Entropies, logarithmic Sobolev inequalities, Differential Harnack inequalities and applications in geometry (Heraklion 2013)

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References:
- Perelman, Nov. 2002 arxiv
- Kleiner – Lott, Notes on Perelman
- Gross, Log. Sob. inequalities Am. J. Math 97 1975
- Ecker, A formula relating entropy monotonicity ...., Comm. Geom. Analysis 15 2008
- ..., Entropy and diff. Harnack type calcul. ..., to apper in Calc. Var. & PDE

website: Klaus Ecker, FU Berlin, Geometric Analysis

Perelman (X, g) closed n-dim. Riem. mfd

$\tau > 0$, $f: X \to \mathbb{R}$ smooth

$$W(g, f, \tau) := \int_X \left[\tau\left(|\nabla f|_g^2 + R_g\right) + f - n\right] u \, dV_g$$

$R_g \mathrel{\hat{=}}$ scalar curvature, $u = \dfrac{e^{-f}}{(4\pi\tau)^{n/2}}$

$$\mu(g, \tau) := \inf\left\{ W(g, f, \tau), \int_X u \, dV_g = 1 \right\}$$

Perelman's entropy

Famous result due to Perelman (consequences explained later)

→

Ricci Flow $\quad \dfrac{\partial g}{\partial t} = -2\,\mathrm{Ric}(g)$

$$\frac{\partial \tau}{\partial t} = -1$$

$$\Longrightarrow \quad \frac{d}{dt}\,\mu\big(g(t), \tau(t)\big) \geq 0$$

If (X, g) fixed, $\mathrm{Ric}(g) \geq 0$ and

$$\left(\frac{\partial}{\partial t} + \Delta_g\right) f = |\nabla f|_g^2 + \frac{n}{2\tau}, \quad \frac{\partial \tau}{\partial t} = -1$$

$$\left(\Longleftrightarrow \left(\frac{\partial}{\partial t} + \Delta_g\right) u = 0 \right)$$

then $\dfrac{d}{dt} W\big(g, f(t), \tau(t)\big) \geq 0$.

Otto, Otto – Villani, Lott – Villani

Understanding $W(g, f, \tau)$:

$X = \mathbb{R}^n$

$g_{ij} = \delta_{ij}$

$R \equiv 0$

$$u = \frac{e^{-f}}{(4\pi\tau)^{n/2}} = \varphi^2 \, \frac{e^{-\frac{|x|^2}{4\tau}}}{(4\pi\tau)^{n/2}}$$

$\xrightarrow[\text{calculation done below}]{}$ $\quad W(\mathbb{R}^n, f, \tau) = \mathcal{E}(\varphi, \tau)$

where

$$\mathcal{E}(\varphi, \tau) = \int_{\mathbb{R}^n} \left(4\tau |\nabla \varphi|^2 - \varphi^2 \log \varphi^2\right) \frac{e^{-\frac{|x|^2}{4\tau}}}{(4\pi\tau)^{n/2}}$$

$\tau = 1$:

$$\mathcal{E}(\varphi) := \mathcal{E}\left(\varphi, \tfrac{1}{2}\right) = \int_{\mathbb{R}^n} \left(2|\nabla \varphi|^2 - \varphi^2 \log \varphi^2\right) \gamma_n$$

where $\gamma_n(x) = \dfrac{e^{-\frac{|x|^2}{2}}}{(2\pi)^{n/2}}$ = Gauß distribution ③

Logarithmic Sobolev inequality (Gross)
(optimal form)

$$\int_{\mathbb{R}^n} \varphi^2 \log \varphi^2 \, \gamma_n \le 2 \int_{\mathbb{R}^n} |\nabla \varphi|^2 \gamma_n$$

$\forall \varphi$ with $\int_{\mathbb{R}^n} \varphi^2 \gamma_n = 1$

i.e. $\mathcal{E}(\varphi) \ge 0$.

"=" holds $\iff$ $\varphi^2 \equiv 1$.

Remark (Exercise) $x = \sqrt{2\tau}\, y$

$\mathcal{E}(\varphi, \tau) \ge 0$ $\forall \varphi$ with $\int_{\mathbb{R}^n} \varphi^2(x) \dfrac{e^{-\frac{|x|^2}{4\tau}}}{(4\pi\tau)^{n/2}} dx = 1$

$\iff$ $\mathcal{E}(\varphi) \ge 0$ $\forall \varphi$ with $\int_{\mathbb{R}^n} \varphi^2(y) \gamma_n(y)\, dy = 1$

This says

$\mu(\mathbb{R}^n, \tau) = 0$ $\forall \tau > 0$

using Perelman's notion of entropy.

Calculation for

$$W(\mathbb{R}^n, f, \tau) = \mathcal{E}(\psi, \tau):$$

$$e^{-f} = \psi^2 e^{-\frac{|x|^2}{4\tau}} \iff f = \frac{|x|^2}{4\tau} - \log \psi^2$$

$$\Rightarrow \nabla f = \frac{x}{2\tau} - \frac{1}{\psi^2} \nabla \psi^2$$

$$|\nabla f|^2 = \frac{|x|^2}{4\tau^2} + \frac{4}{\psi^2} |\nabla \psi|^2 - \frac{x}{\tau \psi^2} \cdot \nabla \psi^2$$

$\Rightarrow$

$$\tau |\nabla f|^2 u = \left(4\tau |\nabla \psi|^2 + \frac{|x|^2}{4\tau} \psi^2 - x \cdot \nabla \psi^2 \right) \frac{e^{-\frac{|x|^2}{4\tau}}}{(4\pi\tau)^{n/2}}$$

$$f u = \left(\frac{|x|^2}{4\tau} \psi^2 - \psi^2 \log \psi^2 \right) \frac{e^{-\frac{|x|^2}{4\tau}}}{(4\pi\tau)^{n/2}}$$

$\Rightarrow$

$$W(\mathbb{R}^n, f, \tau) = \int_{\mathbb{R}^n} \left(\tau |\nabla f|^2 + f - n \right) u$$

$$= \int_{\mathbb{R}^n} \left(4\tau |\nabla \psi|^2 - \psi^2 \log \psi^2 \right) \frac{e^{-\frac{|x|^2}{4\tau}}}{(4\pi\tau)^{n/2}}$$

$$- \int_{\mathbb{R}^n} n \psi^2 \frac{e^{-\frac{|x|^2}{4\tau}}}{(4\pi\tau)^{n/2}} + \int_{\mathbb{R}^n} \frac{|x|^2}{2\tau} \psi^2 \frac{e^{-\frac{|x|^2}{4\tau}}}{(4\pi\tau)^{n/2}}$$

$$- \int_{\mathbb{R}^n} x \cdot \nabla \psi^2 \frac{e^{-\frac{|x|^2}{4\tau}}}{(4\pi\tau)^{n/2}} = -\int_{\mathbb{R}^n} \operatorname{div}\left(\psi^2 x \frac{e^{-\frac{|x|^2}{4\tau}}}{(4\pi\tau)^{n/2}} \right)$$

uses in particular "$\operatorname{div} x = n$" $\quad = 0$

Aim Generalize Perelman's calculation (entropy monotonicity) to domains (or manifolds with boundary). Applications to mean curvature flow are envisaged.

Domains in $\mathbb{R}^{n+1}$, (boundaries are n-dimensional)

$\Omega \subset \mathbb{R}^{n+1}$, $f : \bar{\Omega} \to \mathbb{R}$ smooth

$\beta : \partial\Omega \to \mathbb{R}$ smooth (later $\beta = H_{\partial\Omega}$ = mean curvature of $\partial\Omega$), $\tau > 0$

$$W_\beta(\Omega, f, \tau) := \int_\Omega \left(\tau |\nabla f|^2 + f - (n+1)\right) u + 2\tau \int_{\partial\Omega} \beta u$$

$$u = \frac{e^{-f}}{(4\pi\tau)^{\frac{n+1}{2}}}$$

$$\mu_\beta(\Omega, \tau) := \inf \left\{ W_\beta(\Omega, f, \tau) , \int_\Omega u = 1 \right\}$$

These will be motivated and applied later.

Firstly, let's discuss logarithmic Sobolev inequalities for domains in $\mathbb{R}^{n+1}$: (6)

Standard Sobolev inequality (Gagliardo - Nirenberg), $\partial\Omega \in C^1$

$$\left(\int_\Omega |w|^{\frac{n+1}{n}} dx\right)^{n/n+1} \le c_S(\Omega) \int_\Omega \left(|Dw| + w\right) dx$$

$\forall w \in C^1(\bar{\Omega})$

(Ω not necessarily bounded; $w \in BV(\Omega)$ sufficient; $\partial\Omega \in C^{0,1}$)

Logarithmic Sobolev inequalities usually involve a constant which is independent of n but dx is replaced by $\gamma_{n+1}(x)\,dx$,

$$\gamma_{n+1}(x) = \frac{e^{-\frac{|x|^2}{2}}}{(2\pi)^{\frac{n+1}{2}}}.$$

Many proofs are probabilistic.

<u>Theorem</u> (Gross, Nelson; see 1975 paper by Gross)

If Ω satisfies

$$\left(\int_\Omega |w|^{\frac{n+1}{n}} \, dx \right)^{\frac{n}{n+1}} \le c_S(\Omega) \int_\Omega \left(|\nabla w| + |w| \right) dx$$

$\forall w \in C^1(\bar{\Omega})$ then $\forall \varepsilon > 0$

$\forall \varphi \in C^1(\bar{\Omega})$ with $\int_\Omega \varphi^2 \, dx = 1$

there holds

$$\int_\Omega \left(\varepsilon |\nabla \varphi|^2 - \varphi^2 \log \varphi^2 \right) dx$$

$$\cancel{\ge -c(n)\left(1 + \log c_S(\Omega)\right)}$$

$$\cancel{\ge -c(n, \log c_S(\Omega))} - \frac{1}{\varepsilon}$$

$$\ge -c(n)\left(1 + \log c_S(\Omega)\right)$$

<u>Proof.</u> Interpolation inequality for L^p-spaces:

$$L^p \subset L^q \subset L^p$$

$$\|u\|_q \le \|u\|_p^{\theta} \, \|u\|_r^{1-\theta}$$

if $\frac{\theta}{p} + \frac{1-\theta}{r} = \frac{1}{q}$

$$p = 1, \; r = \frac{n+1}{n} \rightsquigarrow 1 - \theta = n+1 - \frac{n+1}{q}$$

W.l.o.g. $\psi \geq 0$. Set $\|\psi\|_1 = 1$

Then

$$\|\psi\|_q \leq \|\psi\|_{\frac{n+1}{n}}^{n+1-\frac{n+1}{q}}$$

with "=" if $q=1$.

$$\Rightarrow \frac{d}{dq}\Big|_{q=1} \|\psi\|_q \leq \frac{d}{dq}\Big|_{q=1} \|\psi\|_{\frac{n+1}{n}}^{n+1-\frac{n+1}{q}}$$

$$\underset{\text{exercise}}{\Rightarrow} \int_\Omega \psi \log \psi \leq (n+1) \log \|\psi\|_{\frac{n+1}{n}}$$

$$\underset{\substack{\uparrow \\ \text{Gagliardo-} \\ \text{Nirenberg}}}{\leq} (n+1) \log \left(c_S(\Omega) \left(\|\nabla\psi\|_1 + 1 \right) \right)$$

$$= (n+1) \log \left(\frac{1}{n+1} \left(\|\nabla\psi\|_1 + 1 \right) \right)$$

$$+ \overbrace{(n+1) \log \left((n+1)\, c_S(\Omega) \right)}^{=c(n)\left(1+\log c_S(\Omega)\right)}$$

$$\underset{\substack{\uparrow \\ \log x \\ \leq x-1}}{\leq} \|\nabla\psi\|_1 + c(n, \log c_S(\Omega))$$

Set $\psi = \varphi^2$ with $\int_\Omega \varphi^2 \, dx = 1$

Then ...

$$\int_\Omega \varphi^2 \log \varphi^2 \leq \int_\Omega |\nabla \varphi^2| + c(n, \log c_S(\Omega))$$

$$\leq \varepsilon \int_\Omega |\nabla\varphi|^2 + \frac{1}{\varepsilon} \overbrace{\int_\Omega \varphi^2}^{=1} + c(n, \log c_S(\Omega)) \quad //$$

Optimal version for weakly convex domains (not nec. compact)

Bakry – Emery, Otto, Villani, Lott

Here: Diffusion equation proof

Thm. $\Omega \subset \mathbb{R}^{n+1}$ weakly convex, C^2 $(A_{\partial\Omega} \geq 0)$ (also true for less regular (but convex) ~~more general~~ domains but proof here does not carry over easily)

$\forall \, u$ with $\int_\Omega u^2 \gamma_{n+1} = 1$,

$\gamma_{n+1}(x) = \frac{e^{-\frac{|x|^2}{2}}}{(2\pi)^{\frac{n+1}{2}}}$ we have

$$\int_\Omega u^2 \log u^2 \, \gamma_{n+1} + \log\left(\int_\Omega \gamma_{n+1}\right)$$

$$\leq 2 \int_\Omega |\nabla u|^2 \gamma_{n+1}$$

$= 0$

"=" $\iff$ $u^2 = \frac{1}{\int_\Omega \gamma_{n+1}}$.

Remark: $\Omega = \mathbb{R}^{n+1}$

so $0 = \inf \left\{ \mathcal{E}(u), \int_\Omega u^2 \gamma_{n+1} = 0 \right\}$.

Proof. $\mathcal{E}(\psi) = \int_\Omega \Big(2|\nabla\psi|^2 - \underbrace{\psi^2\log\psi^2}_{H(\psi)}\Big)\gamma_{n+1}$

$$0 = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathcal{E}(\psi_0 + \varepsilon\eta)$$

$$= \int_\Omega \big(4\,\nabla\psi_0\cdot\nabla\eta - h(\psi_0)\,\eta\big)\,\gamma_{n+1}$$

with $h = H'$

$$\int_\Omega \operatorname{div}\,(\nabla\psi_0\,\eta\,\gamma_{n+1}) = \int_{\partial\Omega} \nabla\psi_0\cdot\nu\,\eta\,\gamma_{n+1}$$

$$\parallel$$

$$\int_\Omega \Delta\psi_0\,\eta\,\gamma_{n+1} + \int_\Omega \nabla\psi_0\cdot\nabla\eta\,\gamma_{n+1} - \int_\Omega \nabla\psi_0\cdot x\,\eta\,\gamma_{n+1}$$

$$0 = \int_\Omega \Big(\underbrace{-4\,\Delta\psi_0 + 4\,\nabla\psi_0\cdot x - h(\psi_0)}_{-L\psi_0}\Big)\eta\,\gamma_{n+1} + \int_{\partial\Omega} \nabla\psi_0\cdot\nu\,\eta\,\gamma_{n+1}$$

$\rightsquigarrow$ $\nabla\psi_0\cdot\nu = 0$ on $\partial\Omega$ for minimizer (stat. point)

and also $\nabla\psi_0^2\cdot\nu = 0$ on $\partial\Omega$

$L = \Delta - x \cdot \nabla$ Ornstein – Uhlenbeck operator

Property: $\int_\Omega v L w \, \gamma_{n+1} = - \int_\Omega \nabla v \cdot \nabla w \, \gamma_{n+1}$

if $\nabla w \cdot \nu = 0$ on $\partial \Omega$

Let $\psi^2(t)$ be sol. of

$$\left(\frac{\partial}{\partial t} - L\right) \psi^2 = 0 \quad \text{in } \Omega$$

$$\nabla \psi^2 \cdot \nu = 0 \quad \text{on } \partial \Omega$$

$$\psi^2(0) = \psi_0^2 \quad \text{in } \Omega$$

(Later ψ_0 will be an arbitrary minimizer for $\mathcal{E}$.)

$$\frac{d}{dt} \int_\Omega \psi^2 \gamma_{n+1} = \int_\Omega \frac{\partial}{\partial t} \psi^2 \gamma_{n+1}$$

$$= \int_\Omega L \psi^2 \gamma_{n+1} = 0 \quad \text{since } \nabla \psi^2 \cdot \nu = 0 \text{ on } \partial\Omega$$

$$\Rightarrow \boxed{\int_\Omega \psi^2(t) \gamma_{n+1} = \int_\Omega \psi_0^2 \gamma_{n+1} \quad \forall t \geq 0 \qquad (0)}$$

Next, calculate

$$\left(\frac{\partial}{\partial t} - L\right) |\nabla \psi|^2 :$$

Use Bochner formula

$$\Delta |\nabla u|^2 = 2|\nabla^2 u|^2 + 2\nabla u \cdot \nabla(\Delta u).$$

Also, $\frac{\partial}{\partial t}|\nabla u|^2 = 2\nabla u \cdot \nabla\left(\frac{\partial u}{\partial t}\right)$

so

$$\left(\frac{\partial}{\partial t} - \Delta\right)|\nabla u|^2 = -2|\nabla^2 u|^2 + 2\nabla u \cdot \nabla\left(\left(\frac{\partial}{\partial t} - \Delta\right)u\right)$$

for an arbitrary smooth function u.

$$x \cdot \nabla |\nabla u|^2 = 2\nabla^2 u\,(\nabla u, x)$$

~~$2\nabla u \cdot$~~

$$2\nabla u \cdot \nabla(x \cdot \nabla u) = 2\nabla^2 u\,(\nabla u, x) + 2|\nabla u|^2$$

$\Rightarrow$

$$\left(\frac{\partial}{\partial t} - L\right)|\nabla u|^2 = \left(\frac{\partial}{\partial t} - \Delta\right)|\nabla u|^2 + x \cdot \nabla|\nabla u|^2$$

$$= \left(\frac{\partial}{\partial t} - \Delta\right)|\nabla u|^2 + 2\nabla u \cdot \nabla(x \cdot \nabla u) - 2|\nabla u|^2 = -2|\nabla^2 u|^2 - 2|\nabla u|^2 + 2\nabla u \cdot \nabla\left(\left(\frac{\partial}{\partial t} - L\right)u\right)$$

Apply this to $u = \psi^2$ and use (13)

$\left(\frac{\partial}{\partial t} - L\right) \psi^2 = 0$ to obtain

$$\left(\frac{\partial}{\partial t} - L\right) |\nabla \psi^2|^2 = -2\,|\nabla^2 \psi^2|^2 - 2|\nabla \psi^2|^2$$

$$\nabla^2 \psi^2 = 2\,\nabla\psi \otimes \nabla\psi + 2\psi\,\nabla^2\psi$$

$$\left((\nabla\psi \otimes \nabla\psi)_{ij} = \nabla_i \psi\, \nabla_j \psi \right)$$

$\cancel{\Rightarrow |\nabla^2 \psi|^2 = 4(}$ $\Rightarrow$

$$|\nabla^2 \psi^2|^2 = 4\left(\psi^2 |\nabla^2 \psi|^2 + |\nabla\psi|^4 + 2\psi \nabla^2\psi(\nabla\psi, \nabla\psi)\right)$$

$$|\nabla \psi^2|^2 = 4\psi^2 |\nabla\psi|^2$$

$\Rightarrow$

$$\left(\frac{\partial}{\partial t} - L\right) |\nabla \psi^2|^2$$

*

$$= -8\psi^2 |\nabla^2\psi|^2 - 8|\nabla\psi|^4 - 16\psi\nabla^2\psi(\nabla\psi, \nabla\psi) - 8\psi^2 |\nabla\psi|^2$$

$|\nabla\psi^2|^2 = 4\psi^2 |\nabla\psi|^2$ and

$$\left(\frac{\partial}{\partial t} - L\right)(uv) = u\left(\frac{\partial}{\partial t} - L\right)v + v\left(\frac{\partial}{\partial t} - L\right)u - 2\nabla u \cdot \nabla v$$

imply

$$\left(\frac{\partial}{\partial t}-L\right)|\nabla\psi^2|^2=\left(\frac{\partial}{\partial t}-L\right)\left(4\psi^2|\nabla\psi|^2\right)$$

$$=4|\nabla\psi|^2\underbrace{\left(\frac{\partial}{\partial t}-L\right)\psi^2}_{=0}$$

$$+4\psi^2\left(\frac{\partial}{\partial t}-L\right)|\nabla\psi|^2$$

$$-8\nabla\psi^2\cdot\nabla|\nabla\psi|^2 \qquad (**)$$

$$=4\psi^2\left(\frac{\partial}{\partial t}-L\right)|\nabla\psi|^2$$

$$-32\psi\,\nabla^2\psi\,(\nabla\psi,\nabla\psi)$$

Combine (*) and (**):

$$4\psi^2\left(\frac{\partial}{\partial t}-L\right)|\nabla\psi|^2$$

$$=-8\psi^2|\nabla^2\psi|^2-8|\nabla\psi|^4-8\psi^2|\nabla\psi|^2$$

$$+16\psi\nabla^2\psi(\nabla\psi,\nabla\psi)$$

$$\Rightarrow\ \psi^2\left(\frac{\partial}{\partial t}-L\right)|\nabla\psi|^2$$

$$=-2\psi^2|\nabla^2\psi|^2-2|\nabla\psi|^4-2\psi^2|\nabla\psi|^2$$

$$+4\psi\nabla^2\psi(\nabla\psi,\nabla\psi)$$

$$\Rightarrow$$

$$|\psi \nabla^2\psi(\nabla\psi,\nabla\psi)| \leq \frac{1}{2}\psi^2|\nabla^2\psi|^2 + \frac{1}{2}|\nabla\psi|^4$$

$$\Rightarrow$$

$$\psi^2\left(\frac{\partial}{\partial t} - L\right)|\nabla\psi|^2 \leq -2\psi^2|\nabla\psi|^2$$

$$\Rightarrow \boxed{\left(\frac{\partial}{\partial t} - L\right)|\nabla\psi|^2 \underset{(1)}{\leq} -2|\nabla\psi|^2}$$

$$\left(\frac{\partial}{\partial t} - L\right)\varphi(u) = \varphi'(u)\left(\frac{\partial}{\partial t} - L\right)u - \varphi''(u)|\nabla u|^2$$

Use with $u = \psi^2$ and

$\varphi(x) = x\log x$.

$\varphi'(x) = 1 + \log x$; $\varphi''(x) = \frac{1}{x}$

(since $\left(\frac{\partial}{\partial t} - L\right)\psi^2 = 0$)

$$\Rightarrow \boxed{\begin{aligned}\left(\frac{\partial}{\partial t} - L\right)\left(\psi^2\log\psi^2\right) &\underset{(2)}{=} -\frac{1}{\psi^2}|\nabla\psi^2|^2 \\ &= -4|\nabla\psi|^2\end{aligned}}$$

Let $e(\psi) = 2|\nabla\psi|^2 - \psi^2\log\psi^2$.

Then $\boxed{\left(\frac{\partial}{\partial t} - L\right)e(\psi) \underset{(3)}{\leq} 0.}$

$\Longrightarrow$

$$\frac{d}{dt}\mathcal{E}(\psi) = \int_\Omega \frac{\partial}{\partial t} e(\psi)\, \gamma_{n+1}$$

$$\leq \int_\Omega L e(\psi)\, \gamma_{n+1} .$$

Since $\nabla(\psi^2 \log \psi^2) \cdot \nu$

$$= (1 + \log \psi^2)\, \nabla \psi^2 \cdot \nu = 0$$

on $\partial\Omega$

we have

$$\int_\Omega L(\psi^2 \log \psi^2)\, \gamma_{n+1} = 0$$

but for a general fct u

$$\int_\Omega L u\, \gamma_{n+1} = \int_{\partial\Omega} \nabla u \cdot \nu\, \gamma_{n+1} .$$

For $u = |\nabla\psi|^2$ we therefore have

$$\int_\Omega L |\nabla\psi|^2 \gamma_{n+1} = \int_{\partial\Omega} \nabla |\nabla\psi|^2 \cdot \nu\, \gamma_{n+1} .$$

$\Longrightarrow$

$$\frac{d}{dt} \mathcal{E}(u) \le 2 \int_{\partial\Omega} \nabla |\nabla u|^2 \cdot \nu \, \gamma_{n+1} \quad (4)$$

Since $\nabla u \cdot \nu = 0$ on $\partial\Omega$ we have $\nabla u = \nabla^{\partial\Omega} u$ on $\partial\Omega$.

Hence

$$\nabla |\nabla u|^2 \cdot \nu = 2 \nabla^2 u (\nabla u, \nu) \quad (5)$$

$$= 2 \nabla^2 u (\nabla^{\partial\Omega} u, \nu) \quad \text{on } \partial\Omega$$

Differentiating the equation $\nabla u \cdot \nu = 0$ on $\partial\Omega$ in the direction of $\nabla^{\partial\Omega} u$ gives

$$0 = \nabla^{\partial\Omega} u \, (\nabla u \cdot \nu)$$

$$= \nabla^2 u (\nabla^{\partial\Omega} u, \nu) + \nabla u \cdot \nabla_{\nabla^{\partial\Omega} u} \nu$$

$$(6) \quad = \nabla^2 u (\nabla^{\partial\Omega} u, \nu) + \nabla^{\partial\Omega} u \cdot \nabla_{\nabla^{\partial\Omega} u} \nu$$

$$= \nabla^2 u (\nabla^{\partial\Omega} u, \nu) + A_{\partial\Omega}(\nabla^{\partial\Omega} u, \nabla^{\partial\Omega} u)$$

where $A_{\partial\Omega}$ is the second fundamental form of $\partial\Omega$.

(5) & (6) substituted into (4) imply

$$\frac{d}{dt}\,\mathcal{E}(\psi) \leq -4 \int_{\partial\Omega} A_{\partial\Omega}\left(\nabla^{\partial\Omega}\psi,\ \nabla^{\partial\Omega}\psi\right)\gamma_{n+1}.$$

Hence if $A_{\partial\Omega} \geq 0$ we conclude

$$\boxed{\frac{d}{dt}\,\mathcal{E}(\psi) \leq 0} \qquad (7)$$

Let $$c := \frac{\int_\Omega \psi_0^2\,\gamma_{n+1}}{\int_\Omega \gamma_{n+1}} = \frac{1}{\int_\Omega \gamma_{n+1}}.$$

c is the average of ψ_0^2 w.r.t. γ_{n+1}.

By (0) we also have

$$c = \frac{\int_\Omega \psi(t)^2\,\gamma_{n+1}}{\int_\Omega \gamma_{n+1}} \qquad \forall\, t \geq 0.$$

We want to show that

$$\psi^2(t) \xrightarrow[t\to\infty]{} c \quad \text{in } L^2(\gamma_{n+1})$$

at an exponential rate.

Consider Ω with metric

$$g_{ij}(x) = \gamma_{n+1}^{\frac{2}{n+1}}(x)\, \delta_{ij} .$$

$$dV_g(x) = \gamma_{n+1}(x)\, dx .$$

Then $\exists$ Poincaré inequality (at least on smooth domains)

i.e. $\exists\, \delta > 0$ dep. only on Ω and n s.t. $\forall f \in C^1(\bar{\Omega})$

$$\delta \int_\Omega (f - \bar{f})^2 \gamma_{n+1} \leq \int_\Omega |\nabla f|^2 \gamma_{n+1} .$$

where $\bar{f} := \dfrac{\int_\Omega f \gamma_{n+1}}{\int_\Omega \gamma_{n+1}}$.

Then $\dfrac{d}{dt} \displaystyle\int_\Omega (\psi^2 - c)^2 \gamma_{n+1}$

$$= 2 \int_\Omega (\psi^2 - c) \frac{\partial \psi^2}{\partial t} \gamma_{n+1} = 2 \int_\Omega (\psi^2 - c) L\psi^2 \gamma_{n+1}$$

$$\underset{\substack{\nabla\psi^2 \cdot \nu = 0 \\ \text{on } \partial\Omega}}{=} -2 \int_\Omega |\nabla\psi^2|^2 \gamma_{n+1} \underset{\text{Poincaré}}{\leq} -2\delta \int_\Omega (\psi^2 - c)^2 \gamma_{n+1}$$

$\Rightarrow$

$$\int_\Omega (\psi^2(t) - c)^2 \gamma_{n+1} \leq e^{-2\delta t} \int_\Omega (\psi_0^2 - c)^2 \gamma_{n+1}$$

Let ψ_0 be a minimizer for $\mathcal{E}$. Since $\psi^2(t) \to c$ for $t \to \infty$ in $L^2(\gamma_{n+1})$ and $\frac{d}{dt}\mathcal{E}(\psi(t)) \leq 0$ $\sqrt{c}$ is also a minimizer for $\mathcal{E}$. Hence

$$\inf\left\{ \mathcal{E}(\psi),\ \int_\Omega \psi^2 \gamma_{n+1} = 1 \right\}$$

$$= \int_\Omega \left(2|\nabla\sqrt{c}|^2 - c\log c\right)\gamma_{n+1}$$

$$= -\log c = \log \frac{1}{c}.$$

c had to be equal to $\dfrac{1}{\int_\Omega \gamma_{n+1}}$.

Hence

$$\int_\Omega \left(2|\nabla\psi|^2 - \psi^2 \log \psi^2\right)\gamma_{n+1}$$

$$\geq \log \int_\Omega \gamma_{n+1}$$

$\forall \psi \in C^1(\bar{\Omega})$ with $\int_\Omega \psi^2 \gamma_{n+1} = 1$. //

<u>Remark.</u> It is standard to show the existence of a smooth solution w of

$$\left(\frac{\partial}{\partial t} - L\right) w = 0 \quad \text{in } \Omega$$

$$\nabla w \cdot \nu = 0 \quad \text{on } \partial\Omega$$

$$w(0) = w_0 \quad \text{on } \bar{\Omega}$$

under suitable conditions on $\partial\Omega$ and Ω. Since $L = \Delta - x \cdot \nabla$ the parabolic maximum principle implies that $w \geq 0$ for all time in Ω if $w_0 \geq 0$ in Ω. (On non-compact Ω this requires some work.) Moreover, w can only vanish in $\bar{\Omega}$ if $w \equiv 0$. Since we also assume $\int_\Omega w_0 \, \gamma_{n+1} = 1$, a condition which is preserved for all time we have $w(t) > 0$ for all $t > 0$.

We considered $w(t) = \psi^2(t)$.

Since $L\psi^2 = 2\psi L\psi + 2|\nabla\psi|^2$ we have

$$\left(\frac{\partial}{\partial t} - L\right)\psi - \frac{|\nabla\psi|^2}{\psi} = 0.$$

Note that $\psi(t)$ cannot vanish for $t>0$!

Our calculation for $\left(\frac{\partial}{\partial t} - L\right) |D\psi|^2$ becomes a bit easier if we start with this equation for ψ and use the Bochner formula for $\Delta |D\psi|^2$, than doing the calculations first for $\left(\frac{\partial}{\partial t} - L\right) |D\psi^2|^2$.

$\longrightarrow$ see 22a for the calculation

Return to the augmented version of Perelman's functional:

<u>Proposition</u> A smooth function $f : \overline{\Omega} \to \mathbb{R}$ is a minimizer for $\mu_\beta(\Omega, \tau)$ if and only if it satisfies

(a) $$W_\tau(f) := \tau\left(2\Delta f - |Df|^2\right) + f - (n+1) = \mu_\beta(\Omega,\tau)$$ in Ω

(b) $\nabla f \cdot \nu = \beta$ on $\partial\Omega$

Using the Bochner formula we had obtained

$$\left(\frac{\partial}{\partial t} - L\right)|\nabla u|^2 = -2|\nabla^2 u|^2 - 2|\nabla u|^2 + 2\nabla u \cdot \nabla\left(\left(\frac{\partial}{\partial t} - L\right)u\right)$$

for an arbitrary smooth function u.

For $u = \psi$ using

$$\left(\frac{\partial}{\partial t} - L\right)\psi = \frac{|\nabla\psi|^2}{\psi}$$

this gives

$$\left(\frac{\partial}{\partial t} - L\right)|\nabla\psi|^2 = -2|\nabla^2\psi|^2 - 2|\nabla\psi|^2 + 2\nabla\psi \cdot \nabla\left(\frac{|\nabla\psi|^2}{\psi}\right)$$

$$= -2|\nabla^2\psi|^2 - 2|\nabla\psi|^2 - \frac{2}{\psi^2}|\nabla\psi|^4 + \underbrace{\frac{4}{\psi}\nabla^2\psi(\nabla\psi, \nabla\psi)}_{\le 2|\nabla^2\psi|^2 + \frac{2}{\psi^2}|\nabla\psi|^4}.$$

Hence

$$\left(\frac{\partial}{\partial t} - L\right)|\nabla\psi|^2 \le -2|\nabla\psi|^2.$$

(c) $\int_\Omega u = 1 \quad , \; u = \frac{e^{-f}}{(4\pi\tau)^{\frac{n+1}{2}}}$

<u>Remark</u> (i) Any minimizer for $\mu_\beta(\Omega,\tau)$ is a weak solution of (a) − (c) in a suitable function space.

(ii) If $\partial\Omega$ and β are smooth so is any weak solution of (a) − (c). Indeed, if we set $u = \varphi^2$ then (a) − (c) transform to (exercise)

$$-4\tau \Delta\varphi = 2\varphi \log\varphi + \left(\mu_\beta(\Omega,\tau) + (n+1)\left(1+\log\sqrt{4\pi\tau}\right)\right)\varphi$$

in Ω

$$\nabla\varphi \cdot \nu = -\frac{\beta}{2}\varphi \quad \text{on } \partial\Omega$$

$$\int_\Omega \varphi^2 = 1 .$$

For weak solutions of semilinear equations of this type (subcritical) the standard elliptic regularity theory applies.

Proof of Proposition.

(Remark (i) above follows from one of the identities within the proof.)

Vary $W_\beta(\Omega, f, \tau) + \lambda \int_\Omega u$ for a Lagrange multiplier λ (to be determined later) w/o imposing $\int_\Omega u = 1$ during the variation.

Set $\delta f = \eta$ $\left(\delta f = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} f_\varepsilon,\ f_0 = f\right)$.

Then $\delta u = -u\eta$ and

$$\delta\left(W_\beta(\Omega, f, \tau) + \lambda \int_\Omega u\right)$$

extra "η" from $(\delta f)\, u$

$$= \int_\Omega \left(2\tau \nabla f \cdot \nabla\eta\, u - (\tau|\nabla f|^2 + f - (n+2) + \lambda)u\eta\right) - 2\tau \int_{\partial\Omega} \beta u \eta .$$

Use $\nabla u = -u\nabla f$ and integrate by parts ~~(assuming η has compact~~

$$\int_\Omega \nabla f \cdot \nabla \eta\, u = -\int_\Omega \nabla u \cdot \nabla \eta$$

$$= \int_\Omega \Delta u\, \eta - \int_{\partial\Omega} \nabla u \cdot \nu\, \eta .$$

Since also $\Delta u = u(|\nabla f|^2 - \Delta f)$ (exercise)

$\Rightarrow$

$$\int_\Omega \nabla f \cdot \nabla \eta \; u = \int_\Omega \left(|\nabla f|^2 - \Delta f \right) \eta u + \int_{\partial\Omega} \nabla f \cdot \nu \, \eta u$$

$\Rightarrow$

$$\delta \left(W_\beta(\Omega, f, \tau) + \lambda \int_\Omega u \right)$$

$$= \int_\Omega \left(- W_\tau(f) + 1 - \lambda \right) \eta u$$
$$+ 2\tau \int_{\partial\Omega} \left(\nabla f \cdot \nu - \beta \right) \eta u$$

where $W_\tau(f) = \tau \left(2\Delta f - |\nabla f|^2 \right)$
$\qquad + f - (n+1)$

$0 = \delta (\ldots) \quad \forall \eta$

$\Rightarrow \nabla f \cdot \nu = \beta$ on $\partial\Omega$

Note ! If a fct f (arbitrary!) satisfies $\nabla f \cdot \nu = \beta$ on $\partial\Omega$ then

$$W_\beta(\Omega, f, \tau) = \int_\Omega W_\tau(f)\, u \; !$$

$$\Rightarrow \int_\Omega W_\tau(F)\, u = \mu_\beta(\Omega,\tau)$$

for any minimizer F.

$$\delta(\ldots) = 0$$

$$\Rightarrow W_\tau(F) = 1-\lambda \text{ in } \Omega$$

Multiply by u and integrate

$$\Rightarrow \mu_\beta(\Omega,\tau) = 1-\lambda$$

i.e. $W_\tau(F) = \mu_\beta(\Omega,\tau)$.

Opposite direction:

$$DF\cdot\nu = \beta \text{ on } \partial\Omega$$

$$\Rightarrow \int_\Omega W_\tau(F)\, u = W_\beta(\Omega,F,\tau)$$

$$W_\tau(F) = \mu_\beta(\Omega,\tau) \text{ and } \int_\Omega u = 1$$

$$\Rightarrow W_\beta(\Omega,F,\tau) = \mu_\beta(\Omega,\tau). \quad //$$

Lower bound(s) on $\mu_\beta(\Omega,\tau)$:

<u>Proposition</u> Suppose Ω is bounded and open with smooth boundary and $\beta : \partial\Omega \to \mathbb{R}$ is smooth. Then $\exists\ c(\Omega, n)$ s.t. $\forall\ \tau > 0$

$$\mu_\beta(\Omega,\tau) \geq -c(\Omega,n)\Big(1+\log(1+\tau) + \tau \sup_{\partial\Omega}|\beta|\big(1+\sup_{\partial\Omega}|\beta|\big)\Big)$$

<u>Remark</u> (i) The proof also implies

$$\mu_\beta(\Omega,\tau) \geq 2a_\beta(\Omega)\tau - c(\underset{\Omega}{n})\big(1+\log(1+\tau)\big)$$

where

$$a_\beta(\Omega) := \inf\Big\{\int_\Omega|\nabla\varphi|^2 + \int_{\partial\Omega}\beta\varphi^2,\ \int_\Omega\varphi^2 = 1\Big\}.$$

(ii) Setting $u = \varphi^2$ we also obtain

$$\mu_\beta(\Omega,\tau) \geq \tau\int_\Omega|\nabla\varphi|^2 - c(n,\Omega,\tau)$$

by very slightly adapting our proof.

(iii) The proof of the proposition is based on the identity

$$W_\beta(\Omega, f, \tau)$$

$$= \int_\Omega \left(4\tau |\nabla \varphi|^2 - \varphi^2 \log \left((2\tau)^{\frac{n+1}{2}} \varphi^2 \right) \right)$$

$$+ 2\tau \int_{\partial \Omega} \beta \varphi^2 \quad - (n+1) \left(1 + \log \sqrt{\frac{\text{[illegible]}}{2\pi}} \right)$$

which one obtains by setting

$$\frac{e^{-f}}{(4\pi\tau)^{\frac{n+1}{2}}} = u = \varphi^2 .$$

(Derivation as exercise!)

Proof of Proposition.

(i) and (ii) follow ~~[illegible]~~ in the same way as the conclusion of the proposition. We simply do not "use up" the factor 4 above when apply the log-Sobolev inequ. and the trace inequality but leave either a factor 2 or 1 respectively.

Trace inequality $\Rightarrow$

$$\int_{\partial\Omega} \varphi^2 \le c_2 \left(\int_\Omega |\nabla \varphi^2| + \int_\Omega \varphi^2 \right)$$

$c_2 = c_2(\Omega)$
$$= c_2 \left(\int_\Omega 2|\varphi||\nabla\varphi| + 1 \right)$$

$$\le \varepsilon \int_\Omega |\nabla\varphi|^2 + \frac{c_3}{\varepsilon}$$

where c_3 depends on c_2 ~~and~~ .

Hence

$$2\tau \int_{\partial\Omega} \beta\varphi^2 \le \sup_{\partial\Omega} |\beta| \left(2\tau\varepsilon \int_\Omega |\nabla\varphi|^2 + \frac{c_3}{\varepsilon} \right)$$

$$\le 2\tau \int_\Omega |\nabla\varphi|^2 + c(\Omega) \sup_{\partial\Omega}|\beta| \left(1 + \sup_{\partial\Omega}|\beta|\right)$$

$$\varepsilon = \frac{1}{1 + \sup_{\partial\Omega}|\beta|}$$

$\Rightarrow$

$$W_\beta(\Omega, f, \tau) \ge \int_\Omega \left(2\tau|\nabla\varphi|^2 - \varphi^2 \log\left((2\tau)^{\frac{n+1}{2}} \varphi^2 \right) \right)$$

$$- c_4$$

where c_4 depends on n, Ω, τ and $\sup_{\partial\Omega} |\beta|$ (actually on τ · the β-terms) and logarithmically in τ.

To get (ii) in the Remark, we simply skip the above step.

Set $x = \sqrt{2\tau}\, y$, $\Omega_\tau = \frac{1}{\sqrt{2\tau}} \Omega$ and $\varphi_\tau(y) = \tau^{\frac{n+1}{4}} \varphi(\sqrt{2\tau}\, y)$ we obtain

$$W_\beta(\Omega, f, \tau) \geq \int_{\Omega_\tau} \left(|\nabla \varphi_\tau|^2 - \varphi_\tau^2 \log \varphi_\tau^2 \right)$$

$$- c_4$$

where $\int_{\Omega_\tau} \varphi_\tau^2 = 1$.

<u>Note:</u> If $c_S(\Omega)$ is ~~the~~ a Sobolev constant for a domain Ω then $c_S(\Omega)(1+\lambda)$ is a Sobolev constant for $\Omega_\lambda = \frac{1}{\lambda}\Omega$.

(Exercise : Scale)

So ~~the~~ $c_S(\Omega)\left(1+\sqrt{2\cancel{\log}\,\tau}\right)$
is a Sobolev constant for Ω_τ.
The log-Sobolev applied to
Ω_τ yields $(\varepsilon=1)$

$$\int_{\Omega_\tau}\left(|\nabla\varphi_\tau|^2-\varphi_\tau^2\log\varphi_\tau^2\right)$$

$$\geq -c(n)\left(1+\log c_S(\Omega_\tau)\right)$$

$$\geq -\underset{\text{new } c(n)}{c(n)}\left(1+\log c_S(\Omega)+\log(1+\tau)\right)$$

This gives the result.
If we do not care about this
explicit dependence on τ we
could have employed the
log-Sobolev with $\varepsilon=\tau$ to
Ω and to φ above.
Then $\tau\int_\Omega|\nabla\varphi|^2$ will be left
over and in we get (ii)
in the Remark.

(Existence of minimizers)

<u>Proposition</u> $\Omega \subset \mathbb{R}^{n+1}$ bounded domain, $\partial\Omega \in C^1$, $\beta : \partial\Omega \longrightarrow \mathbb{R}$.
Then $\exists$ smooth (not necessarily unique) minimizer f for $\mu_\beta(\Omega,\tau)$ i.e. f s.t.

$$\mu_\beta(\Omega,\tau) = \int_\Omega \left(\tau |\nabla f|^2 + f - (n+1)\right) u + 2\tau \int_{\partial\Omega} \beta u$$

<u>Remark.</u> We will first show the existence of a minimizer in a certain Sobolev class. The smoothness ~~will~~ follows by also using the Euler-Lagrange equation which we ~~will~~ derived ~~later~~ earlier.

<u>Proof of existence theorem:</u>

Proof. By (ii) of the Remark above

$$\cancel{\mu_\beta(\Omega,\tau)}$$

$$W_\beta(\Omega, f, \tau) \geq \tau \int_\Omega |\nabla\varphi|^2 - \text{const.}$$

Let (φ_j) be a minimizing sequence that is one for which the corresponding (f_j) $\left(\frac{e^{-f_j}}{(4\pi\tau)^{\frac{n+1}{2}}} = \varphi_j^2 \right)$ satisfies

$$W_\beta(\Omega, f_j, \tau) \longrightarrow \mu_\beta(\Omega,\tau).$$

Then $\exists\, C > 0$ with

$$\int_\Omega |\nabla\varphi_j|^2 \leq C$$

independent of j.

~~where C is independent of~~ j.
A subsequence of (φ_j) (again called (φ_j)) then converges weakly in $H^1(\Omega)$ to a function $\varphi \in H^1(\Omega)$. In particular,

$$\int_\Omega |\nabla\varphi|^2 \le \underline{\lim_j} \int_\Omega |\nabla\varphi_j|^2 .$$

Rellich compactness $\Longrightarrow$

$$\varphi_j \longrightarrow \varphi \quad \text{in} \quad L^p(\Omega)$$

strongly for any $p < \frac{2(n+1)}{n-1}$.
Moreover, the embedding of $H^1(\Omega)$ into $L^2(\partial\Omega)$ is compact so also

$$\int_{\partial\Omega} \beta\varphi_j^2 \longrightarrow \int_{\partial\Omega} \beta\varphi^2$$

since β is bounded.
The mean value inequality gives

$$|\varphi_j^2 \log\varphi_j^2 - \varphi^2\log\varphi^2| \le 2\sup\Big((\log\theta^2+1)\theta\Big)|\varphi_j -$$

at ~~all $x \in \Omega$~~ everywhere in Ω

$\theta \le \max\{|\varphi_j|, |\varphi|\}$ pointwise ~~at all $x\in\Omega$~~. ~~where~~ everywhere in Ω.

Also $$\theta \log \theta \le \frac{1}{e\gamma}\, \theta^{1+\gamma}$$

for any $\gamma > 0$. Hence by Hölder

$$\int_\Omega \left| \varphi_j^2 \log \varphi_j^2 - \varphi^2 \log \varphi^2 \right|$$

$$\le c(\gamma) \left(\int_\Omega |\varphi_j - \varphi|^p \right)^{1/p} \left(\int_\Omega \max\{|\varphi_j|, |\varphi|\}^{(\gamma+1)q} + \max\{|\varphi_j|, |\varphi|\}^q \right)^{1/q}$$

with $\frac{1}{p} + \frac{1}{q} = 1$.

Choose $2 < p < 2\frac{n+1}{n-1}$ so that Rellich is applicable but then $q < 2$. Then choose γ s.t. $(\gamma+1)q = 2$. Use Hölder on last integral . . .

Hence $\varphi_j^2 \log \varphi_j^2 \longrightarrow \varphi^2 \log \varphi^2$ in $L^1(\Omega)$. This implies altogether that

$$\cancel{\lim \mathcal{A}_\beta(\Omega, \varphi_j, \tau)}$$

$$\varliminf_{j\to\infty} \mathcal{W}_\beta(\Omega, f_j, \tau) \ge \mathcal{W}_\beta(\Omega, f, \tau)$$

so φ ~~is~~ corresponds to a minimizer f for ~~with~~ $\mu_\beta(\Omega, \tau)$ and satisfies $\int_\Omega \varphi^2 = 1$, $\varphi \in H^1(\Omega)$ and $\varphi^2 \log \varphi^2 \in L^1(\Omega)$.

Upper bound for $\mu_\beta(\Omega,\tau)$:

<u>Proposition</u> Let $\Omega \subset \mathbb{R}^{n+1}$ be open with "reasonable" boundary and β be integrable on $\partial\Omega$. Then

$$\mu_\beta(\Omega,\tau) \le \log\left(\frac{|\Omega \cap B_{\sqrt{\tau}}(x_0)|}{\tau^{\frac{n+1}{2}}}\right) + c(n)\,\frac{|\Omega \cap B_{\sqrt{\tau}}(x_0)| + 2\tau \int_{\partial\Omega \cap B_{\sqrt{\tau}}(x_0)} |\beta|}{|\Omega \cap B_{\frac{\sqrt{\tau}}{2}}(x_0)|}$$

for every $\tau > 0$ and every ball $B_{\sqrt{\tau}}(x_0)$ satisfying $|\Omega \cap B_{\frac{\sqrt{\tau}}{2}}(x_0)| > 0$

<u>Proof.</u> Set $e^{-f} = a\,\zeta$. Then $\int_\Omega u = 1$ implies

$$a = \frac{(4\pi\tau)^{\frac{n+1}{2}}}{\int_\Omega \zeta}.$$

Rewrite $W_\beta(\Omega, f, \tau)$ as

$$\frac{a}{(4\pi\tau)^{\frac{n+1}{2}}} \int_\Omega \left(\tau \frac{|\nabla\zeta|^2}{\zeta} - \zeta \log(a\zeta)\right)$$

$$- (n+1) + 2\tau \frac{\int_{\partial\Omega} \beta\zeta}{\int_\Omega \zeta}$$

By approximation we may substitute $\zeta \in C_0^2(\mathbb{R}^{n+1})$ into this expression (by definition $\zeta \geq 0$!).

If we choose ζ as a cut-off function for $B_{\frac{\sqrt{\tau}}{2}}(x_0)$ i.e.

$$\chi_{B_{\frac{\sqrt{\tau}}{2}}(x_0)} \leq \zeta \leq \chi_{B_{\sqrt{\tau}}(x_0)}$$

and also satisfying

$$\tau \frac{|\nabla\zeta|^2}{\zeta} \leq 2\tau \sup|\nabla^2\zeta| \leq C(n).$$

automatic for $0 \leq \zeta \in C_0^2(\mathbb{R}^{n+1})$

Note that

$$\int_\Omega \zeta \geq |\Omega \cap B_{\frac{\sqrt{\tau}}{2}}(x_0)| > 0 .$$

Thus

$$\frac{1}{(4\pi\tau)^{\frac{n+1}{2}}} \int_\Omega \tau a \frac{|D\zeta|^2}{\zeta}$$

$$\leq c(n) \frac{|\Omega \cap \operatorname{spt} \zeta|}{\int_\Omega \zeta} \leq c(n) \frac{|\Omega \cap B_{\sqrt{\tau}}(x_0)|}{|\Omega \cap B_{\frac{\sqrt{\tau}}{2}}(x_0)|}$$

Jensen's inequality :

$\psi : \mathbb{R} \to \mathbb{R}$ convex $\Rightarrow$

$$\psi\left(\fint_\Omega w \right) \leq \fint_\Omega \psi(w)$$

Apply with $\psi(x) = x \log x$ and $w = a\zeta$. Then

$$\left(\frac{1}{|\Omega \cap \operatorname{spt}\zeta|} \int_\Omega a\zeta\right) \log\left(\frac{1}{|\Omega \cap \operatorname{spt}\zeta|} \int_\Omega a\zeta\right)$$

$$\leq \frac{1}{|\Omega \cap \operatorname{spt}\zeta|} \int_\Omega (a\zeta) \log(a\zeta)$$

and therefore

$$-\frac{1}{(4\pi\tau)^{\frac{n+1}{2}}}\int_\Omega (a\zeta)\log(a\zeta)$$

$$\leq -\frac{1}{(4\pi\tau)^{\frac{n+1}{2}}}\int_\Omega a\zeta \cdot \log\left(\frac{1}{|\Omega\cap \operatorname{spt}\zeta|}\int_\Omega a\zeta\right)$$

Since $\operatorname{spt}\zeta = \overline{B_{\sqrt{\tau}}(x_0)}$ and since $\int_\Omega a\zeta = (4\pi\tau)^{\frac{n+1}{2}}$ the RHS equals

$$\log\left(\frac{|\Omega\cap B_{\sqrt{\tau}}(x_0)|}{(4\pi\tau)^{\frac{n+1}{2}}}\right).$$

//

__Corollary__ Suppose Ω is open and bounded. Then

(a) $\sup_{\tau>0}\mu_0(\Omega,\tau) \leq c(n,\Omega) < \infty$

(b) $\inf_{\tau>0}\mu_0(\Omega,\tau) = -\infty$.

__Remark__ One can show that conclusion (b) holds more generally for $\mu_\beta(\Omega,\tau)$ whenever Ω is open and bounded and $\inf\left\{\int_\Omega 4|\nabla\varphi|^2 + 2\int_{\partial\Omega}\beta\varphi^2,\ \int_\Omega\varphi^2=1\right\}\leq 0.$

Proof of Corollary :

(a) Ω open $\Longrightarrow \exists\, B_{\sqrt{\tau_0}}(x_0) \subset \Omega$
for some τ_0 dep. on Ω.
Upper bound for $\mu_0(\Omega, \tau)$

$$\Longrightarrow \mu_0(\Omega, \tau) \leq \log(\omega_{n+1}) + c(n) 2^{n+1}$$

$\forall\, \tau \in (0, \tau_0]$, ω_{n+1} = volume of unit ball in $\mathbb{R}^{n+1}$.

$\forall\, \tau \geq \tau_0$:

$$\mu_0(\Omega, \tau) \leq \log\left(\frac{|\Omega|}{\tau_0^{\frac{n+1}{2}}}\right) + c(n) 2^{n+1} \frac{|\Omega|}{\omega_{n+1}\, \tau_0^{\frac{n+1}{2}}}$$

(b) Ω bounded $\Longrightarrow \Omega \subset B_{\frac{\sqrt{\tau_1}}{2}}(0)$
for some τ_1 dep. on Ω.
Upper bound for $\mu_0(\Omega, \tau)$, $\tau \geq \tau_1$ and $x_0 = 0$ yields

$$\mu_0(\Omega, \tau) \leq \log\left(\frac{|\Omega|}{\tau^{\frac{n+1}{2}}}\right) + c(n) \xrightarrow[\tau \to \infty]{} -\infty \, .$$

<u>Corollary</u> Suppose $\mu_\beta(\Omega, r^2) \geq -c_0$ ($\tau = r^2$ for ease of notation) and for $|\Omega \cap B_{\frac{r}{2}}(x_0)| > 0$ we have

$$\frac{|\Omega \cap B_r(x_0)| + r^2 \int_{\partial\Omega \cap B_r(x_0)} |\beta|}{|\Omega \cap B_{\frac{r}{2}}(x_0)|} \leq c_1 .$$

Then $$\frac{|\Omega \cap B_r(x_0)|}{r^{n+1}} \geq \varkappa > 0$$

where $\varkappa = \varkappa(n, c_0, c_1)$.

<u>Proof.</u> Immediate from upper bound for $\mu_\beta(\Omega, r^2)$.

<u>Examples of unbounded sets Ω for which</u> ~~inf $\mu_\beta(\Omega, \tau)$~~ <u>$\inf_{\tau > 0} \mu_\beta(\Omega, \tau) = -\infty$</u> (these are sets which are also relevant to mean curvature flow):

(1) Slab

$$\Omega = \{x \in \mathbb{R}^{n+1}, -d < x_{n+1} < d\};$$

for some $d > 0$. Choose $\beta = H_{\partial\Omega} = 0$. Let B_R denote $B_R(0)$.

$\forall B_r$ we have $|\Omega \cap B_{\frac{r}{2}}| > 0$

and $\exists c(n)$ s.t. $\forall r > 0$

$$\frac{|\Omega \cap B_r|}{|\Omega \cap B_{\frac{r}{2}}|} \leq c(n).$$

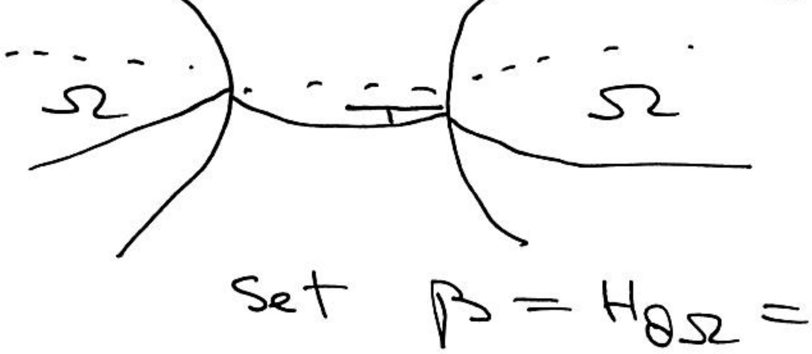
Hence $c_1 = c(n)$.

But
$$\lim_{r\to\infty} \frac{|\Omega \cap B_r|}{r^{n+1}} = 0,$$

Hence

$$\cancel{\#}\ \underline{\lim}_{r\to\infty} \mu_0(\Omega, r^2) = -\infty.$$

(2) $\Omega = \{x = (\hat{x}, x_3) \in \mathbb{R}^3,\ |\hat{x}| \geq 1,$
$|x_3| \leq \cosh^{-1}|\hat{x}|\}$,



$H_{\partial\Omega} = 0$

catenoid
minimal
surface

Set $\beta = H_{\partial\Omega} = 0$

~~Exercise:~~

~~$\exists c_1$ s.t. $\forall r>0$ $|\Omega \cap B$~~

Exercise: (a) $\exists c_1$ $\forall r \geq 2$

$$\frac{|\Omega \cap B_r|}{|\Omega \cap B_{\frac{r}{2}}|} \leq C_1 .$$

(b) $\exists c_2$ $\forall r \geq 2$

$$|\Omega \cap B_r| \leq c_1 r^2 \log(1+r)$$

$$(b) \Longrightarrow \lim_{r \to \infty} \frac{|\Omega \cap B_r|}{r^3} = 0$$

Hence $\lim\limits_{r \to \infty} \mu_0(\Omega, r^2) = -\infty .$

(3) $\Omega = \mathbb{R}^{n-1} \times G$

$\partial G \;\hat{=}\;$ translating solution of curve shortening flow (grim reaper curve; see later)

$$G = \left\{ (x_n, x_{n+1}) \in \mathbb{R}^2 ,\; -\frac{\pi}{2} < x_n < \frac{\pi}{2} ,\; x_{n+1} > -\log \cos x_n \right\}$$

Calculation $\rightsquigarrow$ $H_{\partial\Omega}(x) = e^{-x_{n+1}}$ for any $x \in \partial\Omega$.

Using this, one checks that there exists $\left(B_{r_k}(x_k)\right)$, $r_k \to \infty$ and $|\Omega \cap B_{\frac{r_k}{2}}(x_k)| > 0$

$$\frac{|\Omega \cap B_{r_k}(x_k)|}{|\Omega \cap B_{\frac{r_k}{2}}(x_k)|} \leq c(n)$$

$$\frac{r_k^2 \int_{\partial\Omega \cap B_{r_k}(x_k)} H}{|\Omega \cap B_{\frac{r_k}{2}}(x_k)|} \leq 1 \qquad \left(\begin{array}{l} H \sim e^{-r_k} \\ \text{on } \partial\Omega \cap B_{r_k}(x_k) \end{array}\right)$$

and

$$\frac{|\Omega \cap B_{r_k}(x_k)|}{r_k^{n+1}} \to 0$$

(as for slab example).

Hence $\displaystyle\lim_{\tau \to \infty} \mu_{H_{\partial\Omega}}(\Omega, \tau) = -\infty.$