $q_3\widetilde{\Lambda}$ with vertices $x, x+q_3 v, x+q_3 w, x+q_3(v+w)$, say $\mathcal{P}$. Since $\mathcal{P}$ is a fundamental parallelogram, there is a translate of y by $q_3\widetilde{\Lambda}$ (and hence by $q_3\Lambda$ as well) in $\mathcal{P}$. Without loss of generality, we may assume that $y \in \mathcal{P}$ (if we translate any K_i by an element of $q_3\Lambda$, conditions (1) and (2) still hold). Assume that $y = x + \alpha q_3 v + \beta q_3 w$, where $0 \leqslant \alpha, \beta < 1$. Note that the element

$$y - \beta q_3 w = x + \alpha q_3 v$$

belongs to both $\operatorname{conv}(x, x+q_3 v)$ and $\operatorname{conv}(y, y-q_3 w)$, i.e., the intersection $K_i \cap K_j$ is nonempty. This contradicts condition (2) if $i \neq j$, so we must have $i = j$.

Without loss of generality, assume that $i = 1$, that is, $v, w \in \mathfrak{D}K_1$. Choose v, w so that the index $[\Lambda : \widetilde{\Lambda}]$ is minimal. Assume that $[\Lambda : \widetilde{\Lambda}] > 1$. Then there is a point $q_3 u \in q_3\Lambda$, such that $q_3 u = \mu q_3 v + \nu q_3 w$, with $0 < \mu, \nu < 1$. It is not hard to see that any point in $\mathbb{R}^2$ is congruent modulo $q_3\widetilde{\Lambda}$ to some point in the parallelogram $\operatorname{conv}(\pm q_3 v, \pm q_3 w)$. So, we may assume that $q_3 u \in \operatorname{conv}(\pm q_3 v, \pm q_3 w)$, and by convexity we also have $u \in \mathfrak{D}K_1$. Since $0 < \mu, \nu < 1$, the lattice generated by v, u has strictly smaller index in Λ than $\widetilde{\Lambda}$, contradicting the minimality assumption, therefore we must have $\Lambda = \widetilde{\Lambda}$. By Lemma 2.16 below, there is some $x \in K_1$ such that the boundary of the fundamental parallelogram of $q_3\Lambda$ with vertices $x, x + q_3 v, x + q_3 w, x + q_3(v + w)$ (call it $\mathcal{P}$ again) is a subset of $K_1 + q_3\Lambda$.

By condition (2), all K_j, $j \neq 1$ avoid $K_1 + q_3\Lambda$, and hence the boundary of $\mathcal{P}$. Since one translate of K_j by $q_3\Lambda$ intersects $\mathcal{P}$, as it is a fundamental parallelogram of $q_3\Lambda$, *this translate must lie inside of* $\mathcal{P}$, by convexity since the boundary of $\mathcal{P}$ splits the plane $\mathbb{R}^2$ into two disjoint regions. Thus, all K_j for $j > 1$ satisfy the additional property

$$\mathfrak{D}K_j \cap q_3(\Lambda \setminus \{0\}) = \varnothing.$$

Now, let

$$S = \Big(\bigcup_{j>1} K_j\Big) \cap \Lambda.$$

From the previous identity we get

$$\mathfrak{D}S \cap q_3(\Lambda \setminus \{0\}) = \varnothing,$$

and condition (2) implies

$$(K_1 - S) \cap q_3\Lambda = \varnothing,$$

Therefore, K_1 and S satisfy the conditions of Lemma 2.6, for $r = q_3$, and $d = 2$. So, there is a finite set $S' \subset \Lambda$, obtained from S by translating each element of S with an element of $q_3\Lambda$, satisfying

$$\mathfrak{D}S' \cap q_2(\Lambda \setminus \{0\}) = \varnothing$$

and

$$(K_1 - S') \cap q_2\Lambda = \varnothing,$$

since $q_2 \geqslant q_3$. Then,

$$\sum_{j=1}^{n} G(K_j, \Lambda) = G(K_1, \Lambda) + \#(S') =$$

$$= \sum_{r=0}^{q_2-1} \sum_{t \equiv r (\mathrm{mod} q_2)} G(K_1[t] - te^2, \mathbb{Z}e^1) + \sum_{r=0}^{q_2-1} \sum_{t \equiv r (\mathrm{mod} q_2)} \#(S'[t] - te^2).$$

Here, the notation $K[t]$ refers to the original basis e^1, e^2. It suffices to prove that for fixed r,

$$\sum_{t\equiv r(\mathrm{mod} q_2)} G(K_{1,t}, \mathbb{Z}e^1) + \sum_{t\equiv r(\mathrm{mod} q_2)} \#(S'[t] - te^2) \leqslant q_1.$$

We identify $\mathbb{R}e^1$ with $\mathbb{R}$. Hence, we have a finite collection of nonempty compact intervals, $K_{1,t}$, and some lattice points which come from $S'[t] - te^2$. Assume that $S'[t] - te^2 = \{m_1e^1, \ldots, m_ke^1\}$, where $m_1, m_2, \ldots, m_k$ are distinct integers. Again, we have

$$\mathfrak{D}K_{1,t} \cap q_1(\mathbb{Z}e^1 \setminus \{0\}) \subset \mathfrak{D}K_j \cap q_1(\Lambda \setminus \{0\}) = \varnothing,$$

so condition (1) is satisfied for the intervals $K_{1,t}$ and $m_1e^1, \ldots, m_ke^1$ (it is trivial for a point). If the intersection

$$(K_{1,t} - K_{1,t'}) \cap q_2(\mathbb{Z}e^1)$$

is nonempty for some $t \neq t'$, then there is some $u \in \mathbb{Z}e^1$, such that

$$q_2u + (t - t')e^2 \in K_1[t] - K_1[t'] \subset \mathfrak{D}K_1,$$

which implies (since $q_2|t - t'$)

$$\mathfrak{D}K_1 \cap q_2(\Lambda \setminus \Lambda^1) \neq \varnothing,$$

contradicting condition (1). Furthermore,

$$(K_{1,t} - \{m_ie^1\}) \cap q_2(\mathbb{Z}e^1) \subset (K_1 - S') \cap q_2\Lambda = \varnothing,$$

and for $i \neq j$,

$$\{m_ie^1\} - \{m_je^1\} \cap q_2(\mathbb{Z}e^1) \subset \mathfrak{D}S' \cap q_2\Lambda = \varnothing,$$

so condition (2) holds as well for the intervals $K_{1,t}$ and the points $m_1e^1, m_2e^1, \ldots, m_ke^1$, with respect to the lattice $\mathbb{Z}e^1$ and the integers $q_1 \geqslant q_2$, hence

$$\sum_{t\equiv r(\mathrm{mod} q_2)} G(K_{1,t}, \mathbb{Z}e^1) + \sum_{t\equiv r(\mathrm{mod} q_2)} \#(S'[t] - te^1) \leqslant q_1,$$

as desired, completing the proof.

This implies that inequality (2.2) is true for $d \leqslant 3$. We observe that in order to prove Conjecture 2.15 for $d = 2$, we used the result for $d = 1$. This is exactly the purpose of stating a stronger conjecture than inequality (2.2); we might be able to use induction on the dimension, something that did not seem possible in this inequality. However, when $d > 2$, we need something more than just induction. For $d = 2$, Lemma 2.6 was used, because when $D = 2$, all but one of the K_j must be confined in a fundamental parallelogram. This is not true in higher dimensions in general; perhaps we need a stronger version of Lemma 2.6.

We conclude this section with the following lemma, that was used for the proof of Conjecture 2.15, case $d = 2$:

Lemma 2.16. *Let $K \in \mathcal{K}^2$, and $v^1, v^2 \in \mathbb{R}^2$ two linearly independent vectors such that the intersections $K \cap (K + v^1)$ and $K \cap (K + v^2)$ are nonempty. Then there exists a point $x \in K$ such that the boundary of the parallelogram with vertices $x, x+v^1, x+v^2, x+v^1+v^2$ is contained in $K + \Lambda$, where Λ is the lattice generated by v^1, v^2.*

Proof. From the hypothesis, there is a line parallel to v^1 contained in $K + \mathbb{Z}v^1$, and similarly, a line parallel to v^2 contained in $K + \mathbb{Z}v^2$. Let y be the point of intersection; then the lines parallel to v^1, v^2, passing through y are contained in $K+\Lambda$. The same happens with any lattice translate of y. Pick one such translate that belongs to K, say x. Considering the translates $x + v^1$, $x + v^2$, $x + v^1 + v^2$, we deduce that the union of lines parallel to v^1, v^2 and passing through x, $x+v^1$,

$x+v^2$, $x+v^1+v^2$ is a subset of $K+\Lambda$. It is clear that this union of lines contains the boundary of the fundamental parallelogram with vertices x, $x+v^1$, $x+v^2$, $x+v^1+v^2$, as desired. $\square$

2.5 Reductions of Conjecture 2.2

Two reductions of inequality (2.2) will be given; the first one is a reduction of Conjecture 2.15, while the second one is a certain monotonicity property for the discrete measure that is satisfied by the Lebesgue measure.

2.5.1 A simultaneous translation problem

Observing the proof for the two-dimensional case of Conjecture 2.15, we see that the main technique was projecting onto a certain hyperplane, and then using induction with the result for the one-dimensional case. Can we do this in the general case? In particular, what happens when we consider the projections $K_{j,t} = K_j[t] - te^d$ for $1 \leqslant j \leqslant n$, $t \equiv r(\mathrm{mod} q_d)$, for a fixed r? Do they satisfy conditions (1), (2) of the conjecture, for the lattice Λ^{d-1}, the basis $e^1, \ldots, e^{d-1}$ and the integers $q_1 \geqslant \cdots \geqslant q_d$? Not in general. They do, however, in the special case when q_{d+1} divides q_d. If so, we can replace (2) with the weaker condition

$$(K_j - K_l) \cap q_d\Lambda = \varnothing,$$

simply because $q_d\Lambda$ is a sublattice of $q_{d+1}\Lambda$. Indeed,

$$\mathfrak{D}K_{j,t} \cap q_i(\Lambda^{d-1} \setminus \Lambda^{i-1}) \subset \mathfrak{D}K_j \cap q_i(\Lambda \setminus \Lambda^{i-1}) = \varnothing.$$

For $t \neq t'$, $t \equiv t'(\mathrm{mod} q_d)$, we have

$$\begin{aligned} (K_{j,t} - K_{j,t'}) \cap q_d\Lambda^{d-1} &= (K_j[t] - K_j[t']) \cap (q_d\Lambda^{d-1} + (t-t')e^d) \\ &\subset \mathfrak{D}K_j \cap q_d(\Lambda \setminus \Lambda^{d-1}) = \varnothing, \end{aligned}$$

and for $j \neq l$, $t \equiv t'(\mathrm{mod} q_d)$, we have

$$\begin{aligned}(K_{j,t} - K_{l,t'}) \cap q_d\Lambda^{d-1} &= (K_j[t] - K_l[t']) \cap (q_d\Lambda^{d-1} + (t-t')e^d) \\ &\subset (K_j - K_l) \cap q_d\Lambda = \varnothing.\end{aligned}$$

Hence, as long as q_{d+1} divides q_d, we can apply the induction step, using the projection technique. Given the result of Conjecture 2.15 for $d = 2$, we establish the following:

Theorem 2.17. *Let $K_1, \dots, K_n \subset \mathbb{R}^d$ be convex bodies and $\Lambda \in \mathcal{L}^d$. Also, let $e^1, \dots, e^d$ be a basis of Λ, and denote by Λ^i the $\mathbb{Z}$-span of $0, e^1, \dots, e^i$, and let $q_1 \geqslant q_2 \geqslant \cdots \geqslant q_{d+1}$ be positive integers satisfying*

(1) *$\mathfrak{D}K_j \cap q_i(\Lambda \setminus \Lambda^{i-1}) = \varnothing$ for all $1 \leqslant j \leqslant n$ and $1 \leqslant i \leqslant d$.*

(2) *$(K_j - K_l) \cap q_{d+1}\Lambda = \varnothing$ for all $1 \leqslant j, l \leqslant n$, $j \neq l$.*

(3) *$q_{d+1}|q_d| \cdots |q_3$.*

Then

$$\sum_{j=1}^{n} G(K_j, \Lambda) \leqslant \prod_{i=1}^{d} q_i.$$

Our next objective is to get rid of the successive divisibility property, (3). What happens when q_{d+1} does not divide q_d? We cannot use the same technique anymore, as the projected convex bodies will not always satisfy condition (2). Can we somehow replace q_{d+1} by q_d in condition (2)? We might need to translate the given convex bodies, but we should translate them by a lattice vector, so that the lattice point enumerator remains invariant. We pose the following:

Problem. *Let $K_1, K_2, \dots, K_n$ be convex bodies in $\mathbb{R}^d$, Λ a lattice, and r be a positive integer, such that the following property holds:*

$$(K_i - K_j) \cap r\Lambda = \varnothing,$$

for $i \neq j$, $1 \leqslant i, j \leqslant n$. Given a positive integer $t \geqslant r$, is it true that we can translate each K_i by a lattice vector, thus obtaining the convex bodies $K_1', \dots, K_n'$, so that the following property holds for $i \neq j$, $1 \leqslant i, j \leqslant n$

$$(K_i' - K_j') \cap t\Lambda = \varnothing?$$

It is obvious from the analysis at the beginning of the subsection that if this problem is answered in the affirmative, then it implies Conjecture 2.15, and consequently inequality (2.2) for all dimensions. It should be noted that Lemma 2.6 is a special case of this problem and the case $n = 2$ is covered as a simple consequence of Lemma 2.5. Lastly, the one-dimensional case is trivial, or the case where r divides t. In this case, we do not have to translate the convex bodies at all.

Finally, we state the following corollary to Theorem 2.17, which is a slight improvement of Theorem 2.8:

Corollary 2.18. *Let $K \in \mathcal{K}^d$, $\Lambda \in \mathcal{L}^d$, $q_i = q_i(K, \Lambda)$. Let $n_1, n_2, \dots, n_d$ be a decreasing sequence of positive integers such that*

(1) *$q_i \leqslant n_i$, for $1 \leqslant i \leqslant d$.*

(2) *$n_d | n_{d-1} | \cdots | n_3$.*

Then

$$G(K, \Lambda) \leqslant \prod_{i=1}^{d} n_i.$$

Proof. Let $e^i = e^i(K, \Lambda)$, $\Lambda^i = \Lambda^i(K)$. From the analysis at the beginning of section 2.4, it is clear that the slices $K[t] - te^d$, for $t \equiv r (\mathrm{mod} n_d)$, and numbers $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_d$ satisfy conditions (1), (2), and (3) of Theorem 2.17, whence the desired inequality. □

In particular, inequality (2.2) is verified when $q_d | q_{d-1} | \cdots | q_3$. This shows that the verification of Conjecture 2.15 for $d = 2$ implies that we need not include the first two terms in this successive divisibility property. And it is clear, that if Conjecture 2.15 is proven for, say $d = s$, then inequality (2.2) is verified when $q_d | q_{d-1} | \cdots | q_{s+1}$.

2.5.2 The discrete monotonicity property

In every proof of Minkowski's second theorem, a monotonicity property for the Lebesgue measure is proven in one form or another. For example, Bambah [BWZ65] proves that

$$\mathrm{vol}(tK/L) \geqslant t^{d-i}\,\mathrm{vol}(K/L),$$

where $t \geqslant 1$, $K \in \mathcal{K}^d$, L is a discrete subgroup of $\mathbb{R}^d$ whose rank is equal to i, and $\mathrm{vol}(K/L)$ is the Lebesgue measure of K taken modulo L; i.e., identifying two points of K that are congruent modulo K. The above is equivalent to the assertion that

$$\frac{\mathrm{vol}(K/rL)}{r^i}$$

is decreasing in $r > 0$. This so-called continuous monotonicity property, and holds for all convex bodies K and discrete subgroups L of $\mathbb{R}^d$, unconditionally.

We now state the discrete monotonicity property; we first replace the d-dimensional Lebesgue measure by a discrete measure corresponding to a lattice Λ, so that the measure of a given set A is simply the cardinality of $A \cap \Lambda$. Instead of discrete subgroups of $\mathbb{R}^d$ we consider subgroups of Λ.

Definition. Let $K \in \mathcal{K}^d$, $\Lambda \in \mathcal{L}^d$. We say that K satisfies the **discrete monotonicity property** with respect to Λ, if for any subgroup of Λ, say $\widetilde{\Lambda}$, the following sequence is decreasing in $r > 0$, $r \in \mathbb{Z}$:

$$\frac{D_\Lambda(K, r\widetilde{\Lambda})}{r^i},$$

where i is the rank of $\widetilde{\Lambda}$.

Here $D_\Lambda(K, r\widetilde{\Lambda})$ denotes the cardinality of the set $K \cap \Lambda$ taken modulo $r\widetilde{\Lambda}$. In this setting, we require that r be an integer, because we need $r\widetilde{\Lambda}$ to be a subset of Λ. It is clear that $D_\Lambda(K, r\widetilde{\Lambda})$ is the corresponding quantity of $\mathrm{vol}(K/r\Lambda)$ above. Next we prove the following helpful lemma:

Lemma 2.19. *Let $K \in \mathcal{K}^d$, $\Lambda \in \mathcal{L}^d$, $a^1, \ldots, a^d$ d linearly independent vectors of Λ and*

$$L^i := \mathbb{Z}a^1 \oplus \cdots \oplus \mathbb{Z}a^i.$$

Assume that $\mathfrak{D}K \cap (L^d \setminus L^i) = \varnothing$. Then

$$D_\Lambda(K, L^d) = D_\Lambda(K, L^{d-1}) = \cdots = D_\Lambda(K, L^i).$$

Proof. The hypothesis simply implies that if two points $x, y \in K \cap \Lambda$ are congruent modulo L^d, then they must be congruent modulo L^i, and consequently congruent modulo L^j, for $i \leqslant j \leqslant d$. The lemma then follows from the definition of $D_\Lambda(K, L^i)$. □

Theorem 2.20. *Assume that $K \in \mathcal{K}^d$ satisfies the discrete monotonicity property with respect to $\Lambda \in \mathcal{L}^d$. Then*

$$G(K, \Lambda) \leqslant \prod_{i=1}^{d} q_i(K, \Lambda).$$

Proof. Let $\Lambda^i = \Lambda^i(K)$, for $0 \leqslant i \leqslant d$, and $q_i = q_i(K, \Lambda)$. By Lemma 2.3, we have $\mathfrak{D}K \cap q_i(\Lambda \setminus \Lambda^{i-1})$ for all i, and by the virtue of Lemma 2.19 we have the following series of equalities/inequalities:

$$\begin{aligned}
q_d^d &\geqslant D_\Lambda(K, q_d\Lambda) = D_\Lambda(K, q_d\Lambda^{d-1}) \\
&\geqslant \left(\frac{q_d}{q_{d-1}}\right)^{d-1} D_\Lambda(K, q_{d-1}\Lambda^{d-1}) = \left(\frac{q_d}{q_{d-1}}\right)^{d-1} D_\Lambda(K, q_{d-1}\Lambda^{d-2}) \\
&\vdots \\
&\geqslant \left(\frac{q_d}{q_{d-1}}\right)^{d-1} \left(\frac{q_{d-1}}{q_{d-2}}\right)^{d-2} \cdots \frac{q_2}{q_1} D_\Lambda(K, q_1\Lambda^1) \\
&= \left(\frac{q_d}{q_{d-1}}\right)^{d-1} \left(\frac{q_{d-1}}{q_{d-2}}\right)^{d-2} \cdots \frac{q_2}{q_1} D_\Lambda(K, q_1\Lambda^0) \\
&= \left(\frac{q_d}{q_{d-1}}\right)^{d-1} \left(\frac{q_{d-1}}{q_{d-2}}\right)^{d-2} \cdots \frac{q_2}{q_1} G(K, \Lambda)
\end{aligned}$$

whence

$$G(K, \Lambda) \leqslant \prod_{i=1}^{d} q_i.$$

□

The continuous monotonicity property is proven using the homogeneity of the Lebesgue measure. This property is not valid for the discrete measure, so we expect that it might be very difficult to prove the discrete monotonicity property for all convex bodies and lattices.

CHAPTER 3

Davenport's Problem

We will present a notable generalization of Minkowski's theorems on successive minima, namely Davenport's problem. We expand upon the attempts to solve Davenport's problem to date, in order to introduce some interesting notions, central to the geometry of numbers, as well as to emphasize the similarity between the methods employed herein and those used for Conjecture 2.2. Along the way, we provide an independent proof for some results related to Davenport's problem, originally attributed to Chabauty and Rogers.

3.1 Statement of the problem

Before stating the problem, we need some basic definitions.

Definition. Let $K \in \mathcal{K}_0^d$. A lattice Λ is called a packing lattice for K, if two different translates v_1+K, v_2+K, $v_1 \neq v_2 \in \Lambda$, have no interior points in common, or equivalently, if $\mathrm{int}(2K) \cap \Lambda = \{0\}$. We denote by $\delta(K, \Lambda)$ the density of the non-overlapping arrangement $K + \Lambda$, which is the proportion of space occupied by all the translates of K by points of Λ, given by

$$\delta(K, \Lambda) = \frac{\mathrm{vol}(K)}{d(\Lambda)}.$$

The supremum of all such densities as Λ ranges over the packing lattices of K, is called the density of a densest lattice packing of K, and is denoted by $\delta(K)$.

Since $\delta(K) \leqslant 1$, the following theorem provides a stronger version of Theorem 1.1 originally proven by Minkowski [Min96]:

Theorem 3.1. *Let $K \in \mathcal{K}_0^d$, $\Lambda \in \mathcal{L}^d$. Then*

$$\mathrm{vol}(K) \leqslant \delta(K) d(\Lambda) \left(\frac{2}{\lambda_1(K, \Lambda)} \right)^d. \tag{3.1}$$

Proof. By definition of the first successive minimum, Λ is a packing lattice for the body $(\lambda_1(K, \Lambda)/2)K$. Indeed, since $\mathrm{int}(\lambda_1 K) \cap \Lambda = \{0\}$. Therefore

$$\delta(K) \geqslant \frac{\mathrm{vol}(\frac{\lambda_1}{2} K)}{d(\Lambda)},$$

or equivalently

$$\mathrm{vol}(K) \leqslant \delta(K) d(\Lambda) \left(\frac{2}{\lambda_1(K, \Lambda)} \right)^d.$$

□

Can we replace $(2/\lambda_1)^d$ by the product of $2/\lambda_i$, as with Minkowski's first and second theorem on successive minima? This is the statement of Davenport's problem [Dav49].

Problem. *Let $K \in \mathcal{K}_0^d$, $\Lambda \in \mathcal{L}^d$. Then*

$$\mathrm{vol}(K) \leqslant \delta(K) d(\Lambda) \prod_{i=1}^{d} \frac{2}{\lambda_i(K, \Lambda)}. \tag{3.2}$$

Minkowski [Min96] proved the case $d = 2$, as well as the general result for the special class of ellipsoids whereas Woods [Woo56] tackled the case $d = 3$. We provide proofs for the two-dimensional case, as well as for ellipsoids, in the subsequent sections.

In the general case, Rogers [Rog49] proved that inequality (3.2) holds up to a factor of $2^{(d-1)/2}$, and around the same time, Chabauty [Cha49] improved this factor to $2^{\frac{d-1}{2} - \frac{1}{d}}$.

3.2 The anomaly of a convex body

Inequality (3.2) can also be written as

$$d(L) \leqslant d(\Lambda) \prod_{i=1}^{d} \frac{2}{\lambda_i(K, \Lambda)},$$

where $L \in \mathcal{L}^d$ is a densest packing lattice for K. It is natural to consider the infimum of the right hand side, or equivalently:

Definition. Let $K \in \mathcal{K}_0^d$, $L \in \mathcal{L}^d$, such that L is a packing lattice for K. Then the supremum of the quantity

$$\frac{d(L)}{d(\Lambda)} \prod_{i=1}^{d} \frac{\lambda_i(K, \Lambda)}{2}$$

as Λ ranges over all lattices, is called the anomaly of K, and is denoted by $\alpha(K)$.

A consequence of the above definition is the following corollary.

Corollary 3.2. *Let $K \in \mathcal{K}_0^d$, $\Lambda \in \mathcal{L}^d$. Then*

$$\mathrm{vol}(K) \leqslant \alpha(K)\delta(K)d(\Lambda) \prod_{i=1}^{d} \frac{2}{\lambda_i(K, \Lambda)}.$$

Therefore, inequality (3.2) is equivalent to $\alpha(K) \leqslant 1$. Since we always have $\alpha(K) \geqslant 1$, the following theorem shows that (3.2) is equivalent to $\alpha(K) = 1$.

Theorem 3.3. *Let $K \in \mathcal{K}_0^d$. Then*

$$1 \leqslant \alpha(K) \leqslant 2^{\frac{d-1}{2}}.$$

Proof. Let L be a densest lattice packing for K. Since $\mathrm{int}(2K) \cap L = \{0\}$, we have $\lambda_1(K, L) \geqslant 2$ (one can show that equality holds, but this is unnecessary here). Thus,

$$\alpha(K) \geqslant \prod_{i=1}^{d} \frac{\lambda_i(K, L)}{2} \geqslant 1.$$

Now consider an arbitrary, but fixed, lattice Λ. Let $\mu_1, \mu_2, \ldots, \mu_d$ be a sequence of positive reals such that $\frac{\mu_i}{\mu_{i+1}} \in \mathbb{Z}$, $\mu_i \geqslant 2/\lambda_i$, for all i, where $\lambda_i = \lambda_i(K, \Lambda)$, and

$$\prod_{i=1}^{d} \frac{\mu_i \lambda_i}{2} \leqslant 2^{\frac{d-1}{2}}.$$

Such a choice is possible by Lemma 2.12. Let $e^i = e^i(K, \Lambda)$ form a basis of Λ as in section 2.2. Consider the lattice

$$\widetilde{\Lambda} = \mathbb{Z}\mu_1 e^1 \oplus \cdots \oplus \mathbb{Z}\mu_d e^d.$$

We will prove that $\widetilde{\Lambda}$ is a packing lattice for K. It suffices to prove that

$$\operatorname{int}(2K) \cap \left(\widetilde{\Lambda} \setminus \{0\}\right) = \varnothing.$$

This follows from Lemma 2.3, and the fact that

$$\widetilde{\Lambda} \setminus \{0\} \subset \bigcup_{i=1}^{d} \mu_i \left(\Lambda \setminus \Lambda^{i-1}\right),$$

which follows from the inverse successive divisibility property of the μ_i's, exactly as in the proof of Theorem 2.8. By definition of L we have

$$d(L) \leqslant d(\widetilde{\Lambda}) = d(\Lambda) \prod_{i=1}^{d} \mu_i \leqslant 2^{\frac{d-1}{2}} d(\Lambda) \prod_{i=1}^{d} \frac{2}{\lambda_i}$$

or equivalently,

$$\frac{d(L)}{d(\Lambda)} \prod_{i=1}^{d} \frac{\lambda_i}{2} \leqslant 2^{\frac{d-1}{2}}$$

for all lattices Λ. Taking the supremum of the left hand side, we attain the inequality

$$\alpha(K) \leqslant 2^{\frac{d-1}{2}}.$$

$\square$

Corollary 3.4. *Let $K \in \mathcal{K}_0^d$, $\Lambda \in \mathcal{L}^d$. Then*

$$\mathrm{vol}(K) \leqslant 2^{\frac{d-1}{2}} \delta(K) d(\Lambda) \prod_{i=1}^{d} \frac{2}{\lambda_i(K,\Lambda)}.$$

From the proof of Theorem 3.3 we can also deduce:

Proposition 3.5. *Let $K \in \mathcal{K}_0^d$, $\Lambda \in \mathcal{L}^d$, such that $\lambda_1 | \lambda_2 | \cdots | \lambda_d$. Then K, Λ satisfy inequality (3.2).*

In particular, 3.2 is satisfied when all the successive minima are equal.

3.3 The projective closure property

Definition. Let $K \in \mathcal{K}_0^d$. We say that K satisfies the projective closure property if for each linear subspace V, there is a linear complement W (i.e., $\mathbb{R}^d = V \oplus W$) such that the projection of K on W along V is a subset of K. If for a specific subspace V there exists such a subspace W, we will say that the pair K is projectively closed with respect to V.

We should note that the projective closure property is invariant under non-singular linear transformations. Furthermore, the quantities $\delta(K)$ and $\alpha(K)$ are also invariant under the action of $\mathrm{GL}_n(\mathbb{R})$, however the successive minima are not invariant (i.e., $\lambda_i(K,\Lambda)$ is not necessarily equal to $\lambda_i(TK,\Lambda)$). This means that we can reduce to a more manageable convex body via a linear transformation, in order to produce bounds for the anomaly of a certain convex body.

Theorem 3.6. *Let $K \in \mathcal{K}_0^d$, $\Lambda \in \mathcal{L}^d$. Assume that $K \in \mathcal{K}_0^d$ satisfies the projective closure property. Then K, Λ satisfy inequality (3.2).*

Proof. Let i be the maximal index such that $\lambda_1(K) = \lambda_i(K)$. As noted before, the case $i = d$ is already known, so we may assume $i < d$. Also, let $a^i = a^i(K,\Lambda)$

be defined as in section 2.2. Let V be the linear subspace spanned by $a^1, \dots, a^i$. Since K satisfies the projective closure property, there is a linear complement W of V, such that the projection of K on W along V is a subset of K.

In particular, this implies that for a real $t > 1$ and $T \in \mathrm{GL}_n(\mathbb{R})$ defined by $T(v+w) = v + tw$, when $v \in V$, $w \in W$; then $T \cdot K \subset tK$, for all such t. Indeed, let $v + w \in K$ be arbitrary, where $v \in V$, $w \in W$. By definition of W, we have $w \in K$, therefore, $tv + tw, tw \in tK$, and by convexity of K and the fact that $t > 1$, we have $T(v+w) = v + tw \in K$. Since $v + w$ is arbitrary, we obtain $T \cdot K \subset tK$.

For such T, put $K_t = T \cdot K$. Now, let t be the least positive real, greater than 1 with

$$\dim((\lambda_i(K, \Lambda)K_t) \cap \mathbb{Z}^d) \geqslant i + 1,$$

where we always consider the 0-symmetric convex body K_t, with respect to the decomposition V, W. We remind the reader that $i = \dim V$. Since $\lambda_i(K, \Lambda) < \lambda_{i+1}(K, \Lambda)$ we must have $t > 1$.

Next, we will try to compute the successive minima of K_t, especially the first $i + 1$ minima. The inclusion $K_t \subset tK$ implies the following inequalities for all j:

$$t^{-1}\lambda_j(K, \Lambda) \leqslant \lambda_j(K_t, \Lambda).$$

Now, we will prove that $\lambda_i(K, \Lambda) = \lambda_1(K_t, \Lambda) = \lambda_{i+1}(K_t, \Lambda)$, therefore $\lambda_j(K, \Lambda) = \lambda_j(K_t, \Lambda)$ for all $1 \leqslant j \leqslant i$, and hence

$$\lambda_1(K_t, \Lambda) = \cdots = \lambda_{i+1}(K_t, \Lambda).$$

We do so by proving that $\mathrm{int}(\lambda_i(K, \Lambda)K_t) \cap \mathbb{Z}^d = \{0\}$. Let $y \in \mathbb{Z}^d$ be such that $y \in \mathrm{int}(\lambda_i(K, \Lambda)K_t)$. If $y \in V$, then $y \in \mathrm{int}(\lambda_i(K, \Lambda)K)$, because $K \cap V = K_t \cap V$. Since $\lambda_1(K, \Lambda) = \lambda_i(K, \Lambda)$, we must have $y = 0$. If $y \notin V$, then the result follows from the minimality assumption on t. Assuming otherwise, $y \in \operatorname{int}(\lambda_i(K,\Lambda)K_t)$, there exist $v \in V$, $w \in W$ such that $v + w \in \lambda_i(K,\Lambda)K$ and $y = v + tw$. Since $y \in \operatorname{int}(\lambda_i(K,\Lambda)K_t)$, there is some real r with $t < r < t^2$ such that $v + rw \in \operatorname{int}(\lambda_i(K,\Lambda)K_t)$, hence $y = v + tw \in \operatorname{int}(\lambda_i(K,\Lambda)K_s)$, where $1 < s = t^2/r < t$, contradicting the minimality of t. This proves that $\lambda_i(K,\Lambda) \leqslant \lambda_1(K_t,\Lambda)$. Furthermore, $\lambda_i(K,\Lambda) \geqslant \lambda_{i+1}(K_t,\Lambda)$ follows from the fact that

$$\dim((\lambda_i(K,\Lambda)K_t) \cap \mathbb{Z}^d) \geqslant i+1,$$

concluding that $\lambda_i(K,\Lambda) = \lambda_1(K_t,\Lambda) = \lambda_{i+1}(K_t,\Lambda)$, thus

$$\lambda_1(K_t,\Lambda) = \cdots = \lambda_{i+1}(K_t,\Lambda).$$

The fact that $\lambda_j(K,\Lambda) = \lambda_j(K_t,\Lambda)$ for all $1 \leqslant j \leqslant i$ and (2) implies that

$$t^{-\dim(W)}\delta(K_t)\prod_{j=1}^{d}\frac{2}{\lambda_j(K_t,\Lambda)} \leqslant \delta(K)\prod_{j=1}^{d}\frac{2}{\lambda_j(K,\Lambda)} \tag{3.3}$$

and

$$\operatorname{vol}(K) = t^{-\dim(W)}\operatorname{vol}(K_t). \tag{3.4}$$

We deduce from (3.3) and (3.4) that if inequality (3.2) holds for K_t, then it must also hold for K. However, the maximal index for which $\lambda_1(K_t,\Lambda) = \lambda_j(K_t,\Lambda)$ is strictly greater than that for K (which is equal to i), which shows that the problem reduces to the case where all successive minima are equal, which is covered by Proposition 3.5, so we are done. $\square$

Consequently, any $K \in \mathcal{K}_0^d$ satisfying the projective closure property, also satisfies $\alpha(K) = 1$. The next theorem yields Minkowski's result, which is inequality (3.2) for ellipsoids.

Theorem 3.7. *All ellipsoids centered at the origin satisfy the projective closure property.*

Proof. Since the validity of this property is invariant under the action of $\mathrm{GL}_n(\mathbb{R})$, it suffices to prove the above statement for the unit ball, B. Take V to be any linear subspace of $\mathbb{R}^d$, and take W to be its orthogonal complement. Let $v + w \in B$, where $v \in V$ and $w \in W$. By the Pythagorean theorem,

$$\|w\|^2 \leqslant \|v\|^2 + \|w\|^2 = \|v + w\|^2 \leqslant 1,$$

which shows that $w \in B$ (where $\|\cdot\|$ is the euclidean norm). This completes the proof. $\square$

Corollary 3.8. *All ellipsoids satisfy inequality (3.2), and have anomaly 1.*

3.4 Chabauty's result

Chabauty's result [Cha49] is a consequence of the next proposition and a generalization of Rogers Lemma 2.12 to a more general setting.

Proposition 3.9. *Let $K \in \mathcal{K}_0^d$ and V a linear subspace of $\mathbb{R}^d$ of dimension $d-1$. Then K is projectively closed with respect to V.*

Proof. Pick an arbitrary vector y not in V. Since K is compact, the supremum over all positive t for which $ty + V$ intersects K is finite. Furthermore, if we denote by s this supremum, $sy + V$ must intersect K by the compactness of K. Let w be in the intersection of $sy + V$ and K. Since K is d-dimensional (i.e., 0 lies in its interior), we have $s > 0$ and $w \notin V$. Furthermore, $w + V$ has no point in common with the interior of K; in fact, $tw + V$ intersects K if and only if $|t| \leqslant 1$. Let W be the linear subspace spanned by w. We will prove that W is the desired linear complement of V. To this end, we consider $v+u \in K$, where $v \in V$ and $u \in W$. Since $u + V$ intersects K, we must have $u = tw$ where $|t| \leqslant 1$. By convexity and symmetry of K, we deduce that $w \in K$, completing the proof. $\square$

Letting V be the linear subspace spanned by the vectors $a^i(K,\Lambda)$, $1 \leqslant i \leqslant d-1$, we can use the technique in the proof of Theorem 3.6 to reduce Davenport's problem to the case where $\lambda_{d-1}(K,\Lambda) = \lambda_d(K,\Lambda)$. Now, as in the proof of Theorem 3.3, we consider the packing lattice $\widetilde{\Lambda}$ for K which is spanned by the vectors $\mu_1 e^1, \ldots, \mu_d e^d$, where $\mu_1, \ldots, \mu_d$ are positive reals such that $\mu_d | \mu_{d-1} | \cdots | \mu_1$, $\mu_i \geqslant 2/\lambda_i$ for all i, and the product

$$\prod_{i=1}^{d} \frac{\lambda_i \mu_i}{2} \tag{3.5}$$

is as small as possible. Using Lemma 2.12, we were able to bound this product by $2^{(d-1)/2}$. But since $\lambda_{d-1} = \lambda_d$, we should be able to find a better bound. The value of this improved bound follows from the following generalization of Lemma 2.12.

Lemma 3.10. *Let $m_1, \ldots, m_n$ be fixed positive integers. Let $c(m_1, \ldots, m_n)$ denote the least positive real c with the following property: for each sequence of real numbers $0 < x_1 \leqslant \ldots \leqslant x_n$ there is another sequence $y_1, \ldots, y_n$ such that $x_i \leqslant y_i$, $y_{i+1}/y_i \in \mathbb{Z}$ for all i, and*

$$\prod_{i=1}^{d} \left(\frac{y_i}{x_i}\right)^{m_i} \leqslant c.$$

Then,

$$\log_2 c(m_1, \ldots, m_n) \leqslant \frac{\left(\sum_{i=1}^{n} m_i\right)^2 - \sum_{i=1}^{n} m_i^2}{2\sum_{i=1}^{n} m_i}.$$

Proof. We adapt the proof of Lemma 2.12 in this general case. Again, for each i we construct the sequence $y_1^i, \ldots, y_d^i$ that satisfies

$$y_i^i = x_i, \quad y_j^i = 2^{a_{ij}} x_i, \text{ for } j \neq i \text{ where } a_{ij} = -\left\lfloor \log_2 \frac{x_i}{x_j} \right\rfloor.$$

a_{ij} is the unique integer satisfying

$$x_j \leqslant 2^{a_{ij}} x_i < 2x_j.$$

Therefore,

$$\log_2 \frac{y_j^i}{x_j} = \{\log_2 x_i - \log_2 x_j\},$$

for all j, so

$$m_i \log_2 \prod_{j=1}^{n} \left(\frac{y_j^i}{x_j}\right)^{m_j} = \sum_{j=1}^{n} m_i m_j \{\log_2 x_i - \log_2 x_j\}.$$

Summing over all i, we obtain

$$\sum_{i=1}^{n} m_i \log_2 \prod_{j=1}^{d} \left(\frac{y_j^i}{x_j}\right)^{m_j} = \sum_{i,j=1}^{n} m_i m_j \{\log_2 x_i - \log_2 x_j\}.$$

For any pair, (i,j) with $i \neq j$, $\{\log_2 x_i - \log_2 x_j\} + \{\log_2 x_j - \log_2 x_i\} \leqslant 1$ (for $i = j$ it vanishes). Hence

$$\sum_{i=1}^{n} m_i \log_2 \prod_{j=1}^{n} \left(\frac{y_j^i}{x_j}\right)^{m_j} \leqslant \sum_{1 \leqslant i < j \leqslant n} m_i m_j = \frac{1}{2}\left(\left(\sum_{i=1}^{n} m_i\right)^2 - \sum_{i=1}^{n} m_i^2\right).$$

Hence, there is an index i such that

$$\log_2 \prod_{j=1}^{n} \left(\frac{y_j^i}{x_j}\right)^{m_j} \leqslant \frac{\left(\sum_{i=1}^{n} m_i\right)^2 - \sum_{i=1}^{n} m_i^2}{2\sum_{i=1}^{n} m_i}.$$

Since the increasing sequence $x_1, \ldots, x_n$ is arbitrary, we obtain the desired inequality. $\square$

Now we can establish:

Proposition 3.11. ***(Chabauty)*** *For all $K \in \mathcal{K}_0^d$ we have*

$$\alpha(K) \leqslant 2^{\frac{d-1}{2} - \frac{1}{d}}.$$

Proof. Returning to the argument preceding Lemma 3.10, we can see that we can bound the product in (3.5) by $c(\underbrace{1, \ldots, 1}_{d-2 \text{ times}}, 2)$, which by Lemma 3.10 is bounded above by $2^{(d-1)/2 - 1/d}$. Therefore, as in the proof of Theorem 3.6, we obtain

$$\alpha(K) \leqslant 2^{\frac{d-1}{2} - \frac{1}{d}}.$$

$\square$

This immediately proves inequality (3.2) for the planar case (which is due to Minkowski [Min96]):

Corollary 3.12. *If* $K \in \mathcal{K}_0^2$*. Then* $\alpha(K) = 1$*.*

Proof. The result follows by setting $d = 2$ in Proposition 3.11. $\square$

APPENDIX A

A proof of Minkowski's second theorem on successive minima

We will present a proof of Minkowski's second theorem on successive minima, motivated by the ideas in Chapter 2. From a different point of view, we could say that all the ideas within that chapter, are an attempt to "discretize" this proof.

In the continuous case, many of the arguments presented in Chapter 2 are easier to show. We begin with:

Proposition A.1. *Let $K_1, \dots, K_n \in \mathcal{K}^d$, $\Lambda \in \mathcal{L}^d$, $0 < r < t \in \mathbb{R}$. Assume that*

$$(\mathrm{int}(K_i) - \mathrm{int}(K_j)) \cap r\Lambda = \varnothing \tag{A.1}$$

whenever $1 \leqslant i < j \leqslant n$. Then there are $v^1, \dots, v^d \in \mathbb{R}^d$ such that the convex bodies $K_i' = K_i + v^i$, $1 \leqslant i \leqslant d$, satisfy

$$(\mathrm{int}(K_i') - \mathrm{int}(K_j')) \cap t\Lambda = \varnothing. \tag{A.2}$$

Proof. By hypothesis, we have

$$(\mathrm{int}(\tfrac{t}{r}K_i) - \mathrm{int}(\tfrac{t}{r}K_j)) \cap t\Lambda = \varnothing, \tag{A.3}$$

rescaling (A.1) by t/r. Pick arbitrary points $x^1, \dots, x^n$ in $K_1, \dots, K_n$ respectively. Then,

$$K_i - x^i \subset \tfrac{t}{r}(K_i - x^i),$$

hence

$$K_i + \left(\tfrac{t}{r} - 1\right) x^i \subset \frac{t}{r} K_i,$$

so the vectors $v^i = (\frac{t}{r} - 1)x^i$ satisfy the required properties, as can be seen from (A.3). $\square$

Next, we will prove the continuous counterpart of Conjecture 2.15. We will use the same approach as in the subsection 2.5.1, with the notable difference that this approach works specifically because of Proposition A.1.

Theorem A.2. *Let $K_1, \ldots, K_n \subset \mathbb{R}^d$ be convex bodies. Also, let*

$$e^i = (0, \ldots, 0, \underbrace{1}_{ith\ coordinate}, 0, \ldots, 0)$$

with $1 \leqslant i \leqslant d$ be the standard basis of $\Lambda = \mathbb{Z}^d$, and denote by Λ^i the $\mathbb{Z}$-span of $0, e^1, \ldots, e^i$, and let $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_{d+1}$ be positive real numbers satisfying

(1) $\mathfrak{D}(\mathrm{int}(K_j)) \cap \mu_i(\Lambda \setminus \Lambda^{i-1}) = \varnothing$ *for all* $1 \leqslant j \leqslant n$ *and* $1 \leqslant i \leqslant d$.

(2) $(\mathrm{int}(K_j) - \mathrm{int}(K_l)) \cap \mu_{d+1}\Lambda = \varnothing$ *for all* $1 \leqslant j, l \leqslant n$, $j \neq l$.

Then

$$\sum_{j=1}^{n} \mathrm{vol}(K_j) \leqslant \prod_{i=1}^{d} \mu_i.$$

Proof. By virtue of Proposition A.1 and the fact that $\mu_d \geqslant \mu_{d+1}$, we can substitute condition (2) by

$$(\mathrm{int}(K_j) - \mathrm{int}(K_l)) \cap \mu_{d+1}\Lambda = \varnothing$$

for all $1 \leqslant j, l \leqslant n$, $j \neq l$. In other words, we may assume without loss of generality that $\mu_d = \mu_{d+1}$. Notice that condition (1) still remains true under translation. We now use the notation of section 2 to deduce the formula

$$\sum_{j=1}^{n} \mathrm{vol}(K_j) = \int_0^{\mu_d} \sum_{j=1}^{n} \sum_{t \equiv r (\mathrm{mod} \mu_d)} \mathrm{vol}_{d-1}(K_{j,t}) dr$$

where vol_{d-1} is the $d-1$-dimensional Lebesgue measure, where we identify $\mathbb{R}^{d-1}$ by the vector subspace spanned by $e^1, \ldots, e^{d-1}$. As we did in subsection 2.5.1, it is easy to verify conditions (1) and (2) for the projected slices $K_{j,t}$, where $t \equiv r (\mathrm{mod} \mu_d)$ for a fixed r, the positive reals $\mu_1 \geqslant \cdots \geqslant \mu_d$ and the basis $e^1, \ldots, e^{d-1}$ of $\mathbb{R}^{d-1}$. Hence,

$$\sum_{j=1}^{n} \sum_{t \equiv r (\mathrm{mod} \mu_d)} \mathrm{vol}_{d-1}(K_{j,t}) \leqslant \prod_{i=1}^{d-1} \mu_i$$

for all r, therefore

$$\sum_{j=1}^{n} \mathrm{vol}(K_j) \leqslant \prod_{i=1}^{d} \mu_i,$$

as desired. It only remains to prove the case $d = 1$, but this is trivial, as in the case of the Conjecture 2.15. $\square$

We can deduce Minkowski's second theorem, by applying Theorem A.2 to an arbitrary convex body $K \in \mathcal{K}^d$, and $\Lambda = \mathbb{Z}^d$. We put $n = 1$, $K_1 = K$, and without loss of generality we further assume that we can choose $e^i = e^i(K, \Lambda)$ to be the standard basis of $\mathbb{Z}^d$. Condition (2) is redundant when $n = 1$, and we can verify condition (1) with $\mu_i = 2/\lambda_i(K)$, $1 \leqslant i \leqslant d$, and $\mu_{i+1} = \mu_i$. Theorem A.2 thus yields:

$$\mathrm{vol}(K) \leqslant \prod_{i=1}^{d} \frac{2}{\lambda_i(K)}.$$

Notice that this method deals simultaneously with the non-symmetric setting. Furthermore, we can always assume that the e^i form a standard basis, since the ratio

$$\frac{\mathrm{vol}(K)}{d(\Lambda)}$$

is invariant under the action of $\mathrm{GL}(d, \mathbb{R})$; i.e.,

$$\frac{\mathrm{vol}(K)}{d(\Lambda)} = \frac{\mathrm{vol}(TK)}{d(T\Lambda)}$$

for all $T \in \mathrm{GL}(d, \mathbb{R})$, and so are the successive minima; i.e.,

$$\lambda_i(K, \Lambda) = \lambda_i(TK, T\Lambda),$$

for all i and $T \in \mathrm{GL}(d, \mathbb{R})$.

References

[BHW93] Ulrich Betke, Martin Henk, and Jörg Wills. "Successive-minima-type inequalities." *Discrete Comput. Geom.*, **9**(2):165–175, 1993.

[BWZ65] R. P. Bambah, A. C. Woods, and H. Zassenhaus. "Three proofs of Minkowski's second inequality in the geometry of numbers." *J. Aust. Math. Soc.*, **5**:453–462, 1965.

[Cha49] Claude Chabauty. "Sur les minima arithmétiques des formes." *Ann. Sci. cole Norm. Sup. (3)*, **66**:367–394, 1949.

[Dav49] Harold Davenport. "The product of n homogeneous linear forms." *Indag. Math.*, **8**:525–531, 1949.

[Gau09] Éric Gaudron. "Géometrie des nombres adélique et lemmes de Siegel generalisés." *Manuscripta Math.*, **130**(2):159–182, 2009.

[GL87] P. M. Gruber and C. G. Lekkerkerker. *Geometry of numbers*, volume 37 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, second edition, 1987.

[Hen02] Martin Henk. "Successive minima and lattice points." *Rend. Circ. Mat. Palermo (2) Suppl.*, **70**(part I):377–384, 2002.

[Mal09a] Romanos-Diogenes Malikiosis. "A discrete analogue for Minkowski's second theorem on successive minima." http://arxiv.org/abs/1001.3729, 2009.

[Mal09b] Romanos-Diogenes Malikiosis. "An optimization problem related to Minkowski's successive minima." *Discrete Comput. Geom.*, 2009. available online.

[Min96] Hermann Minkowski. *Geometrie der zahlen.* Teubner, Leipzig-Berlin, 1896.

[Rog49] C.A. Rogers. "The product of the minima and the determinant of a set." *Proc. K. Ned. Akad. Wet.*, **52**:256–263, 1949.

[Woo56] A. C. Woods. "The anomaly of convex bodies." *Proc. Cambridge Philos. Soc.*, **52**:406–423, 1956.