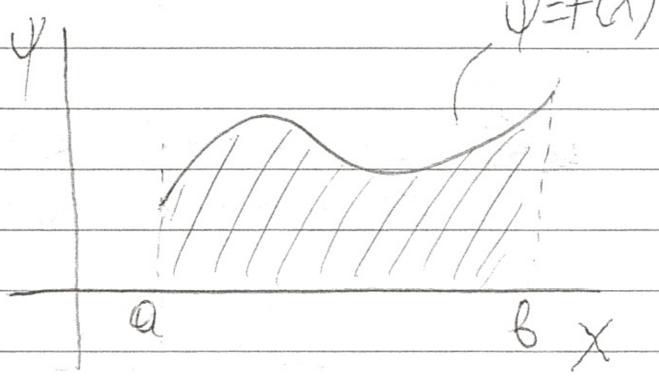
www.math.uoc.gr/~tertikas calculus 3

Πρόοδος 30%

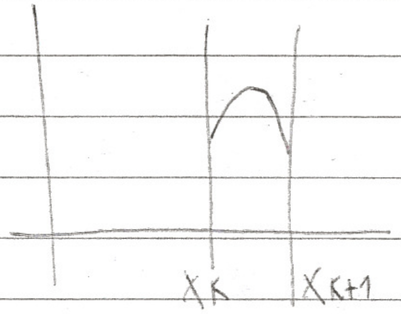
τελικός 70% bonus φυλλάδια 10% (επί της επίδοσης)

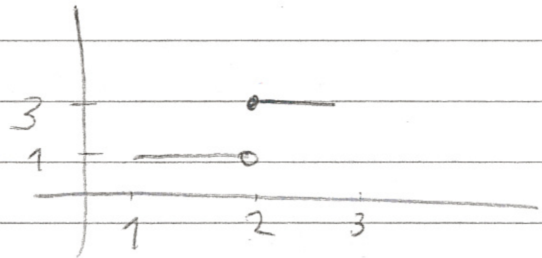
Marsden & Tromba (library genesis)

$$\int_a^b f(x)\,dx \qquad f \geq 0$$



$$P=\{a=x_0<x_1<\dots<x_n=b\}$$





$$m_k=\inf\{f(x) \mid x\in[x_k,x_{k+1}]\}$$

$$M_k=\sup\{f(x) \mid x\in[x_k,x_{k+1}]\}$$

$$\cancel{M_k \leq}$$

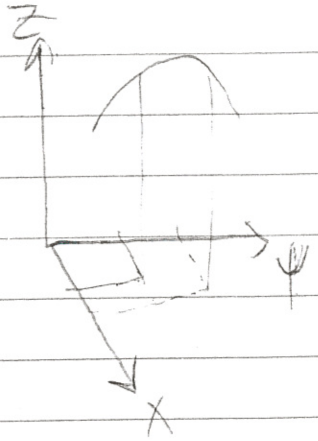
$$\underbrace{m_k(x_{k+1}-x_k)}_{L(f,P)} \leq \mathcal{E}_k \leq M_k(x_{k+1}-x_k)$$

$$\sum_{k=0}^{n-1} m_k(x_{k+1}-x_k) \leq \mathcal{E} \leq \sum_{k=0}^{n-1} M_k(x_{k+1}-x_k)$$

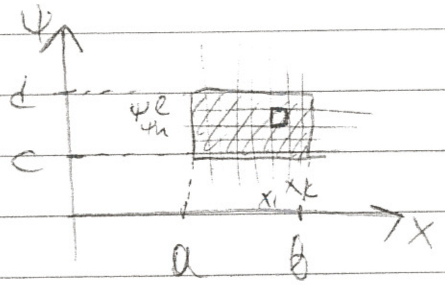
$$\sup_Q L(f,Q) \leq \inf_P U(f,P)$$

$$f(x)=\begin{cases} -1, & x\in\mathbb{Q}\cap[0,1] \\ 1, & x\in[0,1]\setminus\mathbb{Q} \end{cases}$$

$f \geq 0$



$f : [a,b] \times [c,d] \to \mathbb{R}$



$a = x_0 < x_1 < \dots < x_n = b$
$c = \psi_0 < \psi_1 < \dots < \psi_n = d$

$[x_{k-1}, x_k]$
$[\psi_{l-1}, \psi_l]$

$[x_{k-1}, x_k] \times [\psi_{l-1}, \psi_l]$ $\quad 1 \leq k \leq \dots \leq n$
$\quad 1 \leq l \leq \dots \leq m$

$M_{kl} = \sup \{ f(x,\psi), (x,\psi) \in [x_{k-1}, x_k] \times [\psi_{l-1}, \psi_l] \}$

$m_{kl} = \inf$

$m_{kl}(x_k - x_{k+1})(\psi_l - \psi_{l+1}) \leq V_{kl} \leq M_{kl}(x_k - x_{k-1}) \cdot (\psi_l - \psi_{l-1})$

$$\sum m_{kl}(\,)(\,) \leq V \leq \sum_{\substack{1 \leq k \leq n \\ 1 \leq l \leq m}} M_{kl}(x_k - x_{k-1})(\psi_l - \psi_{l-1})$$

$\sup L(f,P) \leq \inf U(f,P)$

$\sup_Q \{L(f,Q), Q\} \leq \inf \{U(f,P), P\}$

$$U(f,P) = \sum_{k=1}^{n} M_k (x_k - x_{k-1})$$

upper

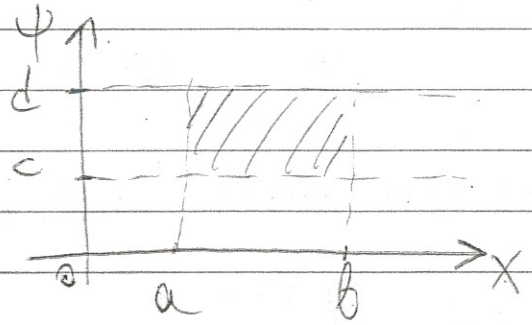
$$\sum_{\substack{1\le k\le n \\ 1\le k\le m}} M_{k\ell}\,(x_k - x_{k-1})(\psi_\ell - \psi_{\ell-1})$$

f ολοκληρωματος

$$f : \underbrace{[a,b] \times [c,d]}_{D \ (\text{Domain})} \to \mathbb{R}$$

$$\iint_D f \, dx \, d\psi$$

<u>Προσανατολισμός</u>



$$D = [a,b] \times [c,d] = \{ (x,\psi) \in \mathbb{R}^2 \ / \ x \in [a,b], \ \psi \in [c,d] \}$$

$$\int_a^b \left(\int_c^d f(x,\psi)\, d\psi \right) dx = \int_c^d \left(\int_a^b f(x,\psi)\, dx \right) d\psi$$

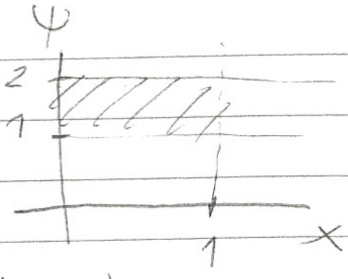
(Fubini)

<u>Άσκηση</u>

Να υπολογιστεί το ολοκλήρωμα

$$\iint_D x\psi \, dx \, d\psi \quad \text{όταν } D = \{(x,\psi)\} \ / \ 0 \le x \le 1, \ 1 \le \psi \le 2$$

Λύση



$$\int_0^1\left(\int_1^2 x\psi\,d\psi\right)dx$$

$$=\int_0^1\left(x\int_1^2\psi\,d\psi\right)dx=\int_0^1\left(x\left(\frac{\psi^2}{2}\right)\Bigg|_{\psi=1}^{\psi=2}\right)dx$$

$$=\int_0^1\left(x\left(2-\frac{1}{2}\right)\right)dx=\frac{3}{2}\int_0^1 x\,dx=\frac{3}{2}\left(\frac{x^2}{2}\right)\Bigg|_0^1=\frac{3}{4}$$

Άσκηση

Να υπολογιστεί το ολοκλήρωμα

$$\iint_D \frac{e^{\frac{x}{\psi}}}{\psi^3}\,dx\,d\psi \quad \text{όταν } D=\{(x,\psi)\mid 0\le x\le 1,\ 1\le\psi\le 2\}$$

$$\int_0^1\left(\int_1^2\frac{e^{\frac{x}{\psi}}}{\psi^3}\,d\psi\right)dx$$

$$\int_1^2\left(\frac{1}{\psi^3}\int_0^1 \frac{e^{\frac{x}{\psi}}}{\cancel{\psi^3}}\,dx\right)d\psi=\int_1^2\left(\frac{1}{\psi^3}\left(\frac{e^{\frac{x}{\psi}}}{\frac{1}{\psi}}\right)\Bigg|_{x=0}^{x=1}\right)d\psi$$

$$=\int_1^2\frac{1}{\psi^2}\left(e^{1/\psi}-1\right)d\psi$$

$$\frac{1}{\psi}=z \Rightarrow dz=-\frac{d\psi}{\psi^2}$$

$$=\int_1^{1/2}\left(e^z-1\right)(-dz)=\int_{1/2}^1\left(e^z-1\right)dz=e^z-z\Bigg|_{1/2}^1$$

$$=e-1-\left(e^{1/2}-\frac{1}{2}\right)=e-e^{1/2}-\frac{1}{2}$$

Να ~~υπολογίζει~~ συσχετίζει το ολοκλήρωμα

$$\int_0^1\left(\int_1^2 f(\psi)\,d\psi\right)dx$$

με (απλό) ολοκλήρωμα

$$\int_1^2 f(\psi)\,d\psi \int_0^1 1\,dx = \int_1^2 f(\psi)\,d\psi$$

Έστω $f\,[1,2]\to \mathbb{R}$

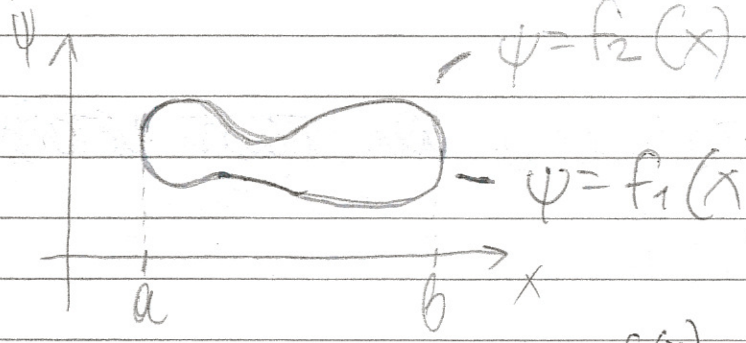
$$\int_1^2\left(\int_0^1 f(\psi)\,dx\right)d\psi =$$

$$\int_1^2\left(f(\psi)\int_0^1 1\,dx\right)d\psi = \int_1^2 1\,f(\psi)\,d\psi = \int_1^2 f(\psi)\,d\psi$$

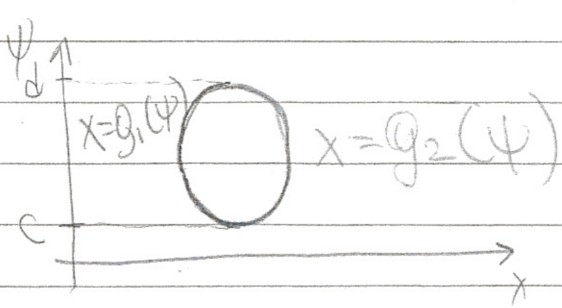
22/9/2016

Ολοκλήρωση σε τυχαίο χωρίο

<u>Απλές</u> περιπτώσεις



$$\iint_D f(x,\psi)\,dx\,d\psi = \int_a^b\left(\int_{f_1(x)}^{f_2(x)} f(x,\psi)\,d\psi\right)dx$$

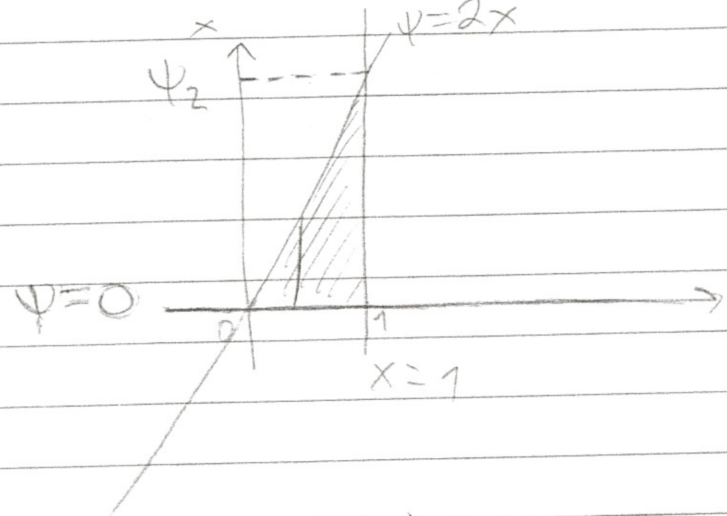


$$\int_c^d\left(\int_{g_1(\psi)}^{g_2(\psi)} f(x,\psi)\,dx\right)d\psi$$

Άσκηση: Να περιγραφεί το χωρίο ολοκλήρωσης όπου το D είναι το φραγμένο χωρίο του επιπέδου που περικλείεται από τις ευθείες $\psi=2x$, $x=1$ & ψ

$$\iint_D f(x,\psi)\,dx\,d\psi.$$

Απ: Αρχικά σχεδιάζουμε το χωρίο



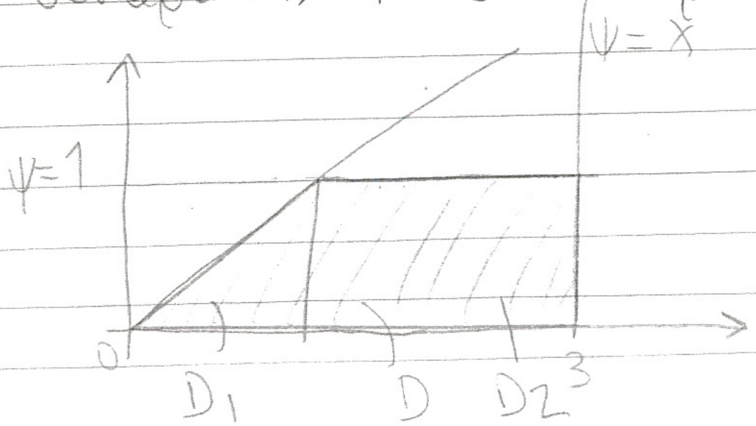
1ος τρόπος: Επιλέγοντας αρχικά τη μεταβλητή x.

$$\iint_D f = \int_0^1\left(\int_0^{2x} f(x,\psi)\,d\psi\right)dx.$$

2ος τρόπος: Επιλέγοντας το ψ σαν πρώτη μεταβλητή στην περιγραφή πάω από το μικρό στο μεγάλο

$$\iint_D f = \int_0^2\left(\int_{\psi/2}^1 f(x,\psi)\,dx\right)d\psi$$

Άσκηση: Να γίνει η περιγραφή της ολοκλήρωσης συνάρτησης f στο χωρίο D, όπως στο σχήμα.



Απ. 1ος τρόπος: Επιλέγουμε πρώτη μεταβλητή το x

$$\iint_D f(x,\psi)\,dx\,d\psi = \iint_{D_1} f(x,\psi)\,dx\,d\psi + \iint_{D_2} f(x,\psi)\,dx\,d\psi$$

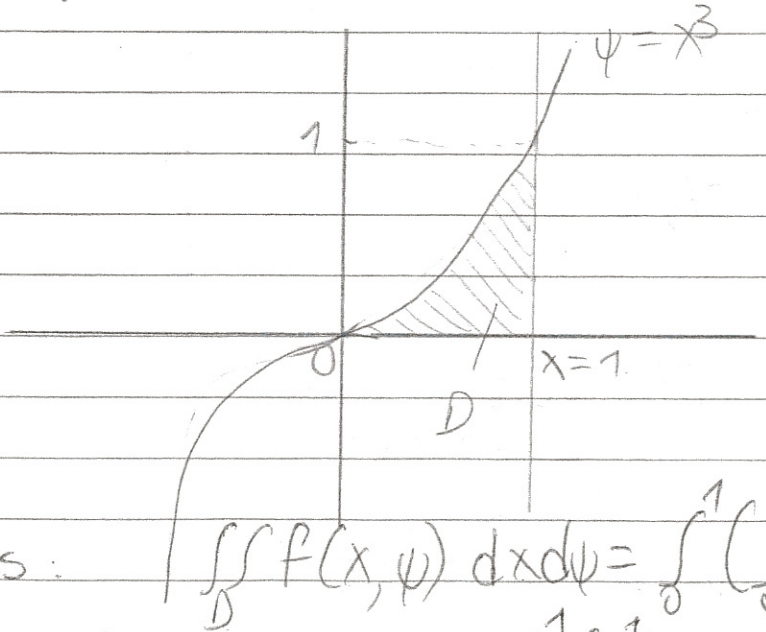
$$= \int_0^1\left(\int_0^x f(x,\psi)\,d\psi\right)dx + \int_1^3\left(\int_0^1 f(x,\psi)\,d\psi\right)dx$$

2ος τρόπος: Επιλέγουμε την μεταβλητή ψ σαν πρώτη μεταβλητή:

$$\int_0^1\left(\int_\psi^3 f(x,\psi)\,dx\right)d\psi$$

Άσκηση: Να περιγραφεί το ολοκλήρωμα στο φραγμένο χωρίο D που ορίζεται από τα γραφήματα της συνάρτησης και των ευθειών. $\psi=x^3$, $x=1$, $\psi=0$. Στη συνέχεια βρείτε το εμβαδόν του χωρίου D.

Γραφική παράσταση του D



$f(x)=x^3$

$f'(x)=3x^2\geq 0$

$f''(x)=6x$

$\Rightarrow$ f γνήσια αύξουσα

f κυρτή στο $(0,+\infty)$

κοίλη στο $(-\infty,0)$

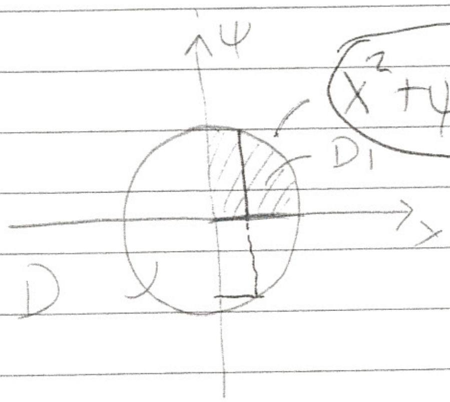
1ος: $\displaystyle\iint_D f(x,\psi)\,dx\,d\psi = \int_0^1\left(\int_0^{x^3} f(x,\psi)\,d\psi\right)dx$

2ος: $\displaystyle\iint_D f(x,\psi)\,dx\,d\psi = \int_0^1\left(\int_{\psi^{1/3}}^1 f(x,\psi)\,dx\right)d\psi$

Το εμβαδόν του χωρίου D είναι

$$\mathcal{E}(D) = \iint_D 1\,dx\,d\psi = \int_0^1 \left(\int_0^{x^3} 1\,d\psi \right) dx = \int_0^1 x^3\,dx = \frac{1}{4}$$

Άσκηση Βρείτε το εμβαδόν του μοναδιαίου κύκλου



$$x^2+\psi^2=1 \Rightarrow \psi^2 = 1-x^2$$
$$\Rightarrow \psi = \pm\sqrt{1-x^2}$$
$$\psi = \sqrt{1-x^2}$$

$$\text{Εμβαδόν κύκλου} = \iint_D 1\,dx\,d\psi$$
$$= 4\iint_{D_1} dx\,d\psi = 4\int_0^1 \left(\int_0^{\sqrt{1-x^2}} d\psi \right) dx$$

$$= 4\int_0^1 (\psi)\Big|_{\psi=0}^{\psi=\sqrt{1-x^2}} dx = 4\int_0^1 \sqrt{1-x^2}\,dx$$

$x = \sin t \Rightarrow$ $x=0 \to t=0$

$0 \le x \le 1$ $\quad x=1 \Rightarrow t=\pi/2$

$dx = \cos t\,dt$

Οπότε $4\int_0^1 \sqrt{1-x^2}\,dx = 4\int_0^{\pi/2} \sqrt{1-\sin^2 t}\,\cos t\,dt$

$$= 4\int_0^{\pi/2} |\cos t| \cos t\,dt = 4\int_0^{\pi/2} \cos^2 t\,dt$$

$$\cos(2t) = \cos^2 t - \sin^2 t$$
$$= -1 + 2\cos^2 t$$

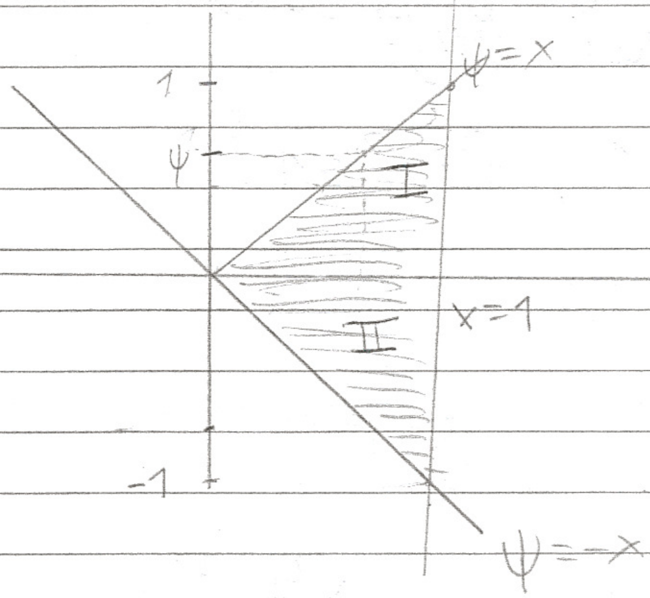
$$\cos^2 t = \frac{1+\cos 2t}{2}$$

$$\text{ολοκλήρωμα} = 4\int_0^{\pi/2} \frac{1+\cos 2t}{2}\,dt = \underbrace{2\int_0^{\pi/2} dt}_{\pi} + 2\int_0^{\pi/2} \cos 2t\,dt$$
$$= \pi \left| 2\left(\frac{\sin 2t}{2}\right)\right|_0^{\pi/2}$$

<u>Άσκηση</u>: Ζωγράφισε το χωρίο ολοκλήρωσης

$$\int_{-1}^{1}\left(\int_{|\psi|}^{1} (x+\psi)^2 dx\right) d\psi$$, και στη συνέχεια υπολογίστε το ολοκλήρωμα

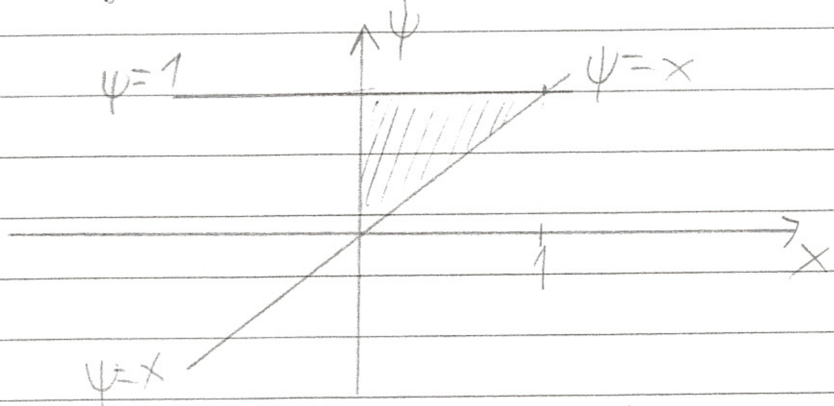
$$= \int_0^1\left(\int_{-x}^{x} (x+\psi)^2 d\psi\right) dx$$



<u>Άσκηση</u>: Να γραφεί το διπλό ολοκλήρωμα

$$\int_0^1\left(\int_x^1 f(\psi)\,d\psi\right)dx$$, σαν απλό.

Σχεδιάζουμε το χωρίο ολοκλήρωσης



$$\int_0^1 \left(\int_x^1 f(\psi)\,d\psi \right) dx = \int_0^1 \left(\int_0^\psi f(\psi)\,dx \right) d\psi$$

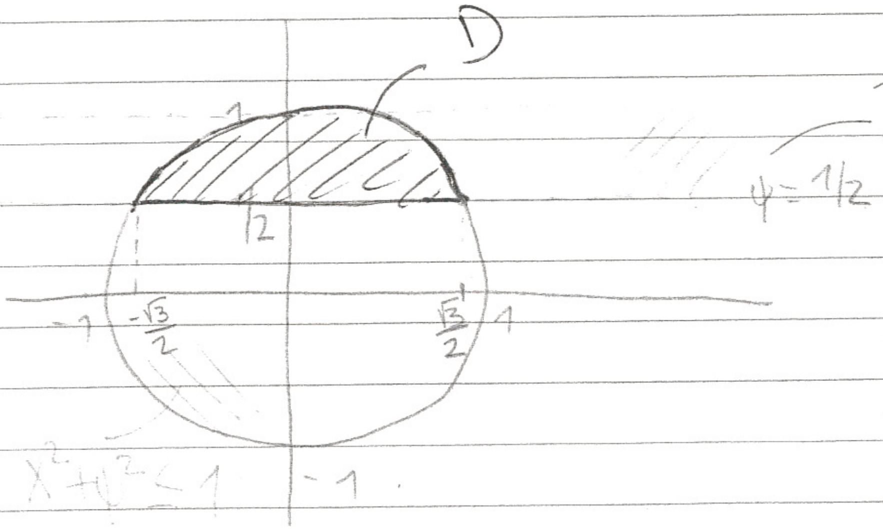
$$= \int_0^1 \left(f(\psi) \int_0^\psi dx \right) d\psi = \int_0^1 \psi f(\psi)\,d\psi \quad //$$

27/09/2016

Άσκηση: Να υπολογίσετε το εμβαδόν του χωρίου D όπου ορίζεται από:

$$D = \{ (x,\psi) \mid 1/2 \leq \psi \leq 1 ,\ x^2+\psi^2 \leq 1 \}$$

τομή 2 σχημάτων

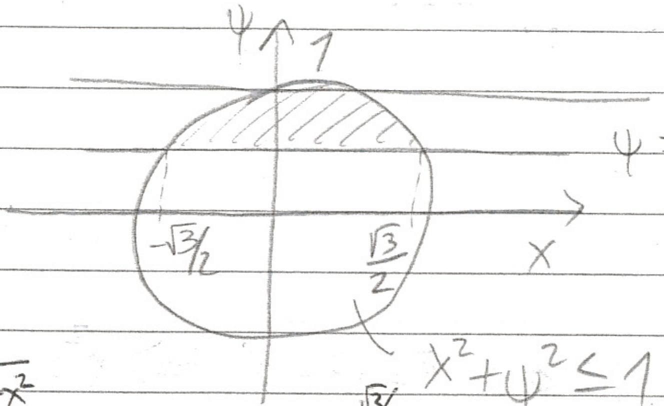


$1/2 \leq \psi \leq 1$

$\psi = 1/2$

$$x^2+\psi^2 = 1$$
$$\psi = 1/2$$
$$x^2 = \frac{3}{4} \Rightarrow \sqrt{x = \pm\frac{3}{2}}$$

Απάντηση. Το χωρίο είναι ως έξης



$x^2+\psi^2=1 \Rightarrow \psi^2=1-x^2$

$\psi = \pm\sqrt{1-x^2}$

$$= \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \left(\int_{1/2}^{\sqrt{1-x^2}} 1 \, d\psi \right) dx = \int_{-\frac{\sqrt{3}}{2}}^{\sqrt{3}/2} \left(\sqrt{1-x^2} - \frac{1}{2} \right) dx$$

$$\int_{-\sqrt{3}/2}^{\sqrt{3}/2} (\psi) \Big|_{\psi=\frac{1}{2}}^{\psi=\sqrt{1-x^2}} dx$$

$$= \int_{-\frac{\sqrt{3}}{2}}^{\sqrt{3}/2} \sqrt{1-x^2} - \frac{1}{2} \int_{-\frac{\sqrt{3}}{2}}^{\sqrt{3}/2} dx \quad \ldots \quad x = \sin t$$

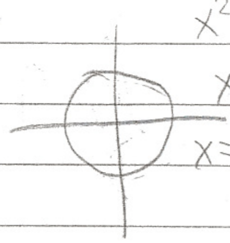
$\frac{\sqrt{3}}{2}$

$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} = \sin t = \cancel{t} = \frac{\pi}{3}$

$\sin\left(\frac{-\pi}{3}\right) = \frac{-\sqrt{3}}{2} = \sin t$

$t = -\frac{\pi}{3}$

2ος τρόπος: $\iint_D dx\,d\psi = \int_{1/2}^{1} \left(\int_{\sqrt{1-\psi^2}}^{\sqrt{1-\psi^2}} 1 \, dx \right) d\psi$



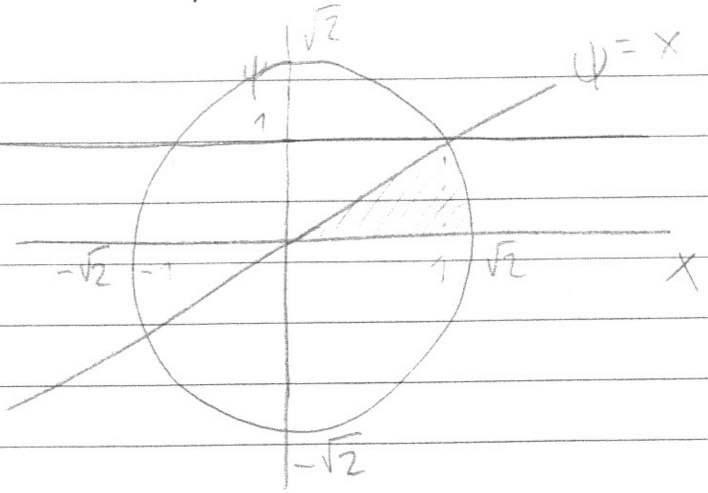
$x^2+\psi^2=1$

$x^2 = 1-\psi^2$

$x = \pm\sqrt{1-\psi^2}$

Άσκηση Δίνεται το διπλό ολοκλήρωμα

$$\int_0^1 \left(\int_\psi^{\sqrt{2-\psi^2}} f(x,\psi)\, dx \right) d\psi$$ Να γίνει αλλαγή σειράς ολοκλήρωσης.



$x = \psi$

$x = \sqrt{2-\psi^2} \Rightarrow x^2 = 2-\psi^2 \Leftrightarrow x^2+\psi^2 = 2 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

$$\left.\begin{array}{l} \psi = 1 \\ \psi = x \end{array}\right| \Leftrightarrow \begin{array}{l} x = 1 \\ \psi = 1 \end{array}$$

$$\iint_{D_1} + \iint_{D_2} = \int_0^1 \left(\int_0^x f(x,\psi)\, d\psi \right) dx + \int_1^{\sqrt{2}} \left(\int_0^{\sqrt{2-x^2}} f(x,\psi)\, d\psi \right.$$

<u>Ιδιότητες</u>

i) $\iint_D \left(f(x,\psi) + g(x,\psi) \right) dx\, d\psi = \iint_D f\, dx\, d\psi + \iint_D g\, dx$

ii) $\lambda \in \mathbb{R}$ $\quad \iint_D \lambda f(x,\psi)\, dx\, d\psi = \lambda \iint_D f\, dx\, d\psi$

$f \geq 0$, f συνεχής

$$\int_0^1 f(t)\, dt = 0 \Rightarrow \boxed{f \equiv 0}$$

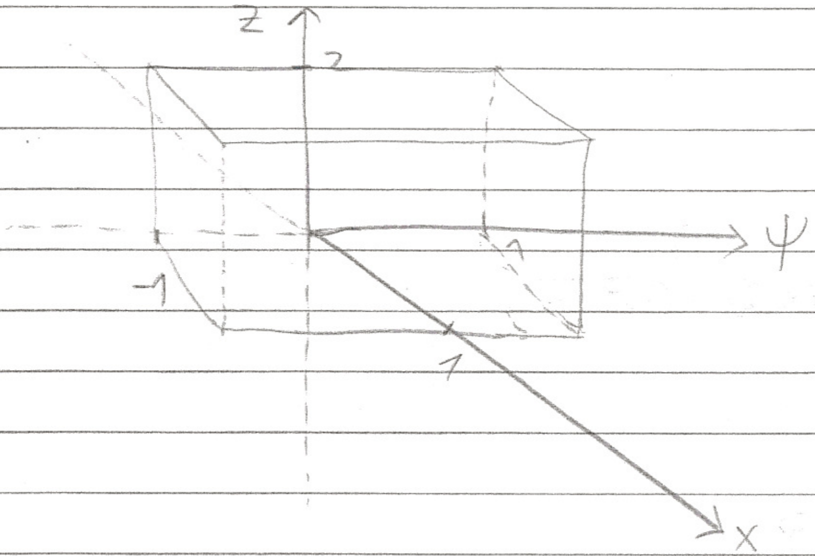
$f(x,\psi) \geq 0$, f συνεχής

$$\iint_D f \, dx d\psi = 0 \Rightarrow f \equiv 0$$

<u>Άσκηση</u>: Να υπολογιστεί το τριπλό ολοκλήρωμα

$$\iiint_D x \, dx d\psi dz$$ όπου

$$D = \{(x,\psi,z) \in R^3 \mid 0 \leq x \leq 1, -1 \leq \psi \leq 1, 0 \leq z \leq 2\}$$



$$\iiint_D x \, dx d\psi dz = \int_0^1 \left(\int_{-1}^1 \left(\int_0^2 x \, dz \right) d\psi \right) dx$$

$$= \int_0^1 \left(\int_{-1}^1 \left[x \int_0^2 dz \right] d\psi \right) dx$$

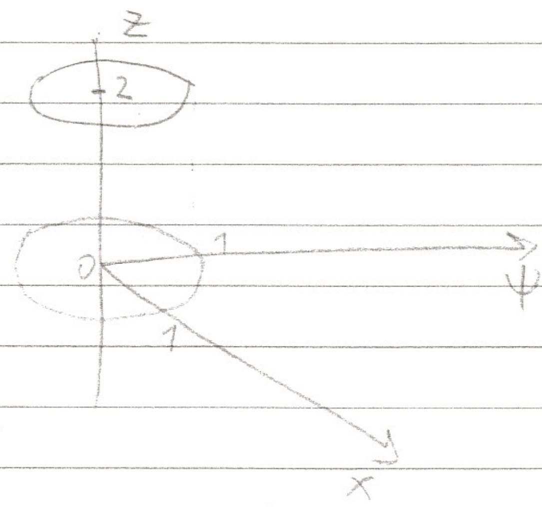
$$= \int_0^1 \left(\int_{-1}^1 (2x) d\psi \right) dx = \int_0^1 (4x) dx = (2x^2) \Big|_0^1 = 2$$

Άσκηση: Να σχεδιάσετε το χωρίο D

$$D = \{(x,\psi,z) \in \mathbb{R}^3 \mid x^2+\psi^2 \le 1,\ 0 \le z \le 2\}$$

και να υπολογίσετε το ολοκλήρωμα:

$$\iiint_D x \, dx\, d\psi\, dz$$



$$\iiint_D x\,dx\,d\psi\,dz = \int_0^2 \Big(\iint_{D(x,\psi,z)} x \, dx\, d\psi \Big) dz =$$

$$= \int_0^2 \Big(\int_{-1}^1 \Big(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x \, d\psi \Big) dx \Big) dz$$

$$= \int_0^2 \Big(\int_{-1}^1 2x\sqrt{1-x^2}\,dx \Big) dz = \int_{-1}^1 2x\sqrt{1-x^2}\,dx \cdot \int_0^2 dz = 0$$

$t = 1-x^2$

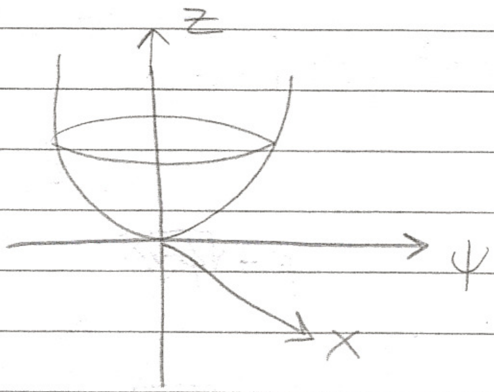
$dt = -2x\,dx$

29/09/2016

<u>Άσκηση</u>: Να σχεδιάσετε το φραγμένο χωρίο που περικλείεται από τις επιφάνειες

$z = x^2 + \psi^2$
$z = 8 - x^2 - \psi^2$

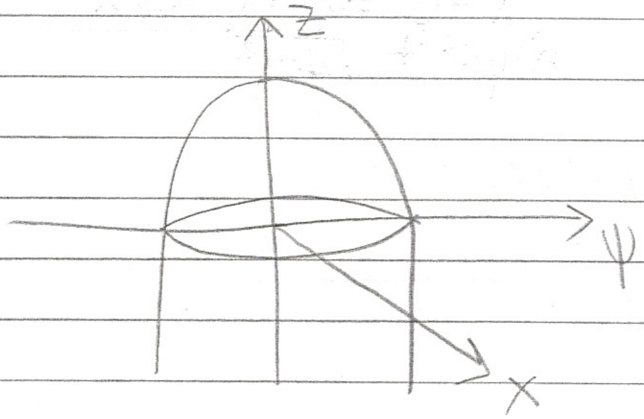
Στη συνέχεια να βρεθεί ο όγκος του χωρίου

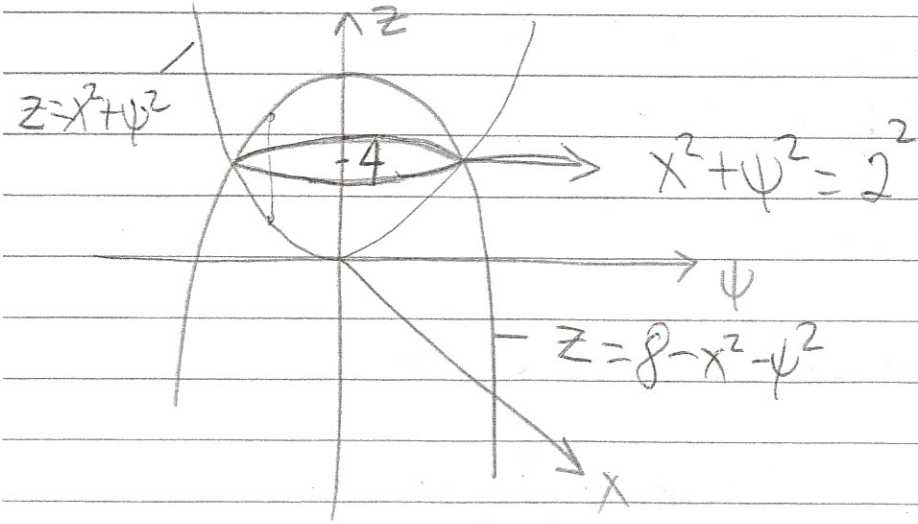


<u>κοινά σημεία</u>

$\left| \begin{array}{l} z = x^2+\psi^2 \\ z = 8-x^2-\psi^2 \end{array} \right| \Leftrightarrow \left| \begin{array}{l} z = x^2+\psi^2 \\ x^2+\psi^2 = 8-x^2-\psi^2 \end{array} \right|$

$\Leftrightarrow \left| \begin{array}{l} z = x^2+\psi^2 \\ x^2+\psi^2 = 4 \end{array} \right| \Leftrightarrow \left| \begin{array}{l} z = 4 \\ x^2+\psi^2 = 2^2 \end{array} \right|$





Όγκος του D $= \iiint_D 1 \, dx\,d\psi\,dz$

$$V_1 = \iint_{D_1} \overset{(8-4)}{(4 - x^2 - \psi^2)}\, dx\,d\psi$$

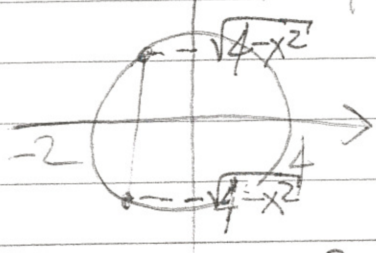
$$= \iint_{D_1} \left(\int_{x^2+\psi^2}^{8-x^2-\psi^2} 1 \, dz \right) dx\,d\psi$$

$$= \iint_{D_1} \left[8 - x^2 - \psi^2 - x^2 - \psi^2 \right] dx\,d\psi$$

$$= 2 \iint_{D_2} (4 - x^2 - \psi^2)\, dx\,d\psi$$

$x^2+\psi^2 \leq 4$

$$D_1 = \{ (x, \psi, 4) \mid x^2 + \psi^2 \leq 4 \}$$



$$2 \int_{-2}^{2} \left(\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - x^2 - \psi^2)\, d\psi \right) dx$$

$x^2 + \psi^2 = 4 \Rightarrow \psi^2 = 4 - x^2$
$\Rightarrow \psi = \pm \sqrt{4 - x^2}$

Παράδειγμα: Δίνεται η απεικόνιση.

$$T : [-1, 1] \times [-1, 1] \rightarrow \mathbb{R}^2 \qquad [-1, 1]^2$$

με $T(x, \psi) = \left(\frac{x+\psi}{2}, \frac{x-\psi}{2} \right)$

i) Να βρεθεί η εικόνα του D, $T(D)$.
ii) είναι η απεικόνιση 1-1;

Πότε η T θα ήταν 1-1;

(ορισμός: $\forall (x_1, \psi_1) \neq (x_2, \psi_2) \Rightarrow$

$T(x_1, \psi_1) \neq T(x_2, \psi_2)$

($\Leftrightarrow$) Με αντιθετοαντιστροφή)

$$\text{Αν } T(x_1,\psi_1)=T(x_2,\psi_2) \Rightarrow (x_1,\psi_1)=(x_2,\psi_2)$$

<u>1-1:</u> Έστω $T(x_1,\psi_1)=T(x_2,\psi_2) \Leftrightarrow$

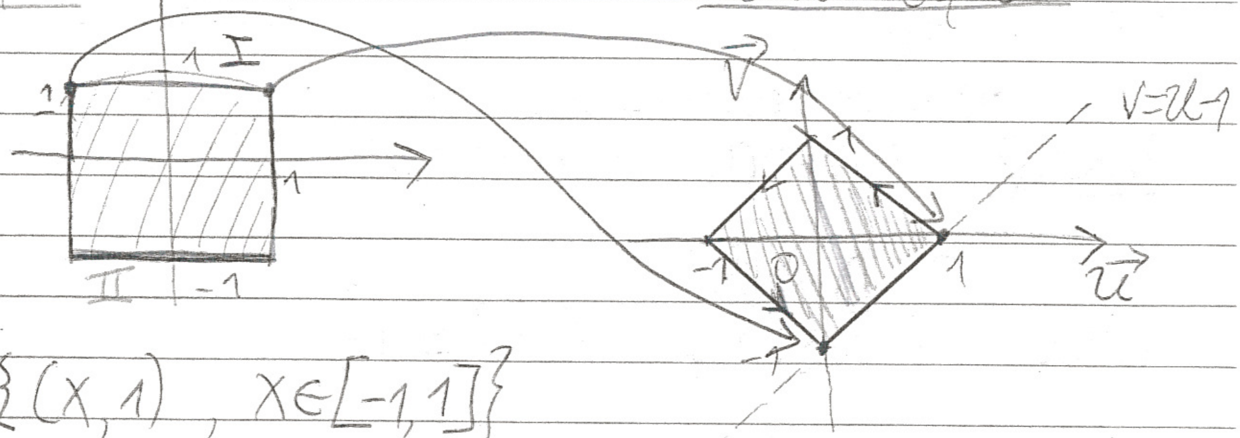
$$\left(\frac{x_1+\psi_1}{2}, \frac{x_1-\psi_1}{2}\right)=\left(\frac{x_2+\psi_2}{2}, \frac{x_2-\psi_2}{2}\right) \Leftrightarrow$$

$$\left.\begin{array}{l} \dfrac{x_1+\psi_1}{2}=\dfrac{x_2+\psi_2}{2} \\ \dfrac{x_1-\psi_1}{2}=\dfrac{x_2-\psi_2}{2} \end{array}\right| \Leftrightarrow \left|\begin{array}{l} x_1+\psi_1=x_2+\psi_2 \\ x_1=x_2 \end{array}\right| \Leftrightarrow \left|\begin{array}{l} x_1=x_2 \\ \psi_1=\psi_2 \end{array}\right.$$

$$\Rightarrow (x_1,\psi_1)=(x_2,\psi_2)$$

<u>Σχεδιασμός της εικόνας</u>

<u>Πεδίο ορισμού της T</u> <u>Πεδίο τιμών</u>



$$I: \{(x,1),\ x\in[-1,1]\}$$

$$T(x,1)=\Big(\underbrace{\frac{x+1}{2}}_{u}, \underbrace{\frac{x-1}{2}}_{v}\Big), \quad x\in[-1,1]$$

$$\left.\begin{array}{l} u=\dfrac{x+1}{2} \\ v=\dfrac{x-1}{2} \end{array}\right\} \Rightarrow \begin{array}{l} x=2u-1 \\ x=2v+1 \end{array} \Bigg| \Leftrightarrow \Bigg| \begin{array}{l} 2u-1=2v+1 \\ x=2v+1 \end{array}$$

$$2u-2v=2 \Leftrightarrow \boxed{v=u-1}$$
$$u-v=1$$

$T(-1,1)=(0,-1)$
$T(1,1)=(1,0)$

$\text{II}: \{(x,-1),\ x\in[-1,1]\}$

$$T(x,-1)=\Big(\underbrace{\frac{x-1}{2}}_{u},\ \underbrace{\frac{x+1}{2}}_{v}\Big)$$

$$\left.\begin{aligned} u&=\frac{x-1}{2}\\ v&=\frac{x+1}{2}\end{aligned}\right\} \Rightarrow \begin{aligned} x&=2u+1\\ x&=2v-1\end{aligned} \ \Big| \Leftrightarrow \Big| \begin{aligned} 2u+1&=2v-1\\ x&=2v+1\end{aligned}$$

$$2u-2v=-2$$
$$u-v=-1$$
$$\Leftrightarrow \boxed{v=u+1}$$

$T(-1,-1)=(-1,0)$

(πολικές συντεταγμένες)

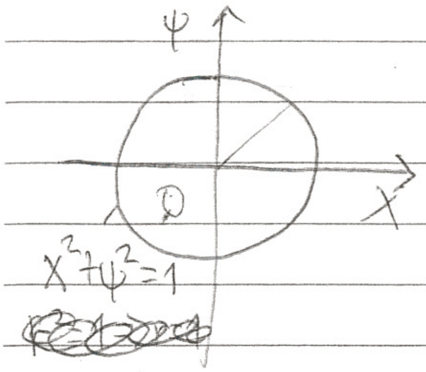
<u>Άσκηση</u>: Δίνεται η άσκηση

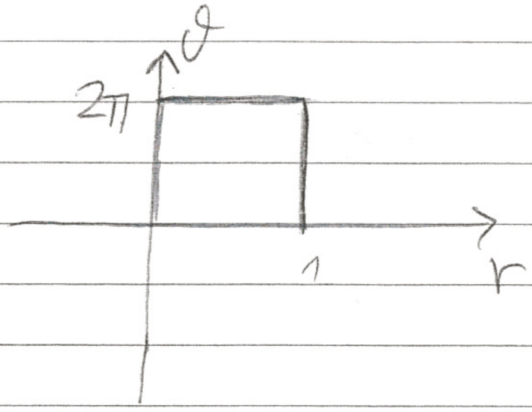
$$T: D \to \mathbb{R}^2$$
όπου $D=\{(x,\psi)\ |\ x^2+\psi^2\le 1\}$

$T(x,\psi)=(r,\vartheta)$

$r=\sqrt{x^2+\psi^2}$
$x=r\cos\vartheta$
$\psi=r\sin\vartheta$

i) Να βρεθεί η εικόνα
ii) Είναι 1-1;





$x^2+\psi^2 \leq 1 \Rightarrow 0 \leq r \leq 1$

$r^2=1 \Rightarrow r=1$

$T(0,\pi)=(0,0)$

$T(0,\frac{\pi}{2})=(0,0)$

$(0,\frac{\pi}{2})$

$(0,\pi)$

$T\ [0,1]^2 \to \mathbb{R}^2,$

$T(x,\psi)=(-x^2+4x,\ \psi)$

i) είναι 1-1;

ii) Να βρεθεί η εικόνα

$\to$ i) 1-1 Αν $T(x_1,\psi_1)=T(x_2,\psi_2) \Rightarrow (x_1,\psi_1)=(x_2,\psi_2)$

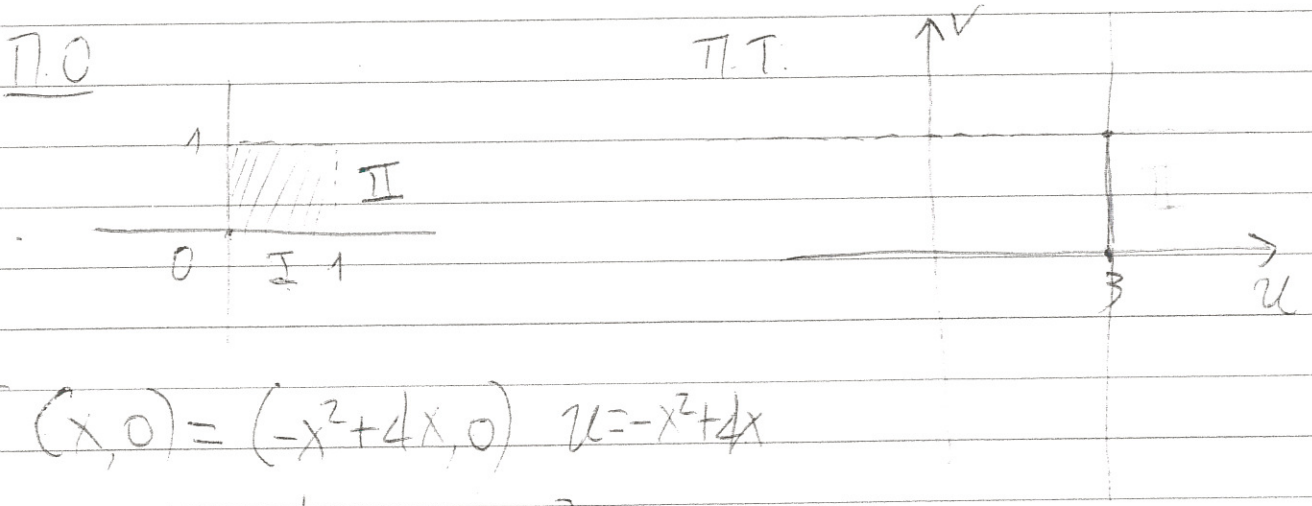
Έστω $T(x_1,\psi_1)=T(x_2,\psi_2) \Leftrightarrow (-x_1^2+4x_1,\psi_1)=(-x_2^2+4x_2$

$\Leftrightarrow \begin{cases} -x_1^2+4x_1=-x_2^2+4x_2 \\ \psi_1=\psi_2 \end{cases} \Leftrightarrow \begin{cases} x_2^2-x_1^2+4(x_1-x_2)=0 \\ \psi_1=\psi_2 \end{cases}$

$$\left.\begin{array}{l}(x_2-x_1)(x_2+x_1)-4(x_2-x_1)=0\\ \psi_1=\psi_2\end{array}\right| \Leftrightarrow \left|\begin{array}{l}(x_2-x_1)(x_1+x_2-4)=0\\ \psi_1=\psi_2\end{array}\right.$$

$$\Leftrightarrow \left|\begin{array}{l}x_1=x_2\\ \psi_1=\psi_2\end{array}\right. \Rightarrow 1\text{-}1$$

$$\begin{array}{l}0\le x_1\le 1\\ 0\le x_2\le 1\end{array} \Rightarrow x_1+x_2\le 2 \Rightarrow \begin{array}{l}x_1+x_2-2\le 0\\ x_1+x_2-4\le -2<0\end{array}$$



$T(x,0)=(-x^2+4x,0)$ $u=-x^2+4x$

I: $\{(x,0)\mid x\in[0,1]\}$

II: $(1,\psi)$, $\psi\in[0,1]$,

$T(1,\psi)=(-1+4,\psi)=(\overset{u}{3},\overset{v}{\psi})$

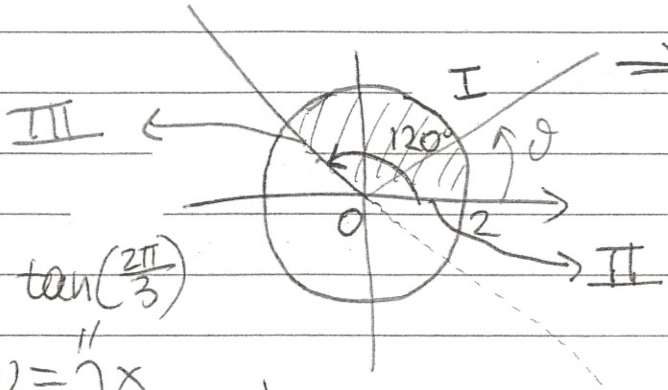
$0\le x\le 1$, $\psi=1$ $T(x,1)=(\underset{u}{-x^2+4x},\underset{v}{1})$

$(x,1)$

Χριστίνα Ευθυμιάδου

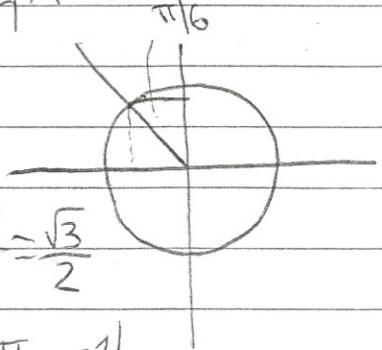
4/10/2016

Άσκηση: Να βρεθεί το D^* ώστε
$T: D^* \to D$ με τύπο
$T(r,\theta) = (r\cos\theta, r\sin\theta)$ όπου D είναι το χωρίο του σχήματος.



$\Rightarrow D = \{(x,\psi) \mid x^2+\psi^2 \leq 2^2$
$\psi \geq -\sqrt{3}x$
$\psi \geq 0$

$\tan\left(\frac{2\pi}{3}\right)$
$\|$
$\psi = \lambda x$

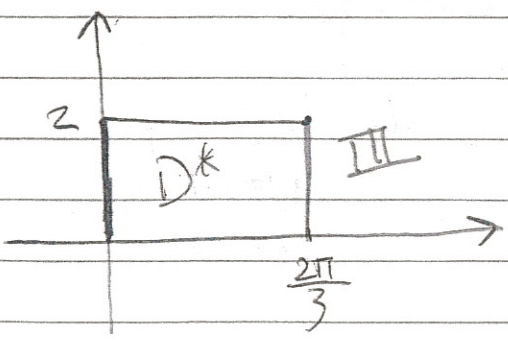


$\tan 2\pi/3 = -\sqrt{3}$
$\psi = -\sqrt{3}x$

$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

$\cos\frac{2\pi}{3} = -1/2$

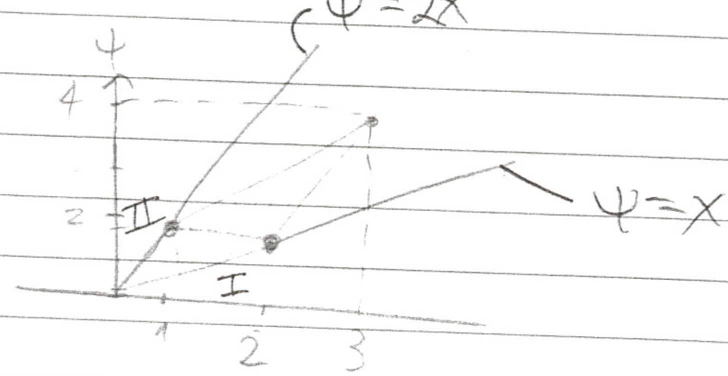
Για το I $\Rightarrow x^2+\psi^2 = 2^2 \Rightarrow r^2 = 2^2 \Rightarrow \underline{r=2}$
$0 \leq \theta \leq \frac{2\pi}{3}$

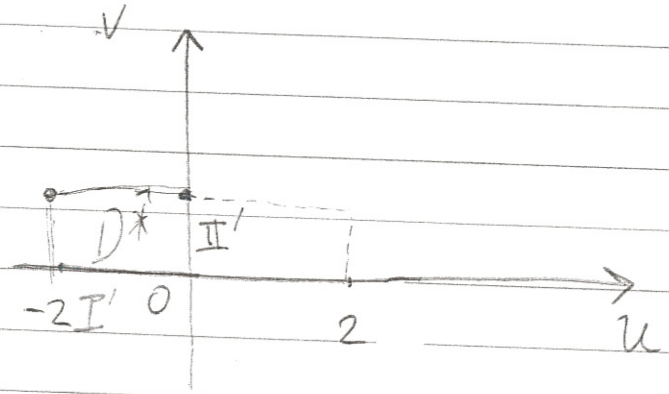


Για το II $\psi = 0$, $0 \leq x \leq 2$
$\Rightarrow \theta = 0$, $0 \leq r \leq 2$

$\psi = -\sqrt{3}x$ $\psi \geq 0 \Rightarrow$
$r\sin\theta = -\sqrt{3}\cos\theta$
$\Rightarrow \theta = \frac{2\pi}{3}$

Άσκηση: Βρείτε την απεικόνιση T ώστε τ ορθογώνιο D^* ώστε $T: D^* \to \Pi$, όπου είναι το χωρίο του σχήματος.





$v = (\psi - x)\,\lambda$

I	$v = (\psi - x)$	$\Leftrightarrow\; u - v = -x \Rightarrow \boxed{\begin{matrix} x = v - u \\ \psi = 2v - u \end{matrix}}$
II	$u = \psi - 2x$	

$\Rightarrow \begin{matrix} x=2 \\ \psi=2 \end{matrix} \Leftrightarrow \begin{matrix} v=0 \\ u=-2 \end{matrix}$

Αντίστοιχα: $x=1, \psi=2$

$v=1$

$u=0$

$x=3 \quad \psi=4$

$\Downarrow$

$v = 4-3 = 1$

$u = 4 - 2\cdot 3 = -2$

Γιατί η προετοιμασία; ⇒ Για τον υπολογισμό ολοκληρωμάτων (όταν κάνουμε αλλαγή συστήματος συντεταγμένων).

<u>Μια μεταβλητή</u>

$$\int_a^b f(x)\,dx \quad , \quad x=g(\psi), \quad dx = g'(\psi)\,d\psi$$

$$= \int_{\psi_1=g^{-1}(a)}^{\psi_2=g^{-1}(b)} f(g(\psi))\cdot g'(\psi)\,d\psi$$

$$a=g(\psi_1) \Leftrightarrow \psi_1=g^{-1}(a)$$
$$b=g(\psi_2) \Leftrightarrow \psi_2=g^{-1}(b)$$

Π

<u>Ανάλογο ΘΕΩΡΗΜΑ</u> $\quad x=x(u,v) \quad \psi=\psi(u,v)$

$$\iint_D f(x,\psi)\,dx\,d\psi = \iint_{D^*} f(x(u,v),\psi(u,v)) \left|\frac{\partial(x,\psi)}{\partial(u,v)}\right| du\,dv.$$

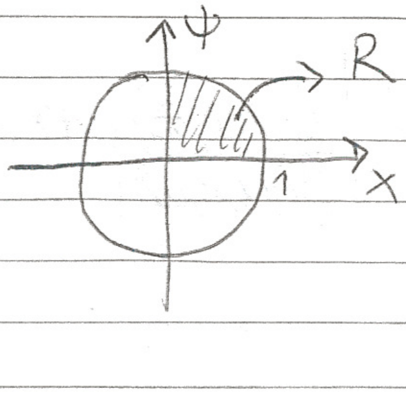
Ιακωβιανή

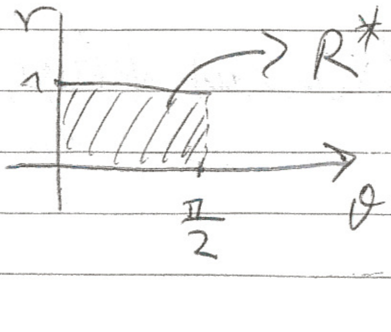
$$\frac{\partial(x,\psi)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial \psi}{\partial u} & \dfrac{\partial \psi}{\partial v} \end{vmatrix}$$

Άσκηση: Να υπολογίσετε το ολοκλήρωμα

$$\iint_R \sqrt{x^2+\psi^2}\,dx\,d\psi \quad \text{όπου } R=\{(x,\psi) \mid x^2+\psi^2\le 1,\ x\ge 0,\ \psi\ge 0\}$$

$$T(r,\theta) = (\overbrace{r\cos\theta}^{x}, \overbrace{r\sin\theta}^{\psi})$$





$x = r\cos\theta$
$\psi = r\sin\theta$

$$\iint_R \sqrt{x^2+\psi^2}\, dx\, d\psi = \iint_{R^*} r \left| \frac{\partial(x,\psi)}{\partial(r,\theta)} \right| dr\, d\theta$$

$x = r\cos\theta, \quad \psi = r\sin\theta$

$\Rightarrow \dfrac{\partial x}{\partial r} = \cos\theta, \quad \dfrac{\partial \psi}{\partial r} = \sin\theta$

$\dfrac{\partial x}{\partial \theta} = -r\sin\theta, \quad \dfrac{\partial \psi}{\partial \theta} = r\cos\theta$

οπότε $\dfrac{\partial(x,\psi)}{\partial(r,\theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial \psi}{\partial r} & \dfrac{\partial \psi}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$

$= r\cos^2\theta + r\sin^2\theta = r$

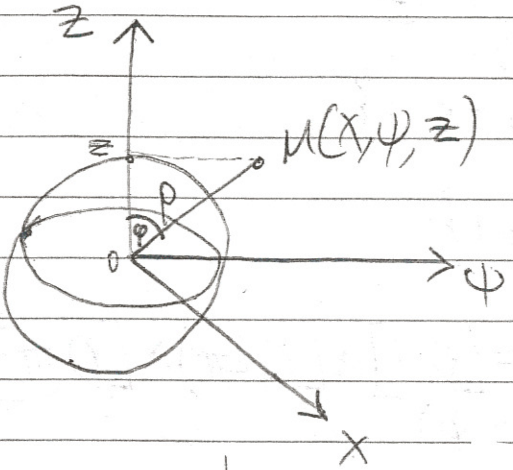
$$= \int_0^{\pi/2} \left(\int_0^1 r \cdot r\, dr \right) d\theta = \int_0^{\pi/2} \left(\int_0^1 r^2 dr \right) d\theta$$

$$= \frac{1}{3} \cdot \frac{\pi}{2} = \frac{\pi}{6}$$

$$\iiint_R f(x,\psi,z)\, dx\, d\psi\, dz$$

$$= \iiint_{R^*} f_1(x(u,v,w), \psi(u,v,w), z(u,v,w)) \left| \frac{\partial(x,\psi,z)}{\partial(u,v,w)} \right| du\, dv\, dw$$

$$\frac{\partial(x,\psi,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial \psi}{\partial u} & \frac{\partial \psi}{\partial v} & \frac{\partial \psi}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$



$\rho = \sqrt{x^2+\psi^2+z^2}$

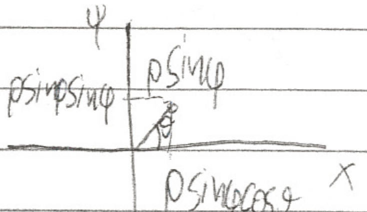
$z = \rho\cos\varphi$

$0 \le \varphi \le \pi$

$x = \rho\sin\varphi\cos\theta$

$\psi = \rho\sin\varphi\sin\theta$

$0 \le \theta \le 2\pi$



$\frac{\partial x}{\partial \rho} = \sin\varphi\cdot\cos\theta$, $\frac{\partial x}{\partial \theta} = \rho\sin\varphi\sin\theta$, $\frac{\partial x}{\partial \phi} = \rho\cos\varphi\cos\theta$

$\frac{\partial \psi}{\partial \rho} = \sin\varphi\sin\theta$, $\frac{\partial \psi}{\partial \theta} = \rho\sin\varphi\cos\theta$, $\frac{\partial \psi}{\partial \phi} = \rho\cos\varphi\sin\theta$

$\frac{\partial z}{\partial \rho} = \cos\varphi$, $\frac{\partial z}{\partial \theta} = 0$, $\frac{\partial z}{\partial \varphi} = -\rho\sin\varphi$

$$\text{Ιακωβιανή} = \begin{vmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta & \rho\cos\varphi\sin\theta \\ \cos\varphi & 0 & -\rho\sin\varphi \end{vmatrix}$$

$$= \rho^2\sin\varphi \begin{vmatrix} \sin\varphi\cos\theta & -\sin\theta & \cos\varphi\cos\theta \\ \sin\varphi\sin\theta & \cos\theta & \cos\varphi\sin\theta \\ \cos\varphi & 0 & -\sin\varphi \end{vmatrix}$$

από την 2η & 3η στήλη, κ.π. το ρ.

$$= \rho^2 \sin\varphi \left(\cos\varphi \begin{vmatrix} -\sin\theta & \cos\varphi\cos\theta \\ \cos\theta & \cos\varphi\sin\theta \end{vmatrix} - \sin\varphi \begin{vmatrix} \sin\varphi\cos\theta & -\sin\theta \\ \sin\varphi\sin\theta & \cos\theta \end{vmatrix} \right.$$

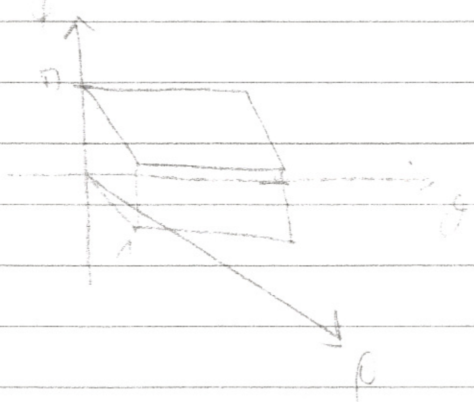
$$= -\rho^2 \sin\varphi$$

<u>Άσκηση</u> Υπολογίστε το $\iiint_D e^{(x^2+\psi^2+z^2)^{3/2}} dx\,d\psi\,dz$

όπου $D = \{(x,\psi,z) \mid x^2+\psi^2+z^2 \leq 1\}$

$B = \{(x,\psi,z) \mid x^2+\psi^2+z^2 \leq 1\}$

$D^* = [0,1] \times [0,2\pi) \times [0,\pi]$
(ρ, θ, φ)



με σφαιρικές συντεταγμένες:

$z = \rho\cos\varphi$
$x = \rho\sin\varphi\cos\theta$ $\quad 0 \leq \rho \leq 1$
$\psi = \rho\sin\varphi\sin\theta$ $\quad 0 \leq \varphi \leq \pi$
$\quad 0 \leq \theta < 2\pi$

$$\int_0^1 \left(\int_0^\pi \left(\int_0^{2\pi} e^{\rho^3} \rho^2 \sin\varphi \, d\theta \right) d\varphi \right) d\rho$$

$$= 2\pi \int_0^1 \left(\int_0^\pi \rho^2 e^{\rho^3} \sin\varphi \, d\varphi \right) d\rho = 2\pi \int_0^1 \left(\rho^2 e^{\rho^3} (-\cos\varphi) \Big|_0^\pi \right) d\rho$$

$$= 4\pi \int_0^1 \rho^2 e^{\rho^3} d\rho = \frac{4\pi}{3}(e-1)$$

5/10/2016

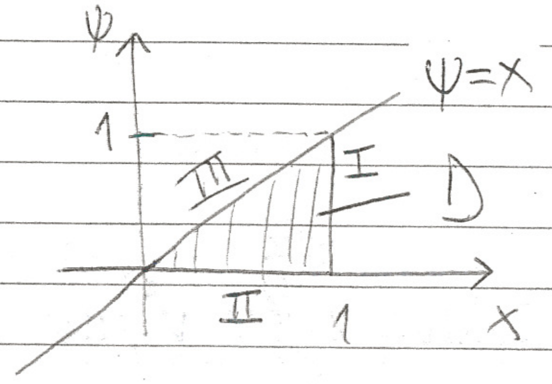
Άσκηση:

Έστω $D=\{(x,\psi) \mid 0\leq \psi \leq x,\ 0\leq x\leq 1\}$

Να υπολογιστεί το:

$\iint_D (x+\psi)\,dx\,d\psi$ με την αλλαγή $x=u+v$, $\psi=u-v$

Ελέγξτε το αποτέλεσμα, υπολογίζοντας ολοκλήρωμα απ' ευθείας.



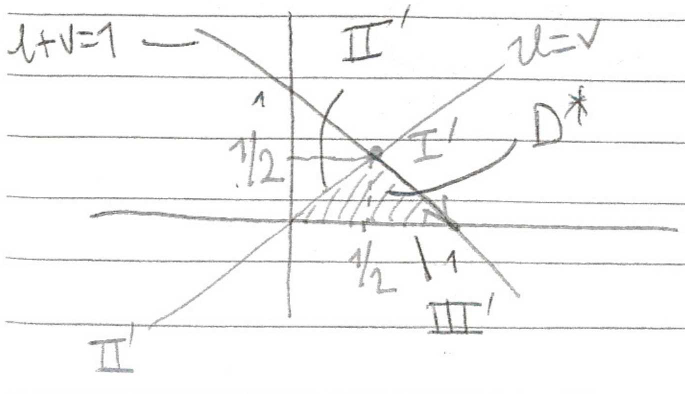
$x=u+v$
$\psi=u-v$

$$\iint_D (x+\psi)\,dx\,d\psi = \iint_{D^*} (u+v+u-v)\left|\frac{\partial(x,\psi)}{\partial(u,v)}\right| du\,dv$$

Ιακωβιανή του μετασχηματισμού:

$$\frac{\partial(x,\psi)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial \psi}{\partial u} & \frac{\partial \psi}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

$\frac{\partial x}{\partial u}=1$, $\frac{\partial x}{\partial v}=1$ $\quad \frac{\partial \psi}{\partial u}=1$, $\frac{\partial \psi}{\partial v}=-1$



$x=1,\ 0\leq\psi\leq 1$
$\Updownarrow$
$u+v=1$

Το $(1,0) \Leftrightarrow u+v=1, u-v=0 \Leftrightarrow u=v=1/2$

Το $(1,1) \Leftrightarrow u+v=1, u-v=1 \Leftrightarrow v=0, u=1$

II. $(x,0)$ $x\in[0,1]$

$u-v=0$

III: $y=x,\ x\in[0,1]\ (x,x)$ $\left.\begin{array}{l} x=u+v \\ x=u-v \end{array}\right| \Rightarrow u+v=u-v \Leftrightarrow v=0.$

$$= \iint_{D^*} 2u\cdot 2\,du\,dv$$

$$= \int_0^{1/2}\left(\int_v^{1-v}(4u)\,du\right)dv$$

$$= \int_0^{1/2}(2u^2)\Big|_{u=v}^{u=1-v}\,dv$$

$$= \int_0^{1/2}\left(2(1-v)^2-2v^2\right)dv = 2\int_0^{1/2}(1-2v)\,dv = 2\int_0^{1/2}(v-v^2)\,dv$$

$$= 2\left(\frac{1}{2}-\frac{1}{4}\right)=\frac{1}{2}$$

Γενικευμένα ολοκληρώματα

1) Αν η συνάρτηση είναι άφραχτη
2) Αν το χωρίο είναι άφρακτο.

$$\int_0^{+\infty}\frac{dx}{x} = \int_0^{+\infty}\frac{dx}{x} + \int_1^{+\infty}\frac{dx}{x}$$

$$\varepsilon>0,\ \lim_{\varepsilon\to 0^+}\int_\varepsilon^1\frac{dx}{x} = \int_0^1\frac{dx}{x} \quad \lim_{\varepsilon\to 0^+}(-\ln\varepsilon)=+\infty$$

$$\| (\ln x)\Big|_\varepsilon^1$$

$$\lim_{A\to+\infty}\int_1^A\frac{dx}{x} = \int_1^{+\infty}\frac{dx}{x}$$

$$\int_1^A \frac{dx}{x} = (\ln x)\Big|_{x=1}^{x=A} = \ln A - \ln 1 = \ln A$$

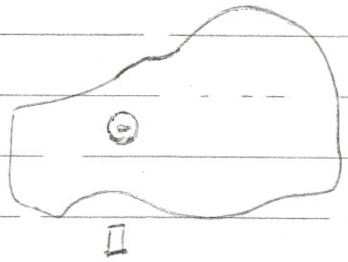
$$\lim_{A\to+\infty} \int_1^A \frac{dx}{x} = \lim_{A\to+\infty} \ln A = +\infty$$

1) Αν η συνάρτηση είναι άφραχτη, $f: D \to \mathbb{R}$
D φραγμένο

$$\lim_{(x,\psi)\to(x_0,\psi_0)} f(x,\psi) = +\infty$$

που επιπρόσθετα

$f: D - B_\varepsilon(x_0,\psi_0)$
φραγμένη



τότε το γενικευμένο ολοκλήρωμα

$\iint_D f(x,\psi)\,dx\,d\psi$ υπάρχει αν

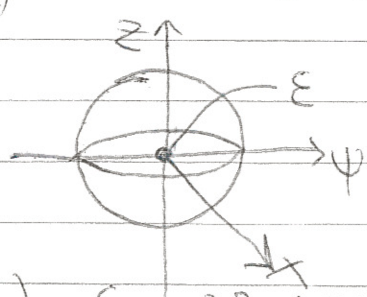
$$\exists \lim_{\varepsilon\to 0^+} \iint_{D-B_\varepsilon(x_0,\psi_0)} f(x,\psi)\,dx\,d\psi = \iint_D f\,dx\,d\psi$$

Άσκηση: Υπάρχει το γενικευμένο ολοκλήρωμα

$$\iiint_B \frac{dx\,d\psi\,dz}{x^2+\psi^2+z^2}$$

και αν ναι να βρεθεί, όπου $B = \{(x,\psi,z),\ x^2+\psi^2+z^2 \le 1\}$

$$\iiint_{B-B_\varepsilon(0)} \frac{dx\,d\psi\,dz}{x^2+\psi^2+z^2}$$



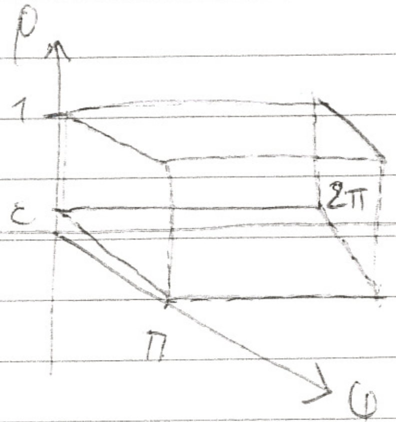
$B_\varepsilon(0) = \{x \in \mathbb{R}^3 \mid \|x\| < \varepsilon\}$

Σε σφαιρικές συντεταγμένες :

$z=\rho\cos\varphi \quad , 0\le\varphi\le\pi$

$x=\rho\sin\varphi\cos\theta \quad , 0\le\theta<2\pi$

$\psi=\rho\sin\varphi\sin\theta$



$$= \iiint_{D^*} \frac{1}{\rho^2}\left|\frac{\partial(x,\psi,z)}{\partial(\rho,\varphi,\theta)}\right| d\rho\, d\varphi\, d\theta$$

$$\iiint_{B-B_\varepsilon} \frac{dx\,d\psi\,dz}{x^2+\psi^2+z^2} = \int_\varepsilon^1\left(\int_0^{2\pi}\left(\int_0^{\pi}\frac{1}{\rho^2}(\rho^2\sin\varphi)\,d\varphi\right)d\theta\right)d\rho$$

$$= 4\pi(1-\varepsilon)$$

$$\lim_{\varepsilon\to0^+}\iiint_{B-B_\varepsilon}\frac{dx\,d\psi\,dz}{x^2+\psi^2+z^2} = \lim_{\varepsilon\to0^+}\left(4\pi(1-\varepsilon)\right)=4\pi$$

$I=\int_{-\infty}^{+\infty} e^{-x^2}dx = \sqrt{\pi}$, Να το αποδείξετε.

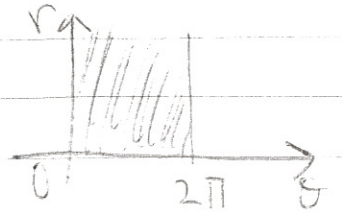
$$I^2=\int_{-\infty}^{+\infty}e^{-x^2}dx\int_{-\infty}^{+\infty}e^{-\psi^2}d\psi$$

$$\iint_{\mathbb{R}^2} e^{-x^2-\psi^2}\,dx\,d\psi$$

Πολικές

$x=r\cos\theta \quad , 0\le r\le+\infty$

$\psi=r\sin\theta, \quad 0\le\theta<2\pi$



$$\left|\frac{\partial(x,\psi)}{\partial(r,\theta)}\right|=r$$

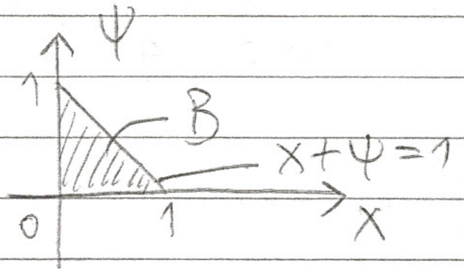
$$\iint_{\mathbb{R}^2} e^{-x^2-\psi^2}\,dx\,d\psi = \int_0^{+\infty}\left(\int_0^{2\pi} e^{-r^2}\cdot r\,d\theta\right)dr = 2\pi\int_0^{+\infty} r e^{-r^2}\,dr = 2\pi\cdot\frac{1}{2} = \pi$$

$$\int_0^A r\cdot e^{-r^2}\,dr = \left(\left(\frac{-e^{-r^2}}{2}\right)\right|_0^A = \frac{-e^{A^2}}{2} + \frac{1}{2}$$

$$\lim_{A\to+\infty}\int_0^A r e^{-r^2}\,dr = \lim_{A\to+\infty}\left(\frac{e^{-A^2}}{2} + \frac{1}{2}\right) = \frac{1}{2}$$

Άσκηση: Να υπολογιστεί το ολοκλήρωμα:

$\iint_B e^{\frac{\psi-x}{\psi+x}}\,dx\,d\psi$, όταν B είναι το εσωτερικό του τριγώνου με κορυφές τα $(0,0)$, $(0,1)$, $(1,0)$



Θέτουμε $u = \psi - x$
$v = \psi + x$

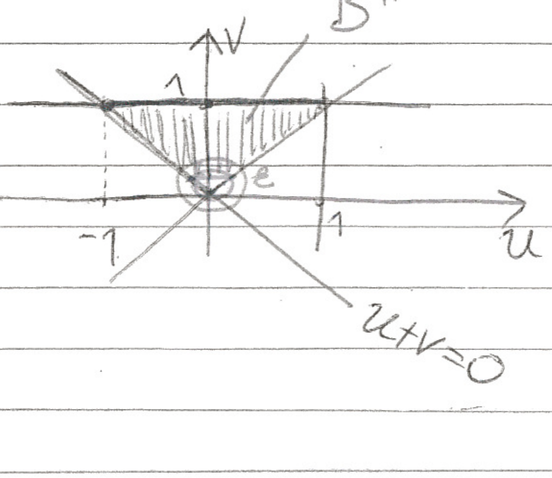
$\left.\begin{array}{l} u = -x \\ v = x \end{array}\right| \Rightarrow u = -v$

$$\iint_B e^{\frac{\psi-x}{\psi+x}}\,dx\,d\psi = \iint_{B^*} e^{\frac{u}{v}}\left|\frac{\partial(x,\psi)}{\partial(u,v)}\right|du\,dv$$

$\Rightarrow \frac{dx}{du} = -\frac{1}{2}$, $\frac{\partial x}{\partial v} = \frac{1}{2}$

$\frac{\partial\psi}{\partial u} = \frac{1}{2}$, $\frac{\partial\psi}{\partial v} = \frac{1}{2}$

$$\frac{\partial(x,\psi)}{\partial(u,v)} = \begin{vmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{vmatrix} = -1/2$$



$$\iint_B e^{\frac{\psi-x}{\psi+x}}\,dx\,d\psi = \iint_{B^*} e^{\frac{u}{v}}\,\frac{1}{2}\,du\,dv$$

$$= \frac{1}{2}\iint_{B^*-B^*_\varepsilon} e^{\frac{u}{v}}\,du\,dv = \frac{1}{2}\int_\varepsilon^1\left(\int_{-v}^{v} e^{\frac{u}{v}}\,du\right)dv$$

B^*_ε τριγωνάκι $= \frac{1}{2}\int_\varepsilon^1\left(\left(v e^{\frac{u}{v}}\Big|_{u=-v}^{u=v}\right)\right)dv$

$$= \frac{1}{2}\int_\varepsilon^1 (ve - ve^{-1})\,dv = \frac{e-e^{-1}}{2}\left(\frac{v^2}{2}\right)\Big|_{v=\varepsilon}^{v=1}$$

$$= \frac{e-e^{-1}}{4}(1-\varepsilon^2)$$

και επειδή $\lim_{\varepsilon\to 0^+}\iint_{B^*-B_\varepsilon} \frac{1}{2} e^{\frac{u}{v}}\,du\,dv = \lim_{\varepsilon\to 0^+}\left(\frac{e-e^{-1}}{4}(1+\varepsilon^2\right.$

$$= \frac{e-e^{-1}}{4}$$

Χριστίνα Ευθυμιάδου

$$\iint_B e^{\frac{\psi-x}{\psi+x}}\,dx\,d\psi = \iint_{B^*} e^{\frac{u}{v}}\,\frac{1}{2}\,du\,dv$$

$$= \frac{1}{2}\iint_{B^*-B^*_\varepsilon} e^{\frac{u}{v}}\,du\,dv = \frac{1}{2}\int_\varepsilon^1\left(\int_{-v}^{v} e^{\frac{u}{v}}\,du\right)dv$$

B^*_ε τριγωνάκι $= \dfrac{1}{2}\displaystyle\int_\varepsilon^1\left(\left(v e^{\frac{u}{v}}\right)\Big|_{u=-v}^{u=v}\right)dv$

$$= \frac{1}{2}\int_\varepsilon^1 (ve - ve^{-1})\,dv = \frac{e-e^{-1}}{2}\left(\frac{v^2}{2}\right)\Bigg|_{v=\varepsilon}^{v=1}$$

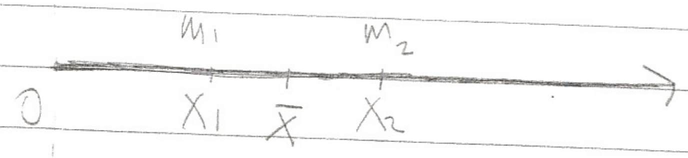
$$= \frac{e-e^{-1}}{4}(1-\varepsilon^2)$$

και επειδή $\displaystyle\lim_{\varepsilon\to 0^+}\iint_{B^*-B_\varepsilon}\frac{1}{2}e^{\frac{u}{v}}\,du\,dv = \lim_{\varepsilon\to 0^+}\left(\frac{e-e^{-1}}{4}\right.$

$$= \frac{e-e^{-1}}{4} \quad //$$

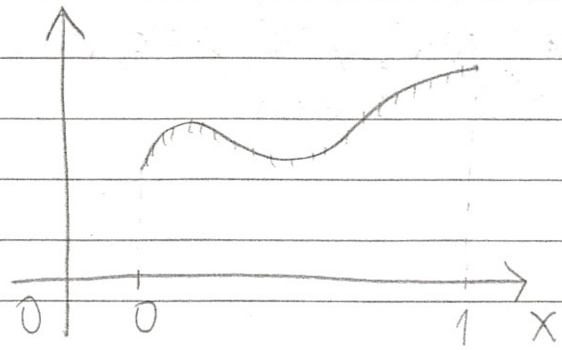
11/10/2016

Κέντρο Μάζας



$$(\bar{X}-X_1)m_1 g = (X_2-\bar{X})m_2 g$$
$$\bar{X}(m_1+m_2) = m_1 X_1 + m_2 X_2$$
$$\bar{X} = \frac{1}{m_1+m_2}(m_1 X_1 + m_2 X_2)$$

$$X = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}$$



$$x \to \rho(x)$$
$$0 \leq x \leq 1$$

$$M_1 = \int_0^1 \rho(x)\,dx$$

$$\bar{x} = \frac{\int_0^1 x\rho(x)\,dx}{\int_0^1 \rho(x)\,dx}$$

$$\rho(x, y, z) \qquad (x, y, z) \in D$$

$$M_{o1} = \iiint_D \rho(x, y, z)\,dx\,dy\,dz$$

$$\bar{X} = (\bar{x}, \bar{y}, \bar{z}) \Rightarrow \bar{x} = \frac{\iiint_D x\rho(x,y,z)\,dx\,dy\,dz}{\iiint_D \rho(x,y,z)\,dx\,dy\,dz}$$

$$\bar{y} = \frac{\iiint_D y\rho(x,y,z)\,dx\,dy\,dz}{\iiint_D \rho(x,y,z)\,dx\,dy\,dz}$$

$$\bar{z} = \frac{\iiint_D z\rho(x,y,z)\,dx\,dy\,dz}{\iiint \rho(x,y,z)\,dx\,dy\,dz}.$$

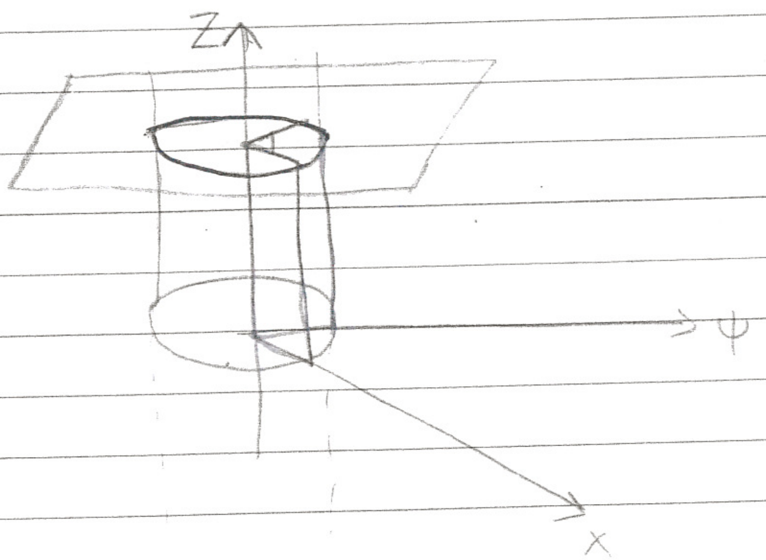
Άσκηση: Να υπολογιστεί το ολοκλήρωμα

$$\iiint_W z\,dx\,d\psi\,dz$$

όπου το W είναι το φραγμένο χωρίο που ορίζεται από τα επίπεδα, $x=0$, $\psi=0$, $z=0$, $z=1$, τους ημίχωρους $x\geq 0$, $\psi \geq 0$ και $x^2+\psi^2 \leq 1$.

στο επίπεδο: εσωτ. του κύκλου

στον χώρο: εσωτ. του κυλίνδρου



$$\iiint_W z\,dx\,d\psi\,dz$$

κυλινδρικές συν/νες:

(x, ψ, z)

$x = r\cos\theta \qquad 0 \leq \theta \leq \pi/2$

$\psi = r\sin\theta \qquad 0 \leq r \leq 1$

$z = z \qquad 0 \leq z \leq 1$

Με αντικατασταση παιρνουμε

$$\iiint_W z\,dx\,d\psi\,dz = \iiint_{W^*} z \left|\frac{\partial(x,\psi,z)}{\partial(r,\theta,z)}\right| dr\,d\theta\,dz$$

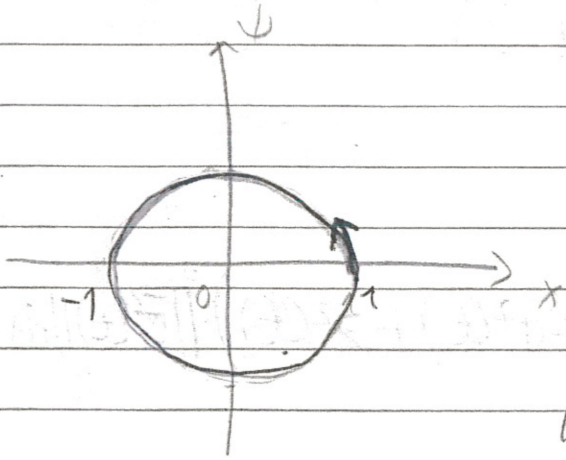
$$= \int_0^1 \left(z \int_0^{\pi/2} \left(\int_0^1 r\,dr\right) d\theta\right) dz = \frac{\pi}{8}$$

Επικαμπύλια ολοκληρώματα

Καμπύλες : $\vec{\sigma} : [a,b] \to \mathbb{R}^2$

$\vec{\sigma}(t) = (x(t), \psi(t)), \quad t \in [a,b]$

$\vec{\sigma}(\theta) = (\cos\theta, \sin\theta), \quad \theta \in [0,\pi]$



$\vec{\sigma} : [a,b] \to \mathbb{R}^3$

α' είδους

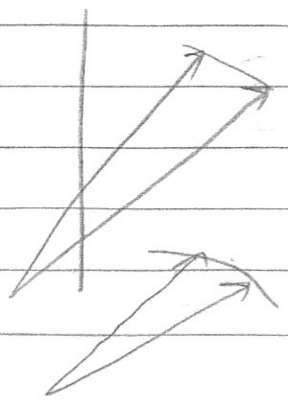
$$\int_C p(x,\psi,z)\,ds$$

$p(x,\psi,z) = 1$

$\int_C 1\,ds =$ συνολικό μήκος

$$\int_C p(x,\psi,z)\,ds = \int_a^b \underbrace{p(\vec{\sigma}(t))}_{= p(x(t),\psi(t),z(t))} \|\vec{\sigma}'(t)\|\,dt$$

$p : \mathbb{R}^3 \to \mathbb{R}$



$$\left|\frac{\vec{\sigma}(t+h)-\vec{\sigma}(t)}{h}\right|$$

$$\|\vec{\sigma}'(t)\|\, dt$$

$$\vec{\sigma}(t)=(x(t),\psi(t)\ z(t))$$
$$\vec{\sigma}'(t)=(x'(t),\psi'(t),z'(t))$$

$$\|\vec{\sigma}'(t)\|=\sqrt{(x'(t))^2+(\psi'(t))^2+(z(t))^2}$$

<u>Άσκηση</u>: Δίνεται η έλικα $\vec{\sigma}:[0,2\pi]\to\mathbb{R}^3$
με τύπο $\sigma(t)=(\cos t,\sin t,t)$, $t\in[0,2\pi]$.
και $f:\mathbb{R}^3\to\mathbb{R}$ με τύπο $f(x,\psi,z)=x^2+\psi^2+z^2$.
Υπολογίστε τό ολοκλήρωμα:

$$\int_C f\, ds \ \left(\int_{\vec{\sigma}} f\, ds\right)$$

$$\int_C (x^2+\psi^2+z^2)\, ds=\int_0^{2\pi}(x^2(t)+\psi^2(t)+z^2(t))\cdot\|\vec{\sigma}'(t)\|dt$$

$$\vec{\sigma}(t)=(\cos t,\sin t,t)\Rightarrow\vec{\sigma}'(t)=(-\sin t,\cos t,1)$$
$$x(t)=\cos t$$
$$\psi(t)=\sin t$$
$$z(t)=t\quad,\ t\in[0,2\pi],\ \|\sigma'(t)\|=\sqrt{(-\sin t)^2+(\cos t)^2+1}$$
$$=\sqrt{2}$$

$$=\int_0^{2\pi}[\cos^2 t+\sin^2 t+t^2]\cdot\sqrt{2}\,dt=\sqrt{2}\int_0^{2\pi}(1+t^2)\,dt=$$

$$=\sqrt{2}\left(t+\frac{t^3}{3}\right)\Big|_0^{2\pi}=\sqrt{2}\left(2\pi+\frac{8\pi^3}{3}\right)$$

Ιδιότητες:

i) $\int_C (f+g)\,ds = \int_C f\,ds + \int_C g\,ds$

ii) $\lambda \in \mathbb{R}$ $\int_C (\lambda f)\,ds = \lambda \int_C f\,ds$

β' είδους

$$\int_C \vec{F}(x,\psi,z)\cdot d\vec{s}$$

$$\int_C \vec{F}(\overbrace{x,\psi,z}^{\vec{r}})\, d\vec{r}$$

$$\int_C \vec{F}(\vec{r})\cdot d\vec{r}$$

$\vec{\sigma}(t) = x(t), \psi(t), z(t)$

$\vec{\sigma}'(t) = (x'(t)), \psi'(t), z'(t))$

$$\int_a^b \vec{F}(\vec{\sigma}(t))\cdot d\vec{\sigma}(t) = \int_a^b \vec{F}(\vec{\sigma}(t))\cdot \vec{\sigma}'(t)\,dt =$$

$$= \int_a^b \left[(P(\vec{\sigma}(t)),\right.$$

$\vec{F} = (P, Q, R)$

$$\left. Q(\vec{\sigma}(t)), R(\vec{\sigma}(t)\cdot (x'(t)), \psi'(t), z'(t)\right)dt$$

$$= \int_a^b \left[P(\vec{\sigma}(t)\cdot x'(t) + Q(\vec{\sigma}(t)\cdot \psi'(t) + R(\vec{\sigma}(t))\cdot z'(t)\right]dt.$$

$$\int_C \vec{F}(x,\psi,z)\cdot d\vec{s} = \int_C (P,Q,R)\cdot(dx, d\psi, dz)$$

$$= \int_C (P\,dx + Q\,d\psi + R\,dz)$$

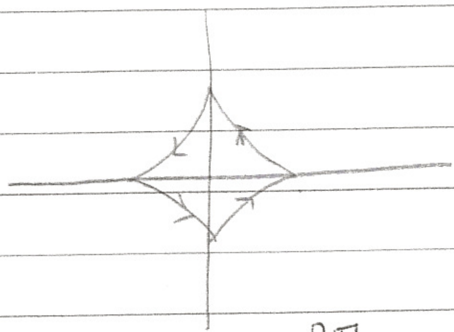
Άσκηση: Να υπολογιστεί το ολοκλήρωμα διανυσματικού πεδίου $\vec{F}(x,\psi) = x\vec{i} + \psi\vec{j} = (x,\psi)$ κατά μήκος της καμπύλης (υποκυκλοειδής) $\vec{\sigma}(t) = (\cos^3 t, \sin^3 t)$, $t \in [0, 2\pi]$.

$(\cos t, \sin t)$

$\cos^3 t = x \qquad \cos t = x^{1/3}$

$\sin^3 t = \psi \qquad \sin t = \psi^{1/3}$

$$\boxed{1 = x^{2/3} + \psi^{2/3}} \qquad 1 = \cos^2 t + \sin^2 t$$



$$\int_C \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F}(\vec{\sigma}(t)) \cdot \vec{\sigma}\,'(t)\,dt$$

$$= \int_0^{2\pi} (\underbrace{x(t)}_{\cos^3 t}, \underbrace{\psi(t)}_{\sin^3 t}) \cdot (-3\cos^2 t \sin t, 3\sin^2 t \cos t)\,dt$$

$$= \int_0^{2\pi} [\cos^3 t(-3\cos^2 t \sin t) + \sin^3 t \cdot 3\sin^2 t \cos t]\,dt$$

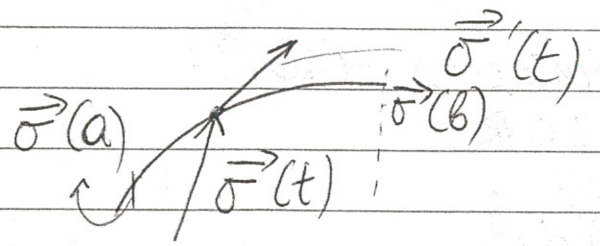
$$= \int_0^{2\pi} [-3\cos^5 t \sin t + 3\sin^5 t \cos t]\,dt$$

$$= 3\int_0^{2\pi} \left[\left(\frac{\cos^6 t}{6}\right)' + \left(\frac{\sin^6 t}{6}\right)'\right] dt = 3 \cdot 0 = 0$$

13/10/2016

$\vec{\sigma}:[a,b] \to \mathbb{R}^3$, ομαλές

$\vec{\sigma}'(t)$ εφαπτόμενο διάνυσμα της καμπύλης στο t_0



$$\lim_{h\to 0} \frac{\vec{\sigma}(t_0+h) - \vec{\sigma}(t_0)}{h} \quad h \neq 0 \quad = \vec{\sigma}'(t_0)$$

α' είδους:
f βαθμωτή

$$\int_{\vec{\sigma}} f(x,\psi,z)\,ds = \int_a^b f(\vec{\sigma}(t))\,\|\vec{\sigma}'(t)\|\,dt$$

$f: \mathbb{R}^3 \to \mathbb{R}$ $\quad \int_C$

Αν $f \equiv 1 \Rightarrow \int_a^b \|\vec{\sigma}'(t)\|\,dt$

μήκος της καμπύλης

(σε διανυσματικές συναρτήσεις)

β' είδους:

$$\int_{\vec{\sigma}} \vec{F}\cdot d\vec{s} = \int_a^b \vec{F}(\vec{\sigma}(t))\cdot \vec{\sigma}'(t)\,dt$$

$$\int_{\vec{\sigma}} \vec{F}\cdot d\vec{r} = \int_{\vec{\sigma}} (P\,dx + Q\,d\psi + R\,dz)$$

$$\vec{F} = (P, Q, R)$$

$$\vec{\sigma}(t) = (x(t), \psi(t), z(t))$$

$$dx = x'(t)\,dt$$

$$\vec{F} = (P, Q, R) = \int_a^b [P(x(t),\psi(t),z(t))\cdot x'(t) + Q(\vec{\sigma}(t))\cdot \psi'(t) + R(\vec{\sigma}(t))\cdot z'(t)]\,dt$$

Άσκηση: Έστω C η καμπύλη $\vec{\sigma}(\theta) = (\cos^3\theta, \sin^3\theta, \theta)$
$0 \le \theta \le \frac{7\pi}{2}$

Να υπολογιστεί το επικαμπύλιο ολοκλήρωμα

$\int_C \vec{F} \cdot d\vec{s}$ όπου $\vec{F}(x,\psi,z) = (\sin z, \cos z, -(x\psi)^{1/3})$,
$x, \psi, z \in \mathbb{R}$.

$$\int_{\vec{\sigma}} \vec{F} \cdot ds = \int_0^{\frac{7\pi}{2}} \vec{F}(\vec{\sigma}(\theta)) \cdot \vec{\sigma}'(\theta)\, d\theta$$

$$\vec{\sigma}'(\theta) = (-3\cos^2\theta\sin\theta, 3\sin^2\theta\cos\theta, 1), \ \theta \in [0, \tfrac{7\pi}{2}]$$

$$\vec{F}(\vec{\sigma}(\theta)) = (\sin\theta, \cos\theta, -(\cos^3\theta\sin^3\theta)^{1/3}), \ \theta \in [0, \pi/2]$$

$$= (\sin\theta, \cos\theta, -\sin\theta\cos\theta), \ \theta \in [0, \tfrac{7\pi}{2}]$$

Οπότε $\int_{\vec{\sigma}} F \cdot d\vec{s} = \int_0^{7\pi/2} (\sin\theta, \cos\theta, -\sin\theta\cos\theta) \cdot (-3\cos^2\theta\sin\theta, 3\sin^2\theta\cos\theta, 1)\, d\theta$

$$= \int_0^{7\pi/2} [-3\cos^2\theta\sin^2\theta + 3\sin^2\theta\cos^2\theta - \sin\theta\cos\theta]\, d\theta$$

$$= \int_0^{7\pi/2} \frac{\sin 2\theta}{2}\, d\theta = -\frac{(-\cos 2\theta)}{4}\Big|_0^{\frac{7\pi}{2}} = \frac{\cos 7\pi - \cos 0}{4}$$

$$= -\frac{1}{2}$$

ΘΕΩΡΗΜΑ: Έστω $\vec{\sigma}: [a,b] \to \mathbb{R}^3$ ομαλή καμπύλη και $f: \mathbb{R}^3 \to \mathbb{R}$ ομαλή $(C^1(\mathbb{R}^3))$ συνάρτηση

Τότε $\int_{\vec{\sigma}} \nabla f \cdot d\vec{s} = f(\vec{\sigma}(b)) - f(\vec{\sigma}(a))$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial \psi}, \frac{\partial f}{\partial z}\right)$$

Αποδ: $\int_{\vec{\sigma}} \nabla f \cdot d\vec{s} = \int_a^b (\nabla f)(\vec{\sigma}(t)) \cdot \vec{\sigma}\,'(t)\,dt$

Αν $\vec{\sigma}(t) = (x(t), \psi(t), z(t))$ · κανόνας αλυσίδας

$$\frac{d}{dt} f(\vec{\sigma}(t)) = \frac{d}{dt} f(x(t), \psi(t), z(t))$$

$$= \frac{\partial f}{\partial x}(x(t), \psi(t), z(t)) \cdot x'(t) +$$

$$\frac{\partial f}{\partial \psi}(x(t), \psi(t), z(t)) \cdot \psi'(t) +$$

$$\frac{\partial f}{\partial z}(x(t), \psi(t), z(t)) \cdot z'(t)$$

$$= \nabla f(\vec{\sigma}(t)) \cdot \vec{\sigma}\,'(t)$$

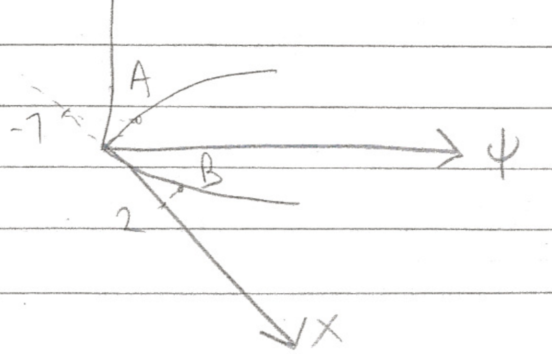
$$\frac{d}{dx}\left(f(\overbrace{g(x)}^{\psi})\right) = \frac{df}{d\psi} \cdot \frac{dg}{dx} = f'(g(x)) \cdot g'(x)$$

$$= \nabla f(\vec{\sigma}(t)) \cdot \vec{\sigma}\,'(t)$$

Επομένως: $\int_{\vec{\sigma}} \nabla f \cdot d\vec{s} = \int_a^b \frac{d}{dt}\left(f(\vec{\sigma}(t))\right) dt$

$$= f(\vec{\sigma}(b)) - f(\vec{\sigma}(a))$$

Άσκηση: Θεωρούμε τη δύναμη $\vec{F}(x, \psi, z) = (x, \psi, z)$. Υπολογίστε το έργο που παράγει για τη μετακίνηση της ενός σωματιδίου κατά μήκος της παραβολής $\psi = x^2$, $z = 0$, από $x = -1$ έως $x = 2$.



$A = (-1, 1, 0)$
$B = (2, 4, 0)$

$\vec{\sigma}'(x) = (1, 2x, 0)$

$\vec{\sigma}(x) = (x, x^2, 0)$
$x \in [-1, 2]$

$$\int_{\vec{\sigma}} \vec{F} \cdot d\vec{s} = \int_{-1}^{2} \vec{F}(\vec{\sigma}(x)) \cdot \vec{\sigma}'(x)\,dx$$

$$= \int_{-1}^{2} (x, x^2, 0) \cdot (1, 2x, 0)\,dx$$

$$= \int_{-1}^{2} (x + x^2(2x) + 0)\,dx = \int_{-1}^{2} (x + 2x^3)\,dx$$

$$= \left(\frac{x^2}{2} + \frac{x^4}{2}\right)\Big|_{-1}^{2} = 2 + 8 - \left(\frac{1}{2} + \frac{1}{2}\right) = 9.$$

<u>2^{os} τρόπος</u> Παρατηρώ ότι:

$$\vec{F} = \nabla f$$

με $f(x, \psi, z) = \frac{1}{2}(x^2 + \psi^2 + z^2)$

από το προηγούμενο θεώρημα

$$\int_{\vec{\sigma}} \vec{F} \cdot d\vec{s} = \int_{\vec{\sigma}} \nabla f \cdot d\vec{s} = f(\vec{\sigma}(2)) - f(\vec{\sigma}(-1))$$

$$= \frac{1}{2}\left(x^2(2) + \psi^2(2) + z^2(2)\right) - \frac{1}{2}\left(x^2(-1) + \psi^2(-1) + z^2(-1)\right)$$

$$= 9$$

$x(2) = 2, \quad x(-1) = -1$
$\psi(2) = 4, \quad \psi(-1) = 1$
$z(2) = 0, \quad z(-1) = 0$

Επιφανειακά ολοκληρώματα

Επιφάνειες

Δύο παραμέτρους: $D \subseteq \mathbb{R}^2$

$\Phi : D \to \mathbb{R}^3$

$\Phi(u,v) = (x(u,v), \psi(u,v), z(u,v))$
$(u,v) \in D$

Αν $D = [0,1] \times [2,5]$

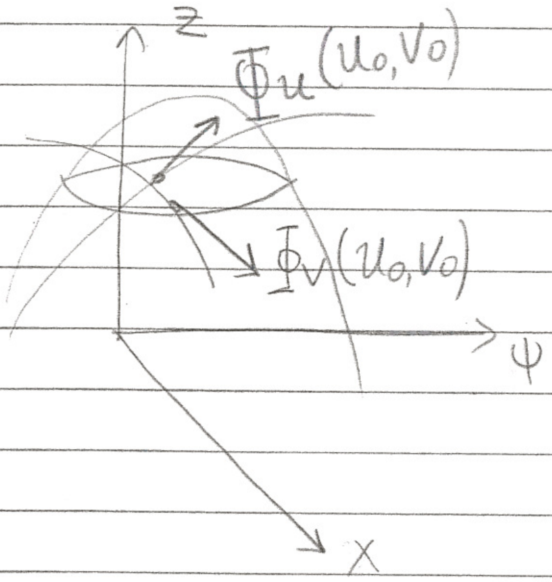
Ομαλές επιφάνειες: C^1 επιφάνειες

λείες επιφάνειες: όταν $\Phi_u \times \Phi_v \neq 0$

$\frac{\partial}{\partial u}\Phi = \Phi_u = (x_u(u,v), \psi_u(u,v), z_u(u,v))$

$\frac{\partial}{\partial v}\Phi = \Phi_v$

κλίση εφαπτόμενης καμπύλης $(v = v_0)$



$\Phi_u(u_0,v_0) = (x_u(u_0,v_0), \psi_u(u_0,v_0), z_u(u_0,v_0))$
T_u //

$\Phi_v(u_0,v_0) = (x_v(u_0,v_0), \psi_v(u_0,v_0), z_v(u_0,v_0))$
// T_v

$T_u(u_0,v_0) \times T_v(u_0,v_0)$ κάθετο διάνυσμα στο εφαπτόμενο επίπεδο στο σημείο $\Phi(u_0,v_0)$.

$\|T_u(u_0,v_0) \times T_v(u_0,v_0)\|$ = εμβαδόν παρ/μα που σχηματίζουν

Χριστίνα Ευθυμιάδου

$\|T_u(u_0,v_0) \times T_v(u_0,v_0)\|$ = εμβαδόν παραλληλ. που σχηματίζουν //

18/10/2016

Λείες επιφάνειες

C^1 επιφάνεια $\Phi: D \to \mathbb{R}^3$

$S = \Phi(D)$

$\|\Phi_u \times \Phi_v\| \neq 0$

Εφαπτόμενο επίπεδο $\Phi(u_0,v_0) \in \mathbb{R}^3$

$Ax + B\psi + \Gamma z = \Delta$

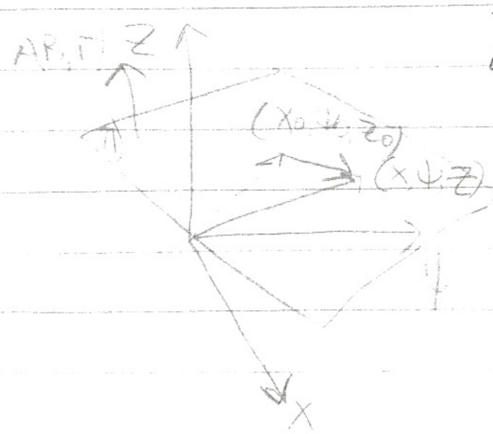
$|A| + |B| + |\Gamma| \neq 0$

- ευθεία και σημείο
- τρία μη συγγραμμικά σημεία
- σημείο του επιπέδου & ένα κάθετο διάνυσμα σ' αυτό

$A x_0 + B\psi_0 + \Gamma z_0 = \Delta$

$\Leftrightarrow Ax + B\psi + \Gamma z = A x_0 + B\psi_0 + \Gamma z_0$

$\Leftrightarrow (A,B,\Gamma)\cdot(x-x_0, \psi-\psi_0, z-z_0) = 0$



$\vec{x}-\vec{x_0} = \vec{u} = (x-x_0, \psi-\psi_0, z-z_0)$

$(A,B,\Gamma)\cdot(x-x_0,\psi-\psi_0,z-z_0) = 0$

$(A,B,\Gamma) \perp (x-x_0, \psi-\psi_0, z-z_0)$

Παραμετρική μορφή

$(x, \psi, z) = \vec{x} = (x_0, \psi_0, z_0) + t\vec{u} + s\vec{v}$

$\vec{u}, \vec{v}$ γραμμικά ανεξάρτητα διανύσματα στο επίπεδο

$\Phi(u_0, v_0) \in \mathbb{R}^3$

(κάθετο διάνυσμα της Φ στο $\Phi(u_0,v_0)$) $\Phi_u(u_0,v_0) \times \Phi_v(u_0,v_0)$ είναι κάθετο στο εφαπτόμενο επίπεδο της επιφάνειας στο $\Phi(u_0,v_0)$

$(A, B, \Gamma) \cdot (x - x_0, \psi - \psi_0, z - z_0) = 0$

$(x_0, \psi_0, z_0) = \Phi(u_0, v_0)$

$(A, B, \Gamma) = \Phi_u(u_0,v_0) \times \Phi_v(u_0,v_0) \neq 0$

$\vec{u} \neq 0 \iff \|\vec{u}\| \neq 0$

Άσκηση: Θεωρούμε την επιφάνεια που δίνεται από την επιφάνεια

$$x = u\cos v \qquad \psi = u\sin v \qquad z = u \qquad u \geq 0,\ v \in [0, 2\pi)$$

Είναι η επιφάνεια διαφορίσιμη; Είναι λεία; Να βρεθεί το εφαπτόμενο επίπεδο της επιφάνειας στο (u_0, v_0), $u_0 \neq 0$

$((x,\psi,z)-(u_0\cos v_0, u_0\sin v_0, u_0))\cdot(-u_0\cos v_0, -u_0\sin v_0, u_0)=$

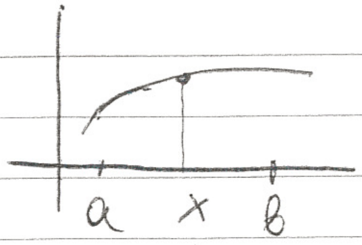
$-u_0\cos v_0(x-u_0\cos v_0)-u_0\sin v_0(\psi-u_0\sin v_0)+u_0(z-u_0)=0$

$-\cos v_0 x+\underbrace{u_0\cos^2 v_0-\sin v_0\psi+u_0\sin^2 v_0}+z-u_0=0$

$\Longleftrightarrow -\cos v_0 x-\sin v_0\psi+z=0$

<u>Άσκηση</u>: Υποθέτουμε ότι η επιφάνεια S είναι γράφημα συνάρτησης $f:\mathbb{R}^2\to\mathbb{R}$. Βρείτε το εφαπτόμενο επίπεδο της επιφάνειας S

<u>Ερώτημα</u> για το επίπεδο: καμπύλη στο επίπεδο $\vec{\sigma}(t)=(x(t),\psi(t))$



$\psi=f(x)$

$\exists\ D\quad (x,\psi)\in D$

$S:=\{(x,\psi,\ f(x,\psi)),\ (x,\psi)\in D\}$

$S:=\{(x(u,v),\psi(u,v),\ z(u,v))\ (u,v)\in D\}$

$\Phi(x,\psi)=(x,\psi,f(x,\psi)),\quad (x,\psi)\in D$

$\Phi_x(x,\psi)=(1,0,\frac{\partial f}{\partial x}(x,\psi))$

$\Phi_\psi(x,\psi)=(0,1,\frac{\partial f}{\partial \psi}(x,\psi))$

$$\Phi_x \times \Phi_\psi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial \psi} \end{vmatrix} = \begin{vmatrix} 0 & \frac{\partial f}{\partial x} \\ 1 & \frac{\partial f}{\partial \psi} \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & \frac{\partial f}{\partial x} \\ 0 & \frac{\partial f}{\partial \psi} \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \vec{k}$$

$$= \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial \psi}, 1\right)$$

εφαπτόμενο επίπεδο της επιφανειας στο (x_0, ψ_0)

$(x_0, \psi_0, f(x_0, \psi_0))$

$$(x - x_0, \psi - \psi_0, z - f(x_0), \psi_0) \cdot \left(-\frac{\partial f}{\partial x}(x_0, \psi_0), -\frac{\partial f}{\partial \psi}(x_0, \psi_0), 1\right) = 0$$

$$-\nabla f(x_0, \psi_0) \cdot (x - x_0, \psi - \psi_0) + z - f(x_0, \psi_0) = 0$$

$$-\left(\frac{\partial f}{\partial x}(x_0, \psi_0), \frac{\partial f}{\partial \psi}(x_0, \psi_0)\right) \cdot (x - x_0, \psi - \psi_0) + z - f(x_0, \psi_0) = 0$$

//

20/10/2016

$\Phi : D \to \mathbb{R}^3, \quad \Phi \in C^1$

$$\Phi_u(u, v) \times \Phi_v(u, v) \neq 0$$

$$\Phi_u(u_0, v_0) \times \Phi_u(u_0, v_0) \cdot ((x, \psi, z) - \Phi(u_0, v_0)) = 0$$

<u>Άσκηση</u>: Βρείτε μια παραμετροποίηση (της επιφάνειας) για το υπερβολοειδές

$$x^2 + \psi^2 - z^2 = 1$$

$\boxed{x^2+\psi^2=w^2}$
$\boxed{w^2-z^2=1}$

$x=\rho\cos\theta$
$\psi=\rho\sin\theta$
$w=\rho$
$\rho=\cosh\phi$
$z=\sinh\phi$

Πρόχειρο

$\rho^2-z^2=1$

$\boxed{\cosh^2 x-\sinh^2 x=1}$

$\cosh x=\dfrac{e^x+e^{-x}}{2}$

$\sinh x=\dfrac{e^x-e^{-x}}{2}$

$\cos x=\dfrac{e^{ix}+e^{-ix}}{2}$

$\sin x=\dfrac{e^{ix}-e^{-ix}}{2i}$

* $e^{ix}=\cos x+i\sin x$

$x=\rho\cos\theta=\cosh\varphi\cos\theta$
$\psi=\rho\sin\theta=\cosh\varphi\sin\theta$

Απάντηση: Θέτουμε $x=\cosh\varphi\cos\theta$
$\psi=\cosh\varphi\sin\theta$
$z=\sinh\varphi$, $\varphi\in\mathbb{R}$, $\theta\in[0,2\pi]$

//

Εμβαδόν Επιφανειας

$\vec{\sigma}:[0,1]\to\mathbb{R}^3$

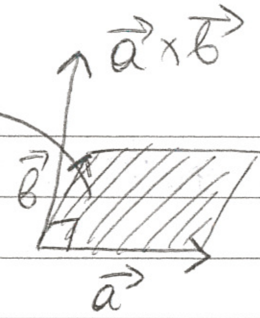
μήκος $=\int_0^1 \|\vec{\sigma}'(t)\|\,dt$

Εστω $\Phi:D\to\mathbb{R}^3$ ομαλή επιφάνεια

όπου $D\subseteq\mathbb{R}^2$

Έστω
$S = \Phi(D)$

$|\vec{a} \times \vec{b}| =$ εμβαδό



$$\iint_D \underbrace{\| \Phi_u(u,v) \times \Phi_v(u,v) \| \, du\, dv}_{dS(u,v)}$$

$\Phi_u(u_0, v_0)$ εφαπτόμενο διάνυσμα της καμπύλης

$\Phi_v(u_0, v_0)$ $\quad u \longmapsto \Phi(u, v_0)$

εφαπτόμενο διάνυσμα της $u \longmapsto \Phi(u_0, v)$

$dS = |\Phi_u(u_0, v_0) \times \Phi_v(u_0, v_0)| \, \delta u \, \delta v$

$\shortparallel$

στοιχειώδες εμβαδό
στο (u_0, v_0)

Έστω $\Phi(u,v) = (x(u,v), \psi(u,v), z(u,v))$

$\vec{\sigma}(t) = (x(t), \psi(t), z(t)) = \int_0^1 \sqrt{(x'(t))^2 + (\psi'(t))^2 + (z'(t))^2} \, dt$

Αν καμπύλη $\vec{\sigma}(t) = (t, \psi(t), z(t)) \Rightarrow = \int_0^1 \sqrt{1 + (\psi'(t))^2 + (z'(t))^2} \, dt$

$\Phi_u(u,v) = (x_u, \psi_u, z_u)$ $\qquad x_u = \frac{\partial}{\partial u} x$

$\Phi_v(u,v) = (x_v, \psi_v, z_v)$

$$\Phi_u \times \Phi_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_u & \psi_u & z_u \\ x_v & \psi_v & z_v \end{vmatrix} = \begin{vmatrix} \psi_u & z_u \\ \psi_v & z_v \end{vmatrix} \vec{i} - \begin{vmatrix} x_u & z_u \\ x_v & z_v \end{vmatrix} \vec{j}$$

$$+ \begin{vmatrix} x_u & \psi_u \\ x_v & \psi_v \end{vmatrix} \vec{k}.$$

$$\|\Phi_u \times \Phi_v\| = \sqrt{\left|\frac{\partial(\psi,z)}{\partial(u,v)}\right|^2 + \left|\frac{\partial(x,z)}{\partial(u,v)}\right|^2 + \left|\frac{\partial(x,\psi)}{\partial(u,v)}\right|^2}$$

Αν η επιφάνεια είναι γράφημα συνάρτησης

$$\Phi(u,v) = (u, v, \underset{\overset{\shortparallel}{z}}{f(u,v)}) \qquad (u,v) \in D$$

Τότε
$$A(D) = \iint_D \sqrt{1 + f_u^2(u,v) + f_v^2(u,v)}\, du\, dv$$

Έστω $\Phi(u,v) = (u, \; v, \; f(u,v))$
$\Phi_u(u,v) = (1, \; 0, \; f_u)$

$\Phi_v(u,v) = (0, 1, f_v)$

$$\Phi_u \times \Phi_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f_u \\ 0 & 1 & f_v \end{vmatrix} = \begin{vmatrix} 0 & f_u \\ 1 & f_v \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & f_u \\ 0 & f_v \end{vmatrix} \vec{j}$$

$$+ \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \vec{k} \qquad //$$

<u>Παράδειγμα</u>: Έστω $D = \{(r,\theta) \mid 0 \le r \le 1, \; 0 \le \theta \le 2\pi\}$ και $\Phi: D \to \mathbb{R}^3$ που ορίζεται από:

$x = r\cos\theta$
$\psi = r\sin\theta \qquad 0 \le r \le 1$
$z = r \qquad 0 \le \theta < 2\pi$

Βρείτε το εμβαδόν της επιφάνειας

Εννοούμε ότι $\Phi(r,\theta) = (r\cos\theta, r\sin\theta, r)$, $0 \le r \le 1$
$0 \le \theta < 2\pi$

$\Phi_r(r,\theta) = (\cos\theta, \sin\theta, 1)$
$\Phi_\theta(r,\theta) = (-r\sin\theta, r\cos\theta, 0)$

$$\Phi_r(r,\theta) \times \Phi_\theta(r,\theta) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} =$$

$$\begin{vmatrix} \sin\theta & 1 \\ r\cos\theta & 0 \end{vmatrix} \vec{i} - \begin{vmatrix} \cos\theta & 1 \\ -r\sin\theta & 0 \end{vmatrix} \vec{j} + \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} \vec{k}$$

$$= -r\cos\theta\,\vec{i} - r\sin\theta\,\vec{j} + r\vec{k}$$

$$\Rightarrow |\Phi_r \times \Phi_\theta| = \sqrt{(-r\cos\theta)^2 + (r\sin\theta)^2 + r^2}$$

$$= \sqrt{r^2(\cos^2\theta + \sin^2\theta) + r^2}$$
$$= \sqrt{2r^2} = r\sqrt{2}$$

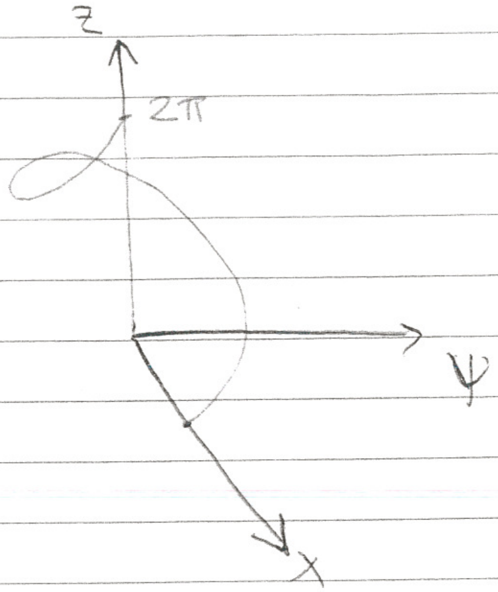
Οπότε $A(D) = \iint_D |\Phi_r \times \Phi_\theta| \, dr\,d\theta$

$$= \int_0^1 \left(\int_0^{2\pi} r\sqrt{2}\, d\theta \right) dr = \sqrt{2} \cdot 2\pi \int_0^1 r\,dr = \sqrt{2}\,\pi.$$

<u>Παράδειγμα:</u> Ένα (ελικοειδές) ορίζεται από την $\Phi: D \to \mathbb{R}^3$

όπου $x = r\cos\theta$
$y = r\sin\theta$
$z = \theta$

$D = \{(r,\theta) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$
Να βρεθεί το εμβαδόν της επιφάνειας



$$\Phi_r(r,\vartheta) = (\cos\vartheta, \sin\vartheta, 0)$$

$$\Phi_\vartheta(r,\vartheta) = (-r\sin\vartheta, r\cos\vartheta, 1)$$

$$\Phi_r \times \Phi_\vartheta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\vartheta & \sin\vartheta & 0 \\ -r\sin\vartheta & r\cos\vartheta & 1 \end{vmatrix} = \begin{vmatrix} \sin\vartheta & 0 \\ r\cos\vartheta & 1 \end{vmatrix} \vec{i} -$$

$$\begin{vmatrix} \cos\vartheta & 0 \\ -r\sin\vartheta & 1 \end{vmatrix} \vec{j} + \begin{vmatrix} \cos\vartheta & \sin\vartheta \\ -r\sin\vartheta & r\cos\vartheta \end{vmatrix} \vec{k} =$$

$$\sin\vartheta \vec{i} - \cos\vartheta \vec{j} + r\vec{k}$$

$$|\Phi_r \times \Phi_\vartheta| = \sqrt{\sin^2\vartheta + \cos^2\vartheta + r^2} = \sqrt{1+r^2}$$

$$A(D) = \int_0^1 \left(\int_0^{2\pi} \sqrt{1+r^2}\, d\vartheta \right) dr = 2\pi \int_0^1 \sqrt{1+r^2}\, dr$$

$$1+r^2 = t^2 \iff t^2 - r^2 = 1$$

Θέτουμε $r=\sinh u$

$\Downarrow$

$dr=\cosh u\, du$

$0=\sinh a \Leftrightarrow a=0$

$1=\sinh b$

$\Leftrightarrow b=\ln(1+\sqrt{2})$

$1=\dfrac{e^b-e^{-b}}{2} \Leftrightarrow e^b=t$

$2=e^b-e^{-b} \Leftrightarrow$

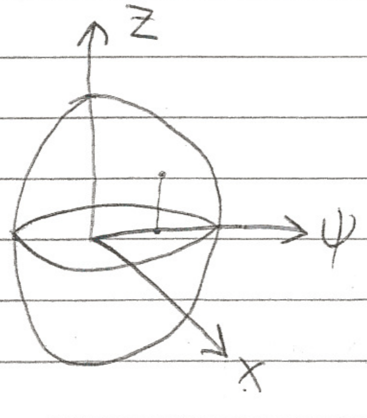
$2=t-\dfrac{1}{t} \Leftrightarrow t^2-2t-1=0$

$t=\dfrac{2\pm\sqrt{8}}{2}$ — $\boxed{1+\sqrt{2}}$, $1-\sqrt{2}<0$ απορρίπτεται

$e^b=1+\sqrt{2} \Leftrightarrow b=\ln(1+\sqrt{2})$

αποτέλεσμα: $\pi(\sqrt{2}+\ln(1+\sqrt{2})$

<u>Παράδειγμα</u>: Υπολογίστε το εμβαδόν της επιφάνειας S που περιγράφεται από την εξίσωση $x^2+\psi^2+z^2=1$



1ος τρόπος: $A(\text{σφαίρας})=$

$=2\iint\limits_{B^+}$

$z=\sqrt{1-x^2-\psi^2}$

$(x,\psi)\in D$

$x^2+\psi^2\leq 1$

$\Phi(x,\psi)=(x,\psi,\sqrt{1-x^2-\psi^2})$

$$|\Phi_x \times \Phi_\psi| = \sqrt{1+|\nabla f|^2}$$

$$f(x,\psi) = \sqrt{1-x^2-\psi^2}$$

$$\frac{\partial f}{\partial x} = \frac{\frac{1}{2}(-2x)}{\sqrt{1-x^2-\psi^2}} \qquad \frac{\partial f}{\partial \psi} = \frac{\frac{1}{2}(-2\psi)}{\sqrt{1-x^2-\psi^2}}$$

$$f_x^2 + f_\psi^2 = \frac{x^2+\psi^2}{1-x^2-\psi^2}$$

$$\iint_{x^2+\psi^2\leq 1} \sqrt{1+\frac{x^2+\psi^2}{1-x^2-\psi^2}}\, dx\, d\psi$$ (γενικευμένο ολοκλήρωμα).

<u>2ος τρόπος:</u> $x^2+\psi^2+z^2=1$

$$x = \sin\varphi\cos\theta$$
$$\psi = \sin\varphi\sin\theta \qquad \varphi\in[0,\pi]$$
$$z = \cos\varphi \qquad \theta\in[0,2\pi)$$

$$\Phi(\varphi,\theta) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$$

$$\Phi_\varphi = (\cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi)$$

$$\Phi_\theta = (-\sin\varphi\sin\theta, \sin\varphi\cos\theta, 0)$$

$$\Phi_\varphi \times \Phi_\theta = \begin{vmatrix} \vec{i} & j & k \\ \cos\varphi\cos\theta & \cos\varphi\sin\theta & -\sin\varphi \\ -\sin\varphi\sin\theta & \sin\varphi\cos\theta & 0 \end{vmatrix}$$

$$= \sin^2\cos\theta\,\vec{i} + \sin^2\varphi\sin\theta\,\vec{j} + (\sin\varphi\cos\varphi\cos^2\theta + \sin\varphi\cos\varphi\sin^2\theta$$

$$|\Phi_\varphi\times\Phi_\theta| = \sqrt{\sin^4\varphi\cos^2\theta + \sin^4\varphi\sin^2\theta + \sin^2\varphi\cos^2\varphi}$$

$$= \sqrt{\sin^4\varphi + \sin^2\varphi\cos^2\varphi} = \sqrt{\sin^2\varphi(\sin^2\varphi + \cos^2\varphi)} = \sin\varphi$$

$$A(S) = \int_0^{\pi}\left(\int_0^{2\pi} \sin\varphi \, d\theta\right)d\varphi = 2\pi\int_0^{\pi}\sin\varphi\, d\varphi = 4\pi$$

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Χριστίνα Ευθυμιάδου

$= \sqrt{\sin^4\varphi + \sin^2\varphi\cos^2\varphi} = \sqrt{\sin^2\varphi(\sin^2\varphi+\cos^2\varphi)} = \sin\varphi$

$A(S) = \int_0^{\pi}\left(\int_0^{2\pi} \sin\varphi \, d\theta\right)d\varphi = 2\pi\int_0^{\pi}\sin\varphi\, d\varphi = 4\pi$ //

25/10/2016

Άσκηση: Να βρεθεί η καρτεσιανή μορφή των επιφανειών:

i) $x=2u,\ \psi=u^2+v,\ z=v^2$, $u,v\in\mathbb{R}$

ii) $x=u^2,\ \psi=u\sin e^v,\ z=\frac{1}{3}u\cos e^v$ $u,v\in\mathbb{R}$

i) $\boxed{u=\frac{x}{2}}$

$\psi=u^2+v \Leftrightarrow \psi=\frac{x^2}{4}+v$

$\Leftrightarrow \boxed{v=\psi-\frac{x^2}{4}}$

οπότε $z=\left(\psi-\frac{x^2}{4}\right)^2$

Αν (x_0,ψ_0,z_0) σημείο της επιφάνειας $z_0=\left(\psi_0-\frac{x_0^2}{4}\right)^2$

Επιλέγω $u_0=\frac{x_0}{2}$, $v_0=\psi_0-u_0^2=\psi_0-\frac{x_0^2}{4}$, τότε όντως $z_0=v_0^2$

ii) $\begin{array}{l}\psi=u\sin e^v\\ 3z=u\cos e^v\end{array} \Rightarrow \psi^2+(3z)^2=u^2\sin^2 e^v+u^2\cos^2 e^v$
$=u^2=x$

(ελλειπτικό παραβολοειδές)

$x=\psi^2+9z^2$

Αν (x_0, ψ_0, z_0) τυχαίο σημείο της επιφάνειας, δηλ. $x_0 = \psi_0^2 + 9z_0^2$

$$u_0^2 = \psi_0^2 + 9z_0^2 \Rightarrow u_0 = \pm\sqrt{\psi_0^2 + 9z_0^2}$$

α) $u_0 = \sqrt{\psi_0^2 + 9z_0^2}$

$$\left.\begin{aligned} \psi_0 &= \sqrt{\psi_0^2 + 9z_0^2} \cdot \sin e^{v_0} \\ 3z_0 &= \sqrt{\psi_0^2 + 9z_0^2} \cdot \cos e^{v_0} \end{aligned}\right\} (*)$$

Αν $\psi_0 = z_0 = 0$ (τότε $x_0 = 0$)

$\Rightarrow u_0 = 0$

Τότε $(u_0, v_0) = (0, v_0)$, $\forall\, v_0 \in \mathbb{R}$ ικανοποιεί

Έστω $\psi_0^2 + 9z_0^2 \neq 0$

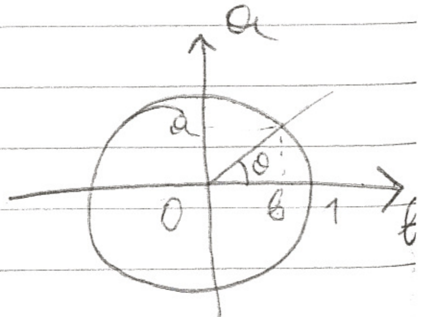
Το σύστημα $(*)$

$$\Leftrightarrow \begin{cases} \sin e^{v_0} = \dfrac{\psi_0}{\sqrt{\psi_0^2 + 9z_0^2}} \\ \cos e^{v_0} = \dfrac{3z_0}{\sqrt{\psi_0^2 + 9z_0^2}} \end{cases}$$

$\exists\, \vartheta_0 \in \mathbb{R}$ $\sin\vartheta_0 = \dfrac{\psi_0}{\sqrt{}}$

$2\kappa\pi + \vartheta_0, \kappa \in \mathbb{Z}$

$\nearrow$ $\cos\vartheta_0 = \dfrac{3z_0}{\sqrt{}}$



$\sin x = \alpha$
$\cos x = \beta$ $\Leftrightarrow$ $x = 2\kappa\pi + \vartheta$
Αν $\alpha^2 + \beta^2 = 1$ $\kappa \in \mathbb{Z}$

$$\boxed{e^{v_0} = 2\kappa\pi + \vartheta_0}$$

$\vartheta_0 \in (0, 2\pi]$

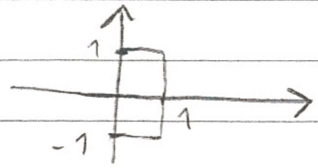
$e^{v_0} = \vartheta_0$

$v_0 = \ln\vartheta_0$

άσκηση Περιγράψτε την επιφάνεια σε καρτεσιανή μορφή.

Αν $u \in [0,1]$ $\quad v \in [-1,1]$

$x = 2u$
$\psi = u^2 + v$
$z = v^2$



$$\Rightarrow z = \left(\psi - \frac{x^2}{4}\right)^2$$

Άσκηση: Να γραφούν σε παραμετρική μορφή οι επιφάνειες:

(α) (κύλινδρος) $\quad (x-1)^2 - 4(\psi-2)^2 = 1$
(β) (κώνος) $\quad (x-1)^2 - 4(\psi-2)^2 = z^2$

$x - 1 = \cosh u$
$2(\psi - 2) = \sinh u$

$x = 1 + \cosh u$
$\psi = 2 + \frac{1}{2}\sinh u$

$\cosh x = \frac{e^x + e^{-x}}{2}$

$\sinh x = \frac{e^x - e^{-x}}{2}$

$\cosh^2 x - \sin^2 h x = 1$

$x - 1 = v\cosh u$
$2(\psi - 2) = v \sinh u \qquad v \in \mathbb{R}$
$z = v \qquad u \in \mathbb{R}$

$(x-1)^2 = z^2 + 4(\psi-2)^2$

$z = \rho\cos\theta$
$2(\psi-2) = \rho\sin\theta$
$x - 1 = \rho \qquad \rho \in \mathbb{R}$
$\theta \in [0, 2\pi)$

Επιφανειακά ολοκληρώματα α' είδους:

Επικαμπύλια α' είδους:

$$\int_\sigma f(x,\psi,z)\,ds \qquad f=1 \quad \sigma:[a,b]\to\mathbb{R}^3$$

$$\int_\sigma ds = \int \|\vec{\sigma}'(t)\|\,dt$$

$$= \int_a^b \sqrt{(x'(t))^2+(\psi'(t))^2+(z'(t))^2}\,dt$$

$S := \{\Phi(u,v),\ (u,v)\in D\}$

Εμβαδό επιφανείας

$$A(S) = \iint_D dS = \iint_D \|\Phi_u \times \Phi_v\|\,du\,dv$$

$\Phi(u,v) = (x(u,v), \psi(u,v), z(u,v))$

$$\iint_S f(x,\psi,z)\,dS =$$

$$= \iint_D f(x(u,v), \psi(u,v), z(u,v)) \cdot \|\Phi_u(u,v) \times \Phi_v(u,v)\|\,du\,dv$$

Άσκηση: Έστω το ελικοειδές

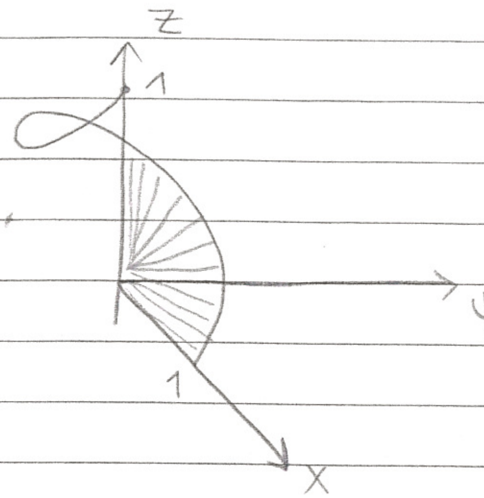
$x = r\cos\theta$

$\psi = r\sin\theta \qquad 0 \le r \le 1$

$z = \theta \qquad 0 \le \theta \le 2\pi$

Να υπολογιστεί το επιφανειακό ολοκλήρωμα

$\iint_S f(x,\psi,z)\,ds$ όταν $f(x,\psi,z) = \sqrt{x^2+\psi^2+1}$



$\Phi_r \times \Phi_\theta = ($s

$\Phi_r = (\cos\theta, \sin\theta, 0)$

$\Phi_\theta = (-r\sin\theta, r\cos\theta, 1)$

$$\Phi_r * \Phi_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 1 \end{vmatrix} = \sin\theta\,\vec{i} - \cos\theta\,\vec{j} + r\vec{k}$$

$$\|\Phi_r \times \Phi_\theta\| = \sqrt{r^2+1}$$

$$\iint_S \sqrt{(x^2+\psi^2+1)}\, dS = \int_0^1\int_0^{2\pi} \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + 1}\cdot\sqrt{r^2+1}\, d\theta\, dr$$

$$= \int_0^1\int_0^{2\pi} (r^2+1)\, d\theta\, dr = 2\pi\int_0^1 (r^2+1)\,dr = 2\pi\left(\frac{r^3}{3}+r\right)\Big|_0^1 = \frac{8\pi}{3}$$

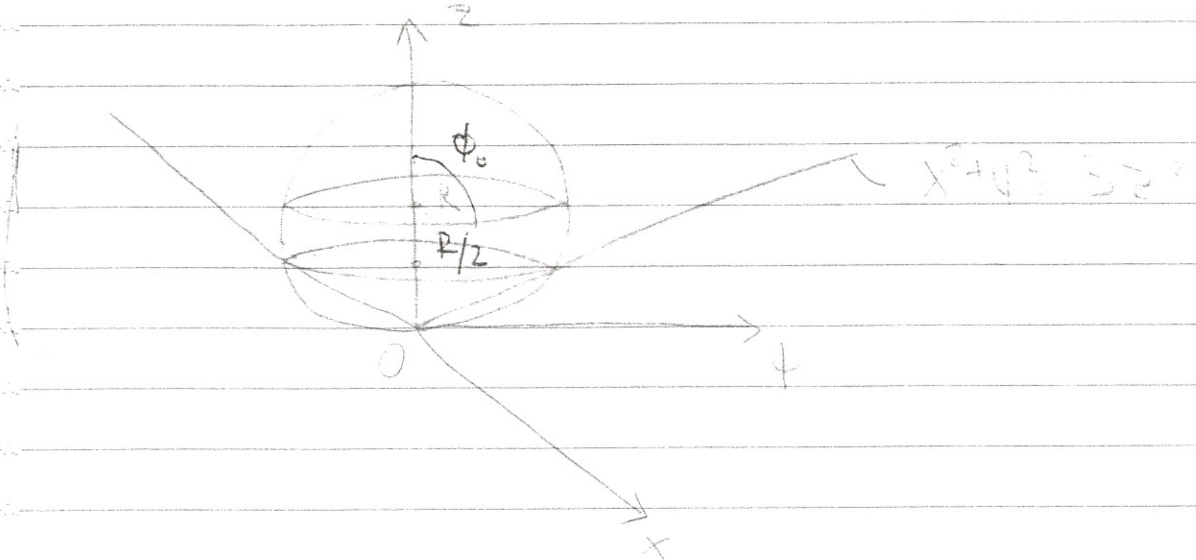
Άσκηση: Να υπολογίσετε το εμβαδό του τμήματος της σφαίρας

$$x^2+\psi^2+z^2 = 2Rz$$

που βρίσκεται στο εσωτερικό του κώνου

$$x^2+\psi^2 = 3z^2$$

$$x^2+\psi^2+z^2-2Rz+R^2=R^2$$
$$x^2+\psi^2+(z-R)^2=R^2$$



Τέμνει τον κώνο: $x^2+\psi^2=3z^2$ $\Leftrightarrow \begin{cases} x^2+\psi^2=3z^2 \\ 4z^2=2Rz \end{cases}$

$x^2+\psi^2+z^2=2Rz$

$$\Leftrightarrow x^2+\psi^2=\frac{3}{4}R^2$$

$$z=\frac{R}{2}$$

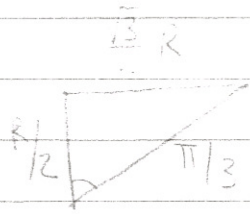
$$z=0 \Rightarrow x=\psi=0$$

Παραμέτρηση της σφαίρας

$x=R\sin\varphi\cos\theta$

$\psi=R\sin\varphi\sin\theta \qquad 0\leq\theta<2\pi$

$z-R=R\cos\varphi \qquad 0\leq\varphi\leq\varphi_0=\frac{2\pi}{3}$

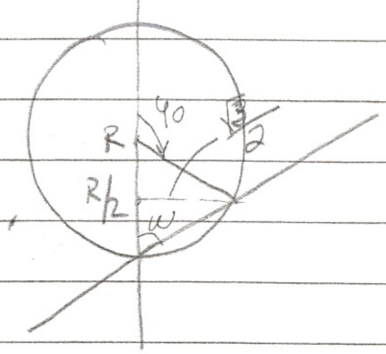


Το κομμάτι της σφαίρας, θα ικανοποιεί:

$x=R\sin\varphi\cos\theta$

$\psi=R\sin\varphi\sin\theta \qquad 0\leq\theta<2\pi$

$z=R+R\cos\varphi \qquad 0\leq\varphi\leq\varphi_0$



$$x^2+\psi^2=\frac{3}{4}R^2$$

$$\tan\omega=\sqrt{3}=\tan\left(\frac{\pi}{3}\right)$$
$$\Rightarrow \varphi_0=\frac{2\pi}{3}$$

Οπότε το εμβαδόν της επιφανειας είναι:

$$A(S)=\iint_S |\Phi_\varphi \times \Phi_\theta|\, d\varphi\, d\theta$$

$$=\int_0^{\frac{2\pi}{3}}\int_0^{2\pi} |\Phi_\varphi \times \Phi_\theta|\, d\theta\, d\varphi$$

$$\Phi(\varphi,\theta)=(R\sin\varphi\cos\theta,\ R\sin\varphi\sin\theta,\ R+R\cos\varphi)$$

$$\Phi_\varphi(\varphi,\theta)=(R\cos\varphi\cos\theta,\ R\cos\varphi\sin\theta,\ -R\sin\varphi) \qquad 0\le\varphi\le\frac{2\pi}{3}$$

$$\Phi_\theta(\varphi,\theta)=(-R\sin\varphi\sin\theta,\ R\sin\varphi\cos\theta,\ 0) \qquad 0\le\theta<2\pi$$

$$\Phi_\varphi\times\Phi_\theta=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ R\cos\varphi\cos\theta & R\cos\varphi\sin\theta & -R\sin\varphi\\ -R\sin\varphi\sin\theta & R\sin\varphi\cos\theta & 0\end{vmatrix}=$$

$$=R^2\sin^2\varphi\cos\theta\,\vec{i}+R^2\sin^2\varphi\sin\theta\,\vec{j}+R^2\sin\varphi\cos\varphi\,\vec{k}$$

$$|\Phi_\varphi\times\Phi_\theta|=R^2\sqrt{\sin^4\varphi\cos^2\theta+\sin^4\varphi\sin^2\theta+\sin^2\varphi\cos^2\varphi}$$

$$=R^2\sqrt{\sin^4\varphi+\sin^2\varphi\cos^2\varphi}$$

$$=R^2\sqrt{\sin^2\varphi(\sin^2\varphi+\cos^2\varphi)}=R^2\sin\varphi$$

$$A(S) = \int_0^{\frac{2\pi}{3}}\int_0^{2\pi} R^2 \sin\varphi \, d\theta \, d\varphi = 2\pi R^2 \int_0^{\frac{2\pi}{3}} \sin\varphi \, d\varphi =$$

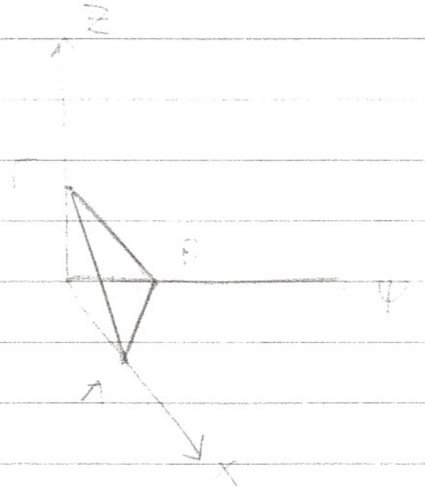
$$= 2\pi R^2 (-\cos\varphi)\Big|_0^{\frac{2\pi}{3}} = 2\pi R^2 \left(\cos\frac{2\pi}{3} + 1\right) = 3\pi R^2 \;//$$

27/10/2016

<u>Άσκηση</u> Να υπολογιστεί το επιφανειακό ολοκλήρωμα

$\iint_S x \, dS$, όπου S είναι η επιφάνεια που ορίζουν οι κορυφές $A(1,0,0)$, $B(0,1,0)$, $\Gamma(0,0,1)$

• Βρίσκουμε αρχικά το επίπεδο που ορίζουν τα σημεία.



$$\vec{A\Gamma} = (0,0,1) - (1,0,0) = (-1,0,1)$$
$$\vec{AB} = (0,1,0) - (1,0,0) = (-1,1,0)$$

$$\vec{AB} \times \vec{A\Gamma} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = \vec{i} + \vec{j} + \vec{k} = (1,1,1)$$

Εξίσωση επιπέδου: $(1,1,1)\cdot(x-1, \psi-0, z-0) = 0$

$$x - 1 + \psi + z = 0$$

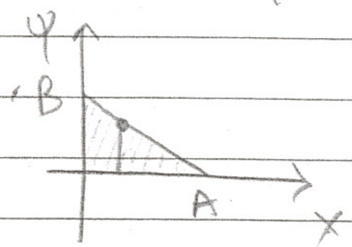
$$\Leftrightarrow \boxed{x + \psi + z = 1} \qquad z = 1 - x - \psi$$

όλα τα σημεία $(x, \psi, 1-\psi)$ $0 \le x \le 1$

$0 \le \psi \le 1$

$(0 \le x+\psi \le 1)$

$\underline{0 \le \psi \le 1-x}$



$$\iint_S x\, dS =$$

$$\Phi(x,\psi) = (x, \psi, 1-x-\psi)$$

$0 \le x \le 1$

$0 \le \psi \le 1-x$

$$= \int_0^1 \int_0^{1-x} x\sqrt{3}\, d\psi\, dx =$$

$$|\Phi_x \times \Phi_\psi| = \sqrt{3}$$

$$= \sqrt{3} \int_0^1 x \int_0^{1-x} d\psi\, dx$$

$$= \sqrt{3} \int_0^1 x(1-x)\, dx = \sqrt{3}\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = \frac{\sqrt{3}}{6}$$

//

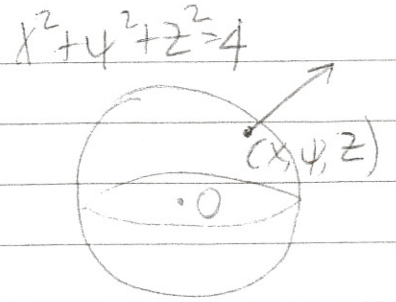
β' είδους	α' είδους
$\iint_S \vec{F} \cdot d\vec{S}$	$\iint_S f\, dS = \iint f(\Phi(u,v))$
$\iint_D \vec{F}(\Phi(u,v)) \cdot \Phi_u \times \Phi_v\, du\, dv$	$\lvert \Phi_u \times \Phi_v \rvert\, du\, dv$
β' επικαμπύλια	α' επικαμπύλια
$\int_C \vec{F} \cdot d\vec{s} = \int_a^b \vec{F}(\vec{\sigma}(t)) \cdot \vec{\sigma}'(t)\, dt.$	$\int_C f\, ds = \int_a^b f(\vec{\sigma}(t)) \lvert \vec{\sigma}'(t) \rvert\, dt$

$$\int_C \vec{F} \cdot d\vec{s} = \int_C \vec{F} \cdot \frac{\vec{\sigma}'(t)}{|\vec{\sigma}'(t)|}\, ds = \int \vec{F} \cdot \frac{\vec{\sigma}'(t)}{|\sigma'(t)|} \cdot |\vec{\sigma}'(t)|\, dt$$

$$\vec{\sigma} : [0,1] \to \mathbb{R}^3$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, ds \qquad \vec{n} = \frac{\Phi_u \times \Phi_v}{|\Phi_u \times \Phi_v|}$$

$$= \iint_S \vec{F} \cdot \frac{\Phi_u \times \Phi_v}{|\Phi_u \times \Phi_v|}\, |\Phi_u \times \Phi_v|\, du\, dv$$



$$\vec{n} = \frac{(x,ψ,z)}{|(x,ψ,z)|} = \frac{1}{2}(x,ψ,z)$$

<u>Ασκηση</u>: Εστω $\vec{F} = \nabla f$, όταν $f(x,ψ,z)=x^2+ψ^2+z$ και S είναι επιφάνεια της μοναδιαίας σφαίρας $x^2+ψ^2+z^2=r$ με θετικό προσανατολισμό προς τα έξω της σφαίρας. Υπολογίστε το $\iint_S \vec{F}\cdot d\vec{S}$

- Παραμετρικοποιούμε τη σφαίρα

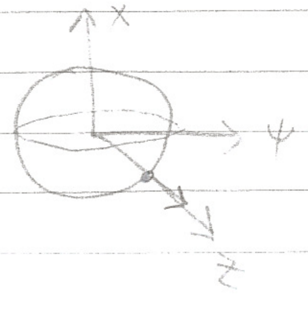
$$x=\sin φ\cos θ$$
$$ψ=\sin φ\sin θ \qquad 0\le φ\le π$$
$$z=\cos φ \qquad 0\le θ< 2π$$

$$\Phi_φ(φ,θ)=(\cos φ\cos θ, \cos φ\sin θ, -\sin φ)$$
$$\Phi_θ(φ,θ)=(-\sin φ\sin θ, \sin φ\cos θ, 0)$$

$$\Phi_φ\times\Phi_θ = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos φ\cos θ & \cos φ\sin θ & -\sin φ \\ -\sin φ\sin θ & \sin φ\cos θ & 0 \end{vmatrix} =$$

$$=\sin^2φ\cos θ\,\vec{i} + \sin^2φ\sin θ\,\vec{j} + \sin φ\cos φ\,\vec{k}$$

Αν $φ=\frac{π}{2}$ και $θ=0 \Rightarrow (1,0,0)$



$$\int_0^{\pi}\int_0^{2\pi}(2\sin\varphi\cos\theta, 2\sin\varphi\sin\theta, 2\cos\varphi)\cdot(\sin^2\varphi\cos\theta, \sin^2\varphi\sin\theta, \sin\varphi\cos\theta)\,d\theta\,d\varphi$$

$$= \int_0^{\pi}\int_0^{2\pi}\left[2\sin^3\varphi\cos^2\theta + 2\sin^3\varphi\sin^2\theta + 2\sin\varphi\cos^2\varphi\right)d\theta\,d\varphi$$

$$= \int_0^{\pi}\int_0^{2\pi}(2\sin^3\varphi + 2\sin\varphi\cos^2\varphi)\,d\theta\,d\varphi = 2\pi\int_0^{\pi}2\sin\varphi\,d\varphi$$

$$= 4\pi(-\cos\varphi)\Big|_0^{\pi} = 8\pi.$$

//

1/11/2016

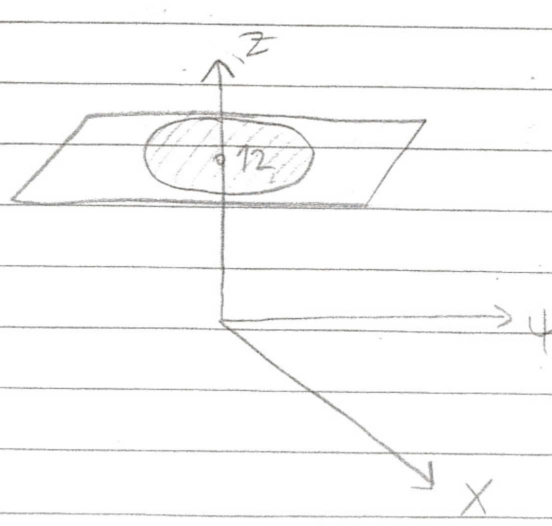
Διαφοροποίηση Επιφανειών

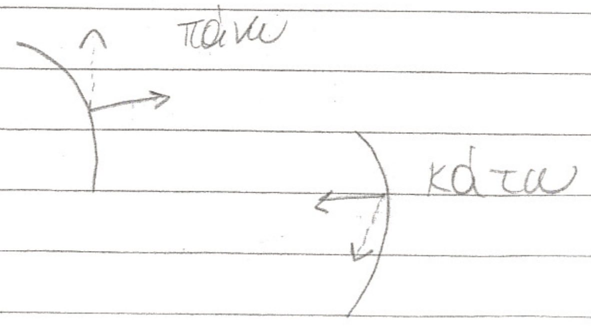
$2^{ου}$ είδους: $\iint \mathbf{F}\cdot d\mathbf{S}$ (ροή διανυσματικού πεδίου)

$1^{ου}$ είδους: $\iint f\cdot d\mathbf{S}$ (εμβαδόν ή μάζα)

Άσκηση: Να υπολογιστεί η ροή του διανυσματικού πεδίου:

$\iint_S \vec{F}\cdot d\vec{S}$, όπου S είναι η επιφάνεια $S=\{(x,\psi,z)\mid x^2+\psi^2\leq 25,\ z=12\}$ με προσανατολισμό (τα μοναδιαία διανύσματος να δείχνει προς τα κάτω (αρνητικό άξονα των z) και $\vec{F}(x,\psi,z)=(x,\psi,z)$.





κάθετο διάνυσμα $(0,0,-1)$

παραμετρικοποίηση $(r\cos\theta, r\sin\theta, 12)$

$x^2+\psi^2 \leq 25 \Leftrightarrow$

$r^2 \leq 25 \Leftrightarrow$

$0 \leq r \leq 5$

$0 \leq \theta < 2\pi$

$\Phi(r,\theta) = (r\cos\theta, r\sin\theta, 12)$

$\Phi_r(r,\theta) = (\cos\theta, \sin\theta, 0)$

$\Phi_\theta(r,\theta) = (-r\sin\theta, r\cos\theta, 0)$

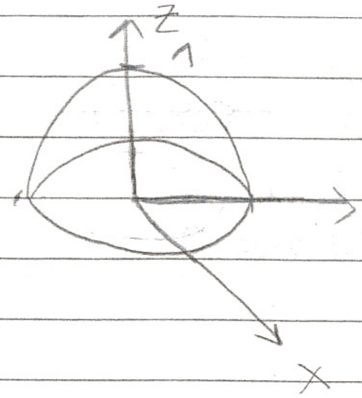
$$\Phi_r \times \Phi_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (r\cos^2\theta + r\sin^2\theta)\vec{k} = r\vec{k}$$

· Οπότε $\iint_S \vec{F}\cdot d\vec{S} = \int_0^5\int_0^{2\pi} (r\cos\theta, r\sin\theta, 12)\cdot(0,0,-r)\,d\theta$

$= \int_0^5\int_0^{2\pi} -12r\,d\theta\,dr = 2\pi\int_0^5 -12r\,dr =$

$= 2\pi(-6r^2)\Big|_0^5 = -2\pi\cdot 6\cdot 25 = -300\pi$ //

<u>Άσκηση</u>. Να υπολογιστεί η ροή του διανυσματικού πεδίου $\iint_S \vec{F}\cdot d\vec{S}$ όπου $\vec{F} = (x, \psi, 2z)$, S είναι η επιφάνεια της συνάρτησης $f(x,\psi) = 1 - x^2 - \psi^2$, $x^2+\psi^2 \leq 1$ με τον θετικό προσανατολισμό.



$$S=\{(x,\psi,1-x^2-\psi^2)\mid x^2+\psi^2\le 1\}$$

$$\Phi(x,\psi)=(x,\psi,1-x^2-\psi^2)$$
$$\Phi_x=(1,0,-2x)$$
$$\Phi_\psi=(0,1,-2\psi)$$

$$\Phi_x\times\Phi_\psi=\begin{vmatrix} \vec i & \vec j & \vec k \\ 1 & 0 & -2x \\ 0 & 1 & -2\psi \end{vmatrix}=2x\vec i+2\psi\vec j+\vec k=(2x,2\psi,1)$$

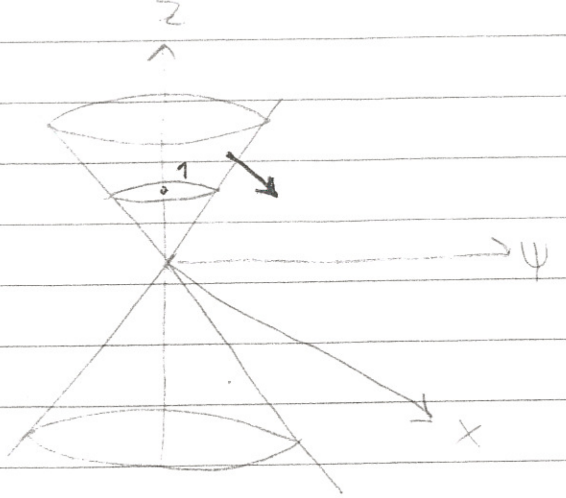
$$\iint \vec F\cdot d\vec S=\iint_{x^2+\psi^2\le 1}\big(x,\psi,2\underbrace{(1-x^2-\psi^2)}_{=z}\big)\cdot(2x,2\psi,1)\,dx\,d\psi$$

$$\iint_{x^2+\psi^2\le 1}\big(2x^2+2\psi^2+2(1-x^2-\psi^2)\big)\,dx\,d\psi=2\iint_{x^2+\psi^2\le 1}dx\,d\psi=2\pi$$

$x=r\cos\theta$ $\quad 0\le r\le 1,\ 0\le\theta<2\pi$
$\psi=r\sin\theta$

$$=2\int_0^1\int_0^{2\pi} r\,d\theta\,dr$$

Άσκηση: Έστω S το μέρος του κώνου
$z^2=x^2+\psi^2,\quad 1\le z\le 2$
προσανατολισμένο ώστε το κάθετο διάνυσμα να δείχνει προς τα έξω του κώνου. Υπολογίστε τη ροή του διανυσματικού πεδίου
$\iint_S \vec F\cdot d\vec S$ όπου $\vec F(x,\psi,z)=(x,\psi,z)$



$x = r\cos\theta \quad r \geq 0$

$\psi = r\sin\theta \quad 0 \leq \theta < 2\pi$

$z^2 = r^2 \Rightarrow z = r$

$\Rightarrow 1 \leq r \leq 2$

$\Phi(r,\theta) = (r\cos\theta, r\sin\theta, r) \quad 1 \leq r \leq 2$

$0 \leq \theta < 2\pi$

$\Phi_r = (\cos\theta, \sin\theta, 1)$

$\Phi_\theta = (-r\sin\theta, r\cos\theta, 0)$

$$\Phi_r \times \Phi_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} =$$

$= -r\cos\theta\,\vec{i} - r\sin\theta\,\vec{j} + r\vec{k}$

$= (-r\cos\theta, -r\sin\theta, r)$

$$\iint_S \vec{F}\cdot d\vec{S} = \int_1^2\int_0^{2\pi} (r\cos\theta, r\sin\theta, r)\cdot(r\cos\theta, r\sin\theta, -r)\,d\theta\,dr$$

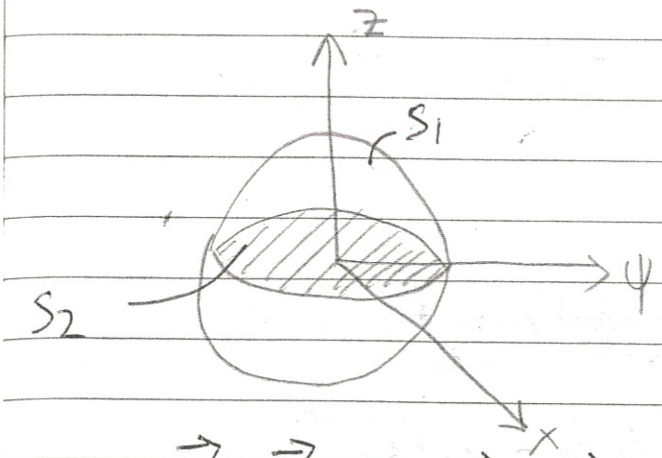
$$= \int_1^2\int_0^{2\pi} [r^2\cos^2\theta + r^2\sin^2\theta - r^2]\,d\theta\,dr$$

$= 0$

Άσκηση: Να υπολογιστεί η ροή του διανυσματικού πεδίου $\vec{F}(x,\psi,z) = (x,\psi,z)$

$\iint_S \vec{F}\cdot d\vec{S}$ όπου S είναι η επιφάνεια της μισής (μοναδιαίας σφαίρας) με προσανατολισμό προς τα έξω ημισφαίρια

$B_+ = \{(x,\psi,z) \mid x^2+\psi^2+z^2 \leq 1,\ z \geq 0\}$



$S_1 = \{(x,\psi,z) \mid x^2+\psi^2+z^2 = 1, \; z \geq 0\}$

$S_2 = \{(x,\psi,0) \mid x^2+\psi^2 \leq 1\}$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_{S_1} \vec{F}\cdot d\vec{S} + \iint_{S_2} \vec{F}\cdot d\vec{S}$$

$S_1 = \{(\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi) \;\; 0 \leq \theta < 2\pi, \; 0 \leq \varphi \leq \frac{\pi}{2}\}$

$S_2 = \{(r\cos\varphi, r\sin\theta, 0), \;\; 0 \leq r \leq 1, \; 0 \leq \theta < 2\pi\}$

3/11/2016

Ορίζουμε τρεις διαφορικούς τελεστές:

• Κλίση βαθμωτής συνάρτησης

$f: \Omega \rightarrow \mathbb{R}$, παραγωγίσιμη $\Omega \subseteq \mathbb{R}^3$

$$\nabla f(x,\psi,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial \psi}, \frac{\partial f}{\partial z}\right) = \left(f_x(x,\psi,z), f_\psi(x,\psi,z), f_z(x,\psi,z)\right)$$

grad $f(x,\psi,z)$

• Απόκλιση διανυσματικού πεδίου $\vec{F}: \Omega \rightarrow \mathbb{R}^3$

$\vec{F} = (F_1, F_2, F_3)$

div $\vec{F}$

$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}F_1(x,\psi,z) + \frac{\partial}{\partial \psi}F_2(x,\psi,z) + \frac{\partial}{\partial z}F_3(x,\psi,z)$$

• Στροβιλισμός διανυσματικού πεδίου $\vec{F}: \Omega \to \mathbb{R}$

$\mathrm{curl}\,\vec{F}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial \psi} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial}{\partial \psi}F_3 - \frac{\partial}{\partial z}F_2\right)\vec{i} + \left(\frac{\partial}{\partial z}F_1 - \frac{\partial}{\partial x}F_3\right)\vec{j} + \left(\frac{\partial}{\partial x}F_2 - \frac{\partial F_1}{\partial \psi}\right)\vec{k}$$

<u>Απλά παραδείγματα:</u>

1) Έστω $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,\psi,z) = x^2 + \psi^3 + x\psi z$

τότε $\nabla f(x,\psi,z) = \left(\frac{\partial}{\partial x}f, \frac{\partial}{\partial \psi}f, \frac{\partial}{\partial z}f\right)$

$$= (2x + \psi z, 3\psi^2 + xz, x\psi)$$

2) Έστω $\vec{F}(x,\psi,z) = (x^2\psi, z, x\psi z)$ τότε $\nabla \times \vec{F} =$;

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial \psi} & \frac{\partial}{\partial z} \\ x^2\psi & z & x\psi z \end{vmatrix} = \left(\frac{\partial}{\partial \psi}(x\psi z) - \frac{\partial}{\partial z}(z)\right)\vec{i} - \left(\frac{\partial}{\partial x}(x\psi z) - \frac{\partial}{\partial z}(x^2\psi)\right)\vec{j} + \left(\frac{\partial}{\partial x}z - \frac{\partial}{\partial \psi}(x^2\psi)\right)\vec{k}$$

$$= (xz - 1)\vec{i} - \psi z\vec{j} - x^2\vec{k}$$

$$= (xz - 1, -\psi z, -x^2)$$

Ιδιότητα 1 : Οι τελεστές ∇, $\nabla\cdot$, $\nabla\times$ είναι γραμμικοί τελεστές.

Π.χ.:

α) $f, g : \Omega \to \mathbb{R}$, $\Omega \subseteq \mathbb{R}^3$ f παραγωγίσιμη
τότε: $\nabla(\lambda f + \mu g) = \lambda \nabla f + \mu \nabla g$
$\lambda, \mu \in \mathbb{R}$

β) Αν $\vec{F}, \vec{G} : \Omega \to \mathbb{R}^3$ διανυσματικά πεδία, $\vec{F}$ $\vec{G}$ παραγωγίσιμα διανυσματικά πεδία
τότε $\nabla\cdot(\lambda\vec{F} + \mu\vec{G}) = \lambda \nabla\cdot\vec{F} + \mu \nabla\cdot\vec{G}$

γ) $\nabla\times(\lambda\vec{F} + \mu\vec{G}) = \lambda \nabla\times\vec{F} + \mu \nabla\times\vec{G}$

Απόδειξη β): Έστω $\vec{F} = (F_1, F_2, F_3)$, $\vec{G} = (G_1, G_2, G_3)$

$$\nabla\cdot(\lambda\vec{F} + \mu\vec{G}) = \nabla\cdot(\lambda(F_1, F_2, F_3) + \mu(G_1, G_2, G_3))$$
$$= \nabla\cdot((\lambda F_1 + \mu G_1, \lambda F_2 + \mu G_2, \lambda F_3 + \mu G_3))$$
$$= \frac{\partial}{\partial x}(\lambda F_1 + \mu G_1) + \frac{\partial}{\partial \psi}(\lambda F_2 + \mu G_2) + \frac{\partial}{\partial z}(\lambda F_3 + \mu G_3)$$
$$= \lambda \frac{\partial}{\partial x} F_1 + \mu \frac{\partial}{\partial x} G_1$$
$$\lambda \frac{\partial}{\partial \psi} F_2 + \mu \frac{\partial}{\partial \psi} G_2$$
$$\lambda \frac{\partial}{\partial z} F_3 + \mu \frac{\partial}{\partial z} G_3$$
$$= \lambda\left(\frac{\partial}{\partial x} F_1 + \frac{\partial}{\partial \psi} F_2 + \frac{\partial}{\partial z} F_3\right) + \mu\left(\frac{\partial}{\partial x} G_1 + \frac{\partial}{\partial \psi} G_2 + \frac{\partial}{\partial z} G_3\right)$$
$$= \lambda \nabla\cdot\vec{F} + \mu \nabla\cdot\vec{G}$$

Ιδιότητα ②: Έστω $f: \Omega \to \mathbb{R}$ τ.ώ $f \in C^2(\Omega)$

Τότε ισχύει:

$$curl(grad\, f) =$$

$$\nabla \times (\nabla f(x,\psi,z)) = \vec{0}$$

Απόδειξη:

$$\nabla \times (\nabla f) = \nabla \times (\nabla f) = \nabla \times \left(\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial \psi}, \frac{\partial f}{\partial z}\right)\right)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial \psi} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial \psi} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial \psi}\left(\frac{\partial f}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \psi}\right)\right)\vec{i} - \left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right)\right)\vec{j}$$

$$+ \left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \psi}\right) - \frac{\partial}{\partial \psi}\left(\frac{\partial f}{\partial x}\right)\right)\vec{k}$$

$$= \left(\frac{\partial^2}{\partial \psi \partial z} f - \frac{\partial^2}{\partial z \partial \psi} f\right)\vec{i} - \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right)\vec{j}$$

$$+ \left(\frac{\partial^2 f}{\partial x \partial \psi} - \frac{\partial^2 f}{\partial \psi \partial x}\right)\vec{k} = \vec{0}$$

Ιδιότητα ③: Έστω $\vec{F}: \Omega \to \mathbb{R}^3$, $\vec{F} \in (C^2(\Omega))^3$

$\vec{F} \in C^2(\Omega) \times C^2(\Omega) \times C^2(\Omega)$

$$div(curl\, \vec{F}) = 0$$

$$\nabla \cdot (\nabla \times \vec{F}) = 0$$

<u>Αποδ.</u> Έστω $\vec{F}=(F_1, F_2, F_3)$ και η υπόθεση δίνει ότι $F_i \in C^2(\Omega)$, $i=1,2,3$

Τότε $\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial \psi} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial \psi} - \frac{\partial F_2}{\partial z}\right)\vec{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial \psi}\right)\vec{k}$

Οπότε $\nabla \cdot (\nabla \times \vec{F}) = \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial \psi} - \frac{\partial F_2}{\partial z}\right) - \frac{\partial}{\partial \psi}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial \psi}\right)$

$$= \frac{\partial^2 F_3}{\partial x \partial \psi} - \frac{\partial^2 F_2}{\partial x \partial z} - \frac{\partial^2 F_3}{\partial \psi \partial x} + \frac{\partial^2 F_1}{\partial \psi \partial z} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial \psi} =$$

$$= \underbrace{\left(\frac{\partial^2 F_3}{\partial x \partial \psi} - \frac{\partial^2 F_3}{\partial \psi \partial x}\right)}_{=0} + \underbrace{\left(\frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_2}{\partial x \partial z}\right)}_{=0} + \left(\frac{\partial^2 F_1}{\partial \psi \partial z} - \frac{\partial^2 F_1}{\partial z \partial \psi}\right) = 0$$

<u>Ιδιότητα ④:</u>

i) f, g παραγωγίσιμες : $\Omega \to \mathbb{R}$, τότε

a) $\nabla(fg) = g\nabla f + f\nabla g$

β) $\nabla\left(\frac{f}{g}\right) = \frac{1}{g}\nabla f - \frac{f}{g^2}\nabla g$ κλίση της $\frac{1}{g}$

ii) $\vec{F}$ παραγωγίσιμο διανυσματικό πεδίο

$$\nabla \cdot (f\vec{F}) = \nabla f \cdot \vec{F} + f\,\nabla \cdot \vec{F}$$

iii) $\nabla\times(f\vec{F}) = (\nabla f)\times\vec{F} + f(\nabla\times\vec{F})$

iv) $\nabla\times(\nabla\times\vec{F}) = \nabla^2\vec{F} + \nabla(\nabla\cdot\vec{F})$

$(\Delta F_1, \Delta F_2, \Delta F_3) = \Delta\vec{F}$ τελεστής Laplace

f βαθμωτή

$\Delta f(x,\psi,z) = \nabla^2 f(x,\psi,z) = \nabla\cdot\nabla f(x,\psi,z)$

$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial \psi^2} + \frac{\partial^2 f}{\partial z^2}$

__Άσκηση:__ Έστω $\vec{F}(x,\psi,z) = (2x\psi e^z, x^2 e^z, x^2\psi e^z + z^2)$

i) Υπολογίστε $\nabla\cdot\vec{F}$, $\nabla\times\vec{F}$

ii) Βρείτε βαθμωτή συνάρτηση f, ώστε $\vec{F} = \nabla f$.

→ i) $\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(2x\psi e^z) + \frac{\partial}{\partial \psi}(x^2 e^z) + \frac{\partial}{\partial z}(x^2\psi e^z + z^2)$

$= 2\psi e^z + x^2\psi e^z + 2z$

- $\nabla\times\vec{F} = \vec{0}$ (Άσκηση)

→ ii) Έστω $f: \mathbb{R}^3 \to \mathbb{R}$ η ζητούμενη βαθμωτή συνάρτηση. Θα έπρεπε

$\nabla f(x,\psi,z) = \vec{F}(x,\psi,z)$

Οπότε $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial \psi}, \frac{\partial f}{\partial z}\right) = (2x\psi e^z, x^2 e^z, x^2\psi e^z + z^2)$

και επομένως προκύπτει το σύστημα

$\frac{\partial f}{\partial x}(x,\psi,z) = 2x\psi e^z$

$\frac{\partial f}{\partial \psi}(x,\psi,z) = x^2 e^z$

$\frac{\partial f}{\partial z}(x,\psi,z) = x^2\psi e^z + z^2$

Αν $f'(x)=0 \Rightarrow f(x)=f(0)$

$f(x,\psi,z)=f(0,\psi,z)=g(\psi,z)$

$\frac{\partial f}{\partial x}(x,\psi,z)=0$

ΘΜΤ

Έστω $f:[a,b]\to\mathbb{R}$

συνεχής f παραγωγίσιμη

τότε $\exists\, \xi \in (a,b)$

$f(b)-f(a)=(b-a)\cdot f'(\xi)$

$$\frac{\partial f}{\partial x}(x,\psi,z)=2x\psi e^{z}=\frac{\partial}{\partial x}(x^2\psi e^{z}) \Rightarrow$$

$$\frac{\partial}{\partial x}\left(f(x,\psi,z)-x^2\psi e^{z}\right)=0$$

$$\Rightarrow \exists\, g=g(\psi,z) \text{ ώστε } f(x,\psi,z)-x^2\psi e^{z}=g(\psi,z)$$

$$f(x,\psi,z)=x^2\psi e^{z}+g(\psi,z)$$

και επειδή:

$$\frac{\partial f}{\partial \psi}=x^2e^{z} \Leftrightarrow \frac{\partial}{\partial \psi}\left(x^2\psi e^{z}+g(\psi,z)\right)=x^2e^{z}$$

$$\Leftrightarrow \cancel{x^2e^{z}}+\frac{\partial g}{\partial \psi}(\psi,z)=\cancel{x^2e^{z}}$$

$$\Rightarrow \frac{\partial g}{\partial \psi}=0$$

$$\Rightarrow \exists\, R=h(z) \text{ ώστε } g(\psi,z)=h(z)$$

οπότε $f(x,\psi,z)=x^2\psi e^{z}+h(z)$

όμως $\frac{\partial f}{\partial z}=x^2\psi e^{z}+z^2 \Leftrightarrow \frac{\partial}{\partial z}\left(x^2\psi e^{z}+h(z)\right)=z^2+x^2\psi e^{z}$

$$h'(z)=z^2 \Rightarrow \left(\frac{z^3}{3}\right)' \Rightarrow \exists\, c\in\mathbb{R} \Rightarrow h(z)=\frac{z^3}{3}+c$$

$$f(x,\psi,z)=x^2\psi e^{z}+\frac{z^3}{3}+c$$

Χριστίνα Ευθυμιάδου

8/11/2016

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial}{\partial \psi} f, \frac{\partial f}{\partial z}\right)$$

$$\vec{F} = (F_1, F_2, F_3)$$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x} F_1 + \frac{\partial}{\partial \psi} F_2 + \frac{\partial}{\partial z} F_3$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial \psi} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial \psi} - \frac{\partial F_2}{\partial z}\right)\vec{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\vec{j}$$

$$+ \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial \psi}\right)\vec{k}$$

$$\nabla^2 f = \Delta f = \nabla \cdot \nabla f = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial \psi}\left(\frac{\partial f}{\partial \psi}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right)$$

$$= \frac{\partial^2}{\partial x^2} f + \frac{\partial^2}{\partial \psi^2} f + \frac{\partial^2}{\partial z^2} f.$$

<u>Άσκηση</u>: Έστω $\vec{F}, \vec{G}$ $\mathbb{R}^{3\times 1} \to \mathbb{R}^3$ τ.ώ.
$\vec{F}, \vec{G} \in C^1(\mathbb{R}^4)$

Αποδείξτε ότι:

α) $\frac{d}{dt}(\vec{F} \cdot \vec{G}) = \left(\frac{d}{dt}\vec{F}\right) \cdot \vec{G} + \vec{F} \cdot \left(\frac{d}{dt}\vec{G}\right)$

β) $\frac{d}{dt}(\vec{F}(t) \times \vec{G}(t)) = \left(\frac{d}{dt}\vec{F}\right) \times \vec{G} + \vec{F} \times \left(\frac{d}{dt}\vec{G}\right)$

γ) Αν $f, g : \mathbb{R}^4 \to \mathbb{R}$

$$\frac{d}{dt}(\nabla(fg)) = \nabla\left(\frac{d}{dt}(fg)\right)$$

$$= \left(\frac{\partial}{\partial \psi}(x^m \psi^m z^{m+n}) - \frac{\partial}{\partial z}(x^m \psi^{m+n} z^n) \right) \vec{i} -$$
$$\left(\frac{\partial}{\partial x}(x^m \psi^m z^{m+n}) - \frac{\partial}{\partial z}(x^{m+n} \psi^m z^m) \right) \vec{j} +$$
$$\left(\frac{\partial}{\partial x}(x^m \psi^{m+n} z^m) - \frac{\partial}{\partial \psi}(x^{m+n} \psi^m z^m) \right) \vec{k}$$

$$= \left(m x^m \psi^{m-1} z^{m+n} - m x^m \psi^{m+n} z^{m-1}, m x^{m+n} \psi^m z^{m-1} - m x^{m-1} \psi^m z^{m+n}, m x^{m-1} \psi^{m+n} z^m - m x^{m+n} \psi^{m-1} z^m \right)$$

$$= m x^{m-1} \psi^{m-1} z^{m-1} \left(x z^{n+1} - x \psi^{n+1}, x^{n+1}\psi - \psi z^{n+1}, \psi^{n+1} z - x^{n+1} z \right)$$

Επειδή $\nabla \times \vec{F} = \vec{0} \iff$

$$\begin{cases} \cancel{m x^{m-1} \psi^{m-1} z^{m-1}}\, x (z^{n+1} - \psi^{n+1}) = 0 \\ \cancel{m x^{m-1} \psi^{m-1} z^{m-1}}\, \psi (x^{n+1} - z^{n+1}) = 0 \\ \cancel{m x^{m-1} \psi^{m-1} z^{m-1}}\, z (\psi^{n+1} - x^{n+1}) = 0 \end{cases} \iff$$

είτε $m = 0$

είτε $m \neq 0$
$z^{n+1} = \psi^{n+1}$
$x^{n+1} = z^{n+1}$ $\forall x, \psi, z$
$\psi^{n+1} = x^{n+1}$

Αν $n+1 = 0 \iff n = -1$ ok.

Αν $n+1 \neq 0 \Rightarrow x = \psi = z$

<u>Άσκηση</u>

α) Έστω $f, g: \mathbb{R}^2 \to \mathbb{R}$ παραγωγίσιμες ώστε $\nabla f(x,\psi) = \nabla g(x,\psi)$ $\forall x, \psi \in \mathbb{R}$. Αποδείξτε ότι $\exists\, c \in \mathbb{R}$ ώστε $f(x,\psi) = g(x,\psi) + c$, $\forall x, \psi \in \mathbb{R}$

<u>Συνεκτικό χωρίο</u>

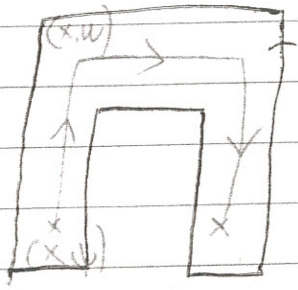
Κάθε 2 σημεία ενώνονται με συνεχή καμπύλη μέσα από το χωρίο.

β) Αν $f, g: \Omega \to \mathbb{R}$, Ω ανοικτό συνεκτικό και $\nabla f = \nabla g$ τότε $\exists\, c \in \mathbb{R}: f(x,\psi) = g(x,\psi) + c$, $\forall x, \psi \in \Omega$

γ) Έστω $f, g : \Omega \subseteq \mathbb{R}^3 \to \mathbb{R}$ παραγωγίσιμες ώστε $\nabla f = \nabla g$ στο Ω και το Ω ανοικτό, συνεκτικ

$\Rightarrow \exists c \in \mathbb{R} : f(x,\psi,z) = g(x,\psi,z) + c \quad \forall (x,\psi,z) \in \Omega$

Ω



Ω

$\nabla(\underbrace{f-g}_{F}) = 0$

$\nabla F(x,\psi) = (0,0)$

$\frac{\partial F}{\partial x}(x,\psi) = 0$

$\forall (x,\psi) \in \Omega$

$\frac{\partial F}{\partial \psi}(x,\psi) = 0$

$\nabla F(x,\psi) = (0,0), \quad \forall\, x\psi \in \mathbb{R}$

$; \Rightarrow \exists c \in \mathbb{R} : F(x,\psi) = c \quad , \forall\, x\psi \in \mathbb{R}$

$f'(t) = 0$

$\Downarrow$

$f(t) \equiv c, \quad t \in I$

<u>ΘΜΤ</u> $[t_0, t] \subseteq I$

$\exists\, \xi \in (t_0, t)$

$f(t) - f(t_0) = (t - t_0) \cdot \underbrace{f'(t_0)}_{0}$

$\Rightarrow f(t) = f(t_0)$

$\forall t \in I$

Έστω $\vec{\sigma} : [0,1] \to \mathbb{R}^2$ καμπύλη που ενώνει το $\vec{\sigma}(0) = (x,\psi)$ με το $\vec{\sigma}(1) = (x_0, \psi_0) \in \Omega$. ορίζουμε την $Q(t) = F(\vec{\sigma}(t))$, $t \in [0,1]$.

$Q'(t) = \nabla F(\vec{\sigma}(t)) \cdot \vec{\sigma}'(t) = 0$

$Q : [0,1] \to \mathbb{R}$

$Q'(t) = 0, \quad t \in [0,1]$

$\Rightarrow Q(1) = Q(0)$

$F(x_0, \psi_0) = F(x, \psi)$

Οπότε τελικά: $f(x,\psi,z) = x\psi + \sin(\psi z) + c$ ■

<u>ΘΕΩΡΗΜΑ</u>: Έστω $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ αστρόβιλο $(C^1(\mathbb{R}^3))^3$ πεδίο. Τότε υπάρχει βαθμωτή συνάρτηση $f: \mathbb{R}^3 \to \mathbb{R}$, $f \in C^1(\mathbb{R}^3)$ τ.ω. $\nabla f = \vec{F}$.

<u>Αποδ.</u> Έστω $\vec{F} = (F_1, F_2, F_3)$ θα βρούμε βαθμωτή συνάρτηση $f: \mathbb{R}^3 \to \mathbb{R}$ ώστε

$$\frac{\partial f}{\partial x}(x,\psi,z) = F_1(x,\psi,z)$$

$$\frac{\partial f}{\partial \psi}(x,\psi,z) = F_2(x,\psi,z)$$

$$\frac{\partial f}{\partial z}(x,\psi,z) = F_3(x,\psi,z)$$

$$\frac{\partial}{\partial x} f(x,\psi,z) = F_1(x,\psi,z) = \frac{\partial}{\partial x}\left(\int_0^x F_1(t,\psi,z)\,dt\right)$$

$$\Rightarrow \frac{\partial}{\partial x}\left[f(x,\psi,z) - \int_0^x F_1(t,\psi,z)\,dt\right] = 0$$

$$\Rightarrow f(x,\psi,z) - \int_0^x F_1(t,\psi,z)\,dt = g(\psi,z)$$

$$\Rightarrow \boxed{f(x,\psi,z) = \int_0^x F_1(t,\psi,z)\,dt + f(0,\psi,z)}$$

<u>Επειδή</u>: $\underbrace{\frac{\partial}{\partial \psi}\left[\int_0^x F_1(t,\psi,z)\,dt + \frac{\partial}{\partial \psi} f(0,\psi,z)\right]}_{} = F_2(x,\psi,z)$

$$\frac{\partial f}{\partial \psi}(x,\psi,z) = F_2(x,\psi,z) \Rightarrow *$$

$$\int_0^x \frac{\partial}{\partial \psi} F_1(t,\psi,z)\,dt + \frac{\partial}{\partial \psi} g(\psi,z) = F_2(x,\psi,z) \Rightarrow$$

$$\frac{\partial}{\partial \psi} f(0,\psi,z) = F_2(x,\psi,z) - \int_0^x \frac{\partial}{\partial \psi} F_1(t,\psi,z)\,dt$$

$$* \Rightarrow \frac{\partial}{\partial \psi} f(0,\psi,z) = F_2(0,\psi,z)$$

$$\frac{\partial}{\partial \psi} f(x,\psi,z) = F_2(x,\psi,z) \quad , \quad x=0$$

$$\frac{\partial}{\partial \psi} f(0,\psi,z) = F_2(0,\psi,z)$$

$$= \frac{\partial}{\partial \psi}\left[\int_0^x F_2(0,t,z)\,dt\right]$$

$$\Rightarrow \frac{\partial}{\partial \psi}\left[f(0,\psi,z) - \int_0^\psi F_2(0,t,z)\,dt\right] = 0$$

$$f(0,\psi,z) - \int_0^\psi F_2(0,t,z)\,dt = f(0,0,z) - \int_0^0 \cancel{F_2(0,t,z)}$$

$$\Rightarrow f(0,\psi,z) = f(0,0,z) + \int_0^\psi F_2(0,t,z)\,dt.$$

$$\frac{\partial}{\partial z} f(x,\psi,z) = F_3(x,\psi,z)$$

$$\overset{x=0,\,\psi=0}{\Longrightarrow} \frac{\partial}{\partial z} f(0,0,z) = F_3(0,0,z)$$

$$= \frac{\partial}{\partial z}\left(\int_0^z F_3(0,0,t)\,dt\right)$$

$$\Rightarrow \frac{\partial}{\partial z}\left(f(0,0,z) - \int_0^z F_3(0,0,t)\,dt\right) = 0$$

$$\Rightarrow f(0,0,z) - \int_0^z F_3(0,0,t)\,dt = f(0,0,0) - 0$$

$$f(x,\psi,z) = \int_0^x F_1(t,\psi,z)\,dt + \int_0^\psi F_2(0,t,z)\,dt + \int_0^z F_3(0,0,t)\,dt + c \quad (*)$$

Θα αποδείξουμε ότι η f πληρεί τις υποθέσεις.

$$\Rightarrow \frac{\partial f}{\partial x}(x,\psi,z) = F_1(x,\psi,z) \quad \checkmark$$

$$\frac{\partial f}{\partial \psi}(x,\psi,z) = \int_0^x \frac{\partial}{\partial \psi} F_1(t,\psi,z)\,dt + F_2(0,\psi,z) = \underbrace{\int_0^x \frac{\partial F_2}{\partial x}(t,\psi,z)\,dt + F_2(0,\psi,z)}$$

όμως το $\vec{F}$ είναι αστρόβιλο $\Rightarrow$ $= F_2(x,\psi,z)$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial \psi} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \vec{0} \Leftrightarrow \left[\begin{array}{l} \frac{\partial F_3}{\partial \psi} = \frac{\partial}{\partial z} F_2 \\ \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial \psi} \end{array}\right]$$

αστρόβιλο $\Rightarrow \exists\, f$ βαθμωτή $\qquad (\nabla \times (\nabla f) = \vec{0})$

$\vec{F} = \nabla f \qquad \nabla \cdot (\nabla \times \vec{F}) = 0.$

ΘΕΩΡΗΜΑ: Αν $\vec{F}$ διανυσματικό πεδίο

$\exists\, \vec{G} : \nabla \times \vec{G} = \vec{F}$

Χριστίνα Ευθυμιάδου

8/11/2016

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial}{\partial \psi} f, \frac{\partial f}{\partial z}\right)$$

$$\vec{F} = (F_1, F_2, F_3)$$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x} F_1 + \frac{\partial}{\partial \psi} F_2 + \frac{\partial}{\partial z} F_3$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial \psi} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial \psi} - \frac{\partial F_2}{\partial z}\right)\vec{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\vec{j}$$

$$+ \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial \psi}\right)\vec{k}$$

$$\nabla^2 f = \Delta f = \nabla \cdot \nabla f = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial \psi}\left(\frac{\partial f}{\partial \psi}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right)$$

$$= \frac{\partial^2}{\partial x^2} f + \frac{\partial^2}{\partial \psi^2} f + \frac{\partial^2}{\partial z^2} f.$$

<u>Άσκηση</u>: Έστω $\vec{F}, \vec{G}$ $\mathbb{R}^{3\times 1} \longrightarrow \mathbb{R}^3$ τ.ώ. $\vec{F}, \vec{G} \in C^1(\mathbb{R}^4)$

Αποδείξτε ότι:

α) $\frac{d}{dt}(\vec{F} \cdot \vec{G}) = \left(\frac{d}{dt}\vec{F}\right) \cdot \vec{G} + \vec{F} \cdot \left(\frac{d}{dt}\vec{G}\right)$

β) $\frac{d}{dt}(\vec{F}(t) \times \vec{G}(t)) = \left(\frac{d}{dt}\vec{F}\right) \times \vec{G} + \vec{F} \times \left(\frac{d}{dt}\vec{G}\right)$

γ) Αν $f, g : \mathbb{R}^4 \longrightarrow \mathbb{R}$

$$\frac{d}{dt}(\nabla(fg)) = \nabla\left(\frac{d}{dt}(fg)\right)$$

$\Rightarrow$ b) $\vec{F} = (F_1, F_2, F_3)$
$\vec{G} = (G_1, G_2, G_3)$

$$\vec{F} \times \vec{G} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ F_1 & F_2 & F_3 \\ G_1 & G_2 & G_3 \end{vmatrix} = (F_2G_3 - F_3G_2)\vec{i} - (F_1G_3 - F_3G_1)\vec{j} + (F_1G_2 - F_2G_1)\vec{k}$$

Κάθε συνιστώσα του διανυσματικού πεδίου $|\vec{F} \times \vec{G}|$ είναι παραγωγίσιμη γιατί κάθε συνιστώσα είναι π.χ. η $F_2G_3 - F_3G_2$ είναι παραγωγίσιμη μιας F_2, G_3, F_3, G_2 είναι παραγωγίσιμη οπότε τα γινόμενα F_2G_3, F_3G_2 είναι παραγωγίσιμη και η διαφορά αυτή θα είναι παραγωγίσιμη συνάρτηση και επομένως:

$$\frac{d}{dt}\vec{F} \times G = \frac{d}{dt}(F_2G_3 - F_3G_2)\vec{i} - \frac{d}{dt}(F_1G_3 - F_3G_1)\vec{j} + \frac{d}{dt}(F_1G_2 - F_2G_1)\vec{k}$$

Οπότε:

$$\frac{d}{dt}\vec{F} \times \vec{G} = \left(\frac{d}{dt}F_2G_3 + F_2\frac{d}{dt}G_3 - \frac{d}{dt}F_3G_2 - F_3\frac{d}{dt}G_2\right)\vec{i} - (\qquad)\vec{j} + (\qquad)\vec{k}$$

$$= \left(\frac{d}{dt}F_2G_3 - \frac{d}{dt}F_3G_2\right)\vec{i} + \left(F_2\frac{d}{dt}G_3 - F_3\frac{d}{dt}G_2\right)\vec{i}$$

$$= \left(\frac{d}{dt}\vec{F} \times \vec{G} \qquad + \vec{F} \times \left(\frac{d}{dt}\vec{G}\right)\right.$$

Χριστίνα Ευθυμιάδου

8/11/2016

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial}{\partial \psi} f, \frac{\partial f}{\partial z}\right)$$

$$\vec{F} = (F_1, F_2, F_3)$$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x} F_1 + \frac{\partial}{\partial \psi} F_2 + \frac{\partial}{\partial z} F_3$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial \psi} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial \psi} - \frac{\partial F_2}{\partial z}\right)\vec{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\vec{j}$$

$$+ \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial \psi}\right)\vec{k}$$

$$\nabla^2 f = \Delta f = \nabla \cdot \nabla f = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial \psi}\left(\frac{\partial f}{\partial \psi}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right)$$

$$= \frac{\partial^2}{\partial x^2} f + \frac{\partial^2}{\partial \psi^2} f + \frac{\partial^2}{\partial z^2} f.$$

<u>Άσκηση</u> Έστω $\vec{F}, \vec{G}$ $\mathbb{R}^{3\times 1} \to \mathbb{R}^3$ τ.ώ. $\vec{F}, \vec{G} \in C^1(\mathbb{R}^4)$

Αποδείξτε ότι:

α) $\frac{d}{dt}(\vec{F} \cdot \vec{G}) = \left(\frac{d}{dt}\vec{F}\right) \cdot \vec{G} + \vec{F} \cdot \left(\frac{d}{dt}\vec{G}\right)$

β) $\frac{d}{dt}(\vec{F}(t) \times \vec{G}(t)) = \left(\frac{d}{dt}\vec{F}\right) \times \vec{G} + \vec{F} \times \left(\frac{d}{dt}\vec{G}\right)$

γ) Αν $f, g : \mathbb{R}^4 \to \mathbb{R}$

$$\frac{d}{dt}(\nabla(fg)) = \nabla\left(\frac{d}{dt}(fg)\right)$$

$$= \left(\frac{\partial}{\partial \psi} (x^m \psi^m z^{m+n}) - \frac{\partial}{\partial z} (x^m \psi^{m+n} z^n) \right) \vec{i} -$$
$$\left(\frac{\partial}{\partial x} (x^m \psi^m z^{m+n}) - \frac{\partial}{\partial z} (x^{m+n} \psi^m z^m) \right) \vec{j} +$$
$$\left(\frac{\partial}{\partial x} (x^m \psi^{m+n} z^m) - \frac{\partial}{\partial \psi} (x^{m+n} \psi^m z^m) \right) \vec{k}$$

$$= \left(m x^m \psi^{m-1} z^{m+n} - m x^m \psi^{m+n} z^{m-1}, m x^{m+n} \psi^m z^{m-1} - m x^{m-1} \psi^m z^{m+n}, m x^{m-1} \psi^{m+n} z^m - m x^{m+n} \psi^{m-1} z^m \right)$$

$$= m x^{m-1} \psi^{m-1} z^{m-1} \left(x z^{n+1} - x\psi^{n+1}, x^{n+1}\psi - \psi z^{n+1}, \psi^{n+1} z - x^{n+1} z \right)$$

Επειδή $\nabla \times \vec{F} = \vec{0} \iff$

$\cancel{m} x^{m-1} \psi^{m-1} z^{m-1} x (z^{n+1} - \psi^{n+1}) = 0$

$\cancel{m} x^{m-1} \psi^{m-1} z^{m-1} \psi (x^{n+1} - z^{n+1}) = 0$

$\cancel{m} x^{m-1} \psi^{m-1} z^{m-1} z (\psi^{n+1} - x^{n+1}) = 0$

$\iff$ είτε $m = 0$

είτε $m \neq 0$

$z^{n+1} = \psi^{n+1}$

$x^{n+1} = z^{n+1}$ $\forall x, \psi, z$

$\psi^{n+1} = x^{n+1}$

Αν $n+1 = 0 \iff n = -1$ ok.

Αν $n+1 \neq 0 \Rightarrow x = \psi = z$

Άσκηση

α) Έστω $f, g: \mathbb{R}^2 \to \mathbb{R}$ παραγωγίσιμες ώστε $\nabla f(x, \psi) = \nabla g(x, \psi)$ $\forall x, \psi \in \mathbb{R}$. Αποδείξτε ότι $\exists\, c \in \mathbb{R}$ ώστε $f(x, \psi) = g(x, \psi) + c$, $\forall x, \psi \in \mathbb{R}$

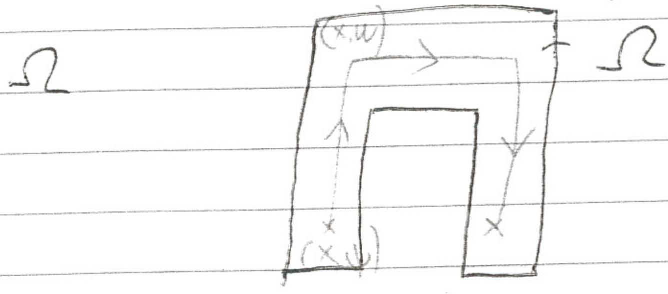
Συνεκτικό χωρίο

Κάθε 2 σημεία ενώνονται με συνεχή καμπύλη μέσα από το χωρίο.

β) Αν $f, g: \Omega \to \mathbb{R}$, Ω ανοικτό συνεκτικό και $\nabla f = \nabla g$ τότε $\exists\, c \in \mathbb{R}: f(x, \psi) = g(x, \psi) + c$, $\forall x, \psi \in \Omega$

γ) Έστω $f, g: \Omega \subseteq \mathbb{R}^3 \to \mathbb{R}$ παραγωγίσιμες ώστε $\nabla f = \nabla g$ στο Ω και το Ω ανοικτό, συνεκτικ

$\Rightarrow \exists c \in \mathbb{R}: f(x,\psi,z) = g(x,\psi,z) + c \quad \forall (x,\psi,z) \in \Omega$



$\underbrace{\nabla(f-g)}_{F} = 0$

$\nabla F(x,\psi) = (0,0)$

$\frac{\partial F}{\partial x}(x,\psi) = 0$

$\forall (x,\psi) \in \Omega$

$\frac{\partial F}{\partial \psi}(x,\psi) = 0$

$\nabla F(x,\psi) = (0,0), \quad \forall x\psi \in \mathbb{R}$

$\Rightarrow \exists c \in \mathbb{R}: F(x,\psi) = c, \quad \forall x\psi \in \mathbb{R}$

$f'(t) = 0$

$\Downarrow$

$f(t) \equiv c, \quad t \in I$

<u>ΘΜΤ</u> $[t_0, t] \subseteq I$

$\exists \xi \in (t_0, t)$

$f(t) - f(t_0) = (t - t_0) \cdot \underbrace{f'(t_0)}_{0}$

$\Rightarrow f(t) = f(t_0)$

$\forall t \in I$

Έστω $\vec{\sigma}: [0,1] \to \mathbb{R}^2$ καμπύλη που ενώνει το $\vec{\sigma}(0) = (x,\psi)$ με το $\vec{\sigma}(1) = (x_0,\psi_0) \in \Omega$. ορίζουμε την $Q(t) = F(\vec{\sigma}(t))$, $t \in [0,1]$.

$Q'(t) = \nabla F(\vec{\sigma}(t)) \cdot \vec{\sigma}'(t) = 0$

$Q: [0,1] \to \mathbb{R}$

$Q'(t) = 0, \quad t \in [0,1]$

$\Rightarrow Q(1) = Q(0)$

$F(x_0,\psi_0) = F(x,\psi)$

$$F(x,w)-F(x,\psi)= \quad \forall (x,\psi)\in\Omega$$
$$(w-\psi)\frac{\partial F}{\partial \psi}(x,\xi)=0$$

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10/11/2016

<u>Άσκηση</u>: Δίνεται το διανυσματικό πεδίο
$$\vec{F}(x,\psi,z)=(\psi,\ z\cos(\psi z)+x,\ \psi\cos(\psi z)).$$
Αποδείξτε ότι $\nabla\times\vec{F}=\vec{0}$ (αστρόβιλο πεδίο) και βρείτε βαθμωτή συνάρτηση $f:\mathbb{R}^3\to\mathbb{R}$ ώστε
$$\vec{F}=\nabla f$$

<u>Λύση</u>:

Το πεδίο είναι αστρόβιλο διότι:
$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial \psi} & \frac{\partial}{\partial z}\\ \psi & z\cos(\psi z)+x & \psi\cos(\psi z)\end{vmatrix}=\left(\frac{\partial}{\partial\psi}(\psi\cos(\psi z))-\frac{\partial}{\partial z}(z\cos(\psi z)+x)\right)\vec{i}$$
$$+\left(\frac{\partial}{\partial z}(\psi)-\frac{\partial}{\partial x}(\psi\cos(\psi z))\right)\vec{j}+$$
$$\left(\frac{\partial}{\partial x}(z\cos(\psi z)+x)-\frac{\partial}{\partial\psi}(\psi)\right)\vec{k}$$
$$=(\cancel{\cos(\psi z)}-\cancel{\psi z\sin(\psi z)}-\cancel{\cos(\psi z)}+\cancel{\psi z\sin(\psi z)})\vec{i}+0\vec{j}+(1-1)\vec{k}=$$
$$(0,0,0).$$

Έστω f η βαθμωτή συνάρτηση, τότε θα έπρεπε:
$$\nabla f(x,\psi,z)=(\psi,\ z\cos(\psi z)+x,\ \psi\cos(\psi z)).$$

$$\frac{\partial}{\partial x}f(x,\psi,z)=\psi$$
$$\frac{\partial}{\partial \psi}f(x,\psi,z)=z\cos(\psi z)+x$$
$$\frac{\partial}{\partial z}f(x,\psi,z)=\psi\cos(\psi z)$$

$$\frac{\partial}{\partial x} f(x,\psi,z)=\psi=\frac{\partial}{\partial x}(x\psi)$$

$$\Longrightarrow \frac{\partial}{\partial x}\left(f(x,\psi,z)-x\psi\right)=0 \Longrightarrow \exists\, g=g(\psi,z)$$

$f(0,\psi,z)-0\psi$

ώστε $f(x,\psi,z)-x\psi=g(\psi,z)$

$$\Longrightarrow f(x,\psi,z)=x\psi+g(\psi,z)$$

$$\frac{\partial}{\partial \psi} f(x,\psi,z)=z\cos(\psi z)+x \Longleftrightarrow$$

$$\frac{\partial}{\partial \psi}\left(x\psi+g(\psi,z)\right)=z\cos(\psi z)+x$$

$$x+\frac{\partial}{\partial \psi} g(\psi,z)=z\cos(\psi z)+x \Longleftrightarrow$$

$$\frac{\partial}{\partial \psi} g(\psi,z)=z\cos(\psi z)=\frac{\partial}{\partial \psi}\left(\sin(\psi z)\right)$$

$$\Longleftrightarrow \frac{\partial}{\partial \psi}\left(g(\psi,z)-\sin(\psi z)\right)=0$$

$\Longrightarrow \exists\, h=h(z)$ ώστε:

$$g(\psi,z)-\sin(\psi z)=h(z)$$

$$\Longrightarrow g(\psi,z)=\sin(\psi z)+h(z)$$

οπότε $f(x,\psi,z)=x\psi+\sin(\psi z)+h(z)$

Επομένως: $\frac{\partial}{\partial z}\left(x\psi+\sin(\psi z)+h(z)\right)=\psi\cos(\psi z) \Leftarrow$

$$\cancel{\psi\cos(\psi z)}+h'(z)=\cancel{\psi\cos(\psi z)}$$

$\exists c\in\mathbb{R}$ $\quad \Leftrightarrow h'(z)=0$

$\Leftrightarrow h(z)=h(0)=$ σταθερά $=c$

$\forall z\in\mathbb{R}$

Οπότε τελικά: $f(x,\psi,z) = x\psi + \sin(\psi z) + c$ ■

ΘΕΩΡΗΜΑ: Έστω $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ αστρόβιλο $(C^1(\mathbb{R}^3))^3$ πεδίο. Τότε υπάρχει βαθμωτή συνάρτηση $f: \mathbb{R}^3 \to \mathbb{R}$, $f \in C^1(\mathbb{R}^3)$ τ.ω. $\nabla f = \vec{F}$.

Αποδ. Έστω $\vec{F} = (F_1, F_2, F_3)$ θα βρούμε βαθμωτή συνάρτηση $f: \mathbb{R}^3 \to \mathbb{R}$ ώστε

$$\frac{\partial f}{\partial x}(x,\psi,z) = F_1(x,\psi,z)$$

$$\frac{\partial f}{\partial \psi}(x,\psi,z) = F_2(x,\psi,z)$$

$$\frac{\partial f}{\partial z}(x,\psi,z) = F_3(x,\psi,z)$$

$$\frac{\partial}{\partial x} f(x,\psi,z) = F_1(x,\psi,z) = \frac{\partial}{\partial x}\left(\int_0^x F_1(t,\psi,z)\,dt\right)$$

$$\Rightarrow \frac{\partial}{\partial x}\left[f(x,\psi,z) - \int_0^x F_1(t,\psi,z)\,dt\right] = 0$$

$$\Rightarrow f(x,\psi,z) - \int_0^x F_1(t,\psi,z)\,dt = g(\psi,z)$$

$$\Rightarrow \boxed{f(x,\psi,z) = \int_0^x F_1(t,\psi,z)\,dt + f(0,\psi,z)}$$

Επειδή: $\underbrace{\frac{\partial}{\partial \psi}\left[\int_0^x F_1(t,\psi,z)\,dt + \frac{\partial}{\partial \psi} f(0,\psi,z)\right]}_{} = F_2(x,\psi,z)$

$\frac{\partial f}{\partial \psi}(x,\psi,z) = F_2(x,\psi,z) \Rightarrow *$

$$\int_0^x \frac{\partial}{\partial \psi} F_1(t,\psi,z)\,dt + \frac{\partial}{\partial \psi} g(\psi,z) = F_2(x,\psi,z) \Rightarrow$$

$$\frac{\partial}{\partial \psi} f(0,\psi,z) = F_2(x,\psi,z) - \int_0^x \frac{\partial}{\partial \psi} F_1(t,\psi,z)\,dt$$

$$* \Rightarrow \frac{\partial}{\partial \psi} f(0,\psi,z) = F_2(0,\psi,z)$$

$$\frac{\partial}{\partial \psi} f(x,\psi,z) = F_2(x,\psi,z) \quad , \quad x=0$$

$$\frac{\partial}{\partial \psi} f(0,\psi,z) = F_2(0,\psi,z)$$

$$= \frac{\partial}{\partial \psi}\left[\int_0^x F_2(0,t,z)\,dt\right]$$

$$\Rightarrow \frac{\partial}{\partial \psi}\left[f(0,\psi,z) - \int_0^\psi F_2(0,t,z)\,dt\right] = 0$$

$$f(0,\psi,z) - \int_0^\psi F_2(0,t,z)\,dt = f(0,0,z) - \cancel{\int_0^0 F_2(0,t,z)}$$

$$\Rightarrow f(0,\psi,z) = f(0,0,z) + \int_0^\psi F_2(0,t,z)\,dt.$$

$$\frac{\partial}{\partial z} f(x,\psi,z) = F_3(x,\psi,z)$$

$$\xRightarrow{x=0,\ \psi=0} \frac{\partial}{\partial z} f(0,0,z) = F_3(0,0,z)$$

$$= \frac{\partial}{\partial z}\left(\int_0^z F_3(0,0,t)\,dt\right)$$

$$\Rightarrow \frac{\partial}{\partial z}\left(f(0,0,z) - \int_0^z F_3(0,0,t)\,dt\right) = 0$$

$$\Rightarrow f(0,0,z) - \int_0^z F_3(0,0,t)\,dt = f(0,0,0) - 0$$

$$f(x,\psi,z) = \int_0^x F_1(t,\psi,z)\,dt + \int_0^\psi F_2(0,t,z)\,dt + \int_0^z F_3(0,0,t)\,dt + c \quad (*)$$

Θα αποδείξουμε ότι η f πληρεί τις υποθέσεις.

$$\Rightarrow \frac{\partial f}{\partial x}(x,\psi,z) = F_1(x,\psi,z) \quad \checkmark$$

$$\frac{\partial f}{\partial \psi}(x,\psi,z) = \int_0^x \frac{\partial}{\partial \psi} F_1(t,\psi,z)\,dt + F_2(0,\psi,z) = \underbrace{\int_0^x \frac{\partial F_2}{\partial x}(t,\psi,z)\,dt + F_2(0,\psi,z)}$$

όμως το $\vec{F}$ είναι αστρόβιλο $\Rightarrow$ $= F_2(x,\psi,z)$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial \psi} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \vec{0} \Leftrightarrow \begin{bmatrix} \frac{\partial F_3}{\partial \psi} = \frac{\partial}{\partial z} F_2 \\ \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial \psi} \end{bmatrix}$$

αστρόβιλο $\Rightarrow \exists\, f$ βαθμωτή $\vec{F} = \nabla f$ $\qquad (\nabla \times (\nabla f) = \vec{0})$

$\nabla \cdot (\nabla \times \vec{F}) = 0$

<u>ΘΕΩΡΗΜΑ</u>: Αν $\vec{F}$ διανυσματικό πεδίο $\exists\, \vec{G}: \nabla \times \vec{G} = \vec{F}$

Άσκηση Έστω $\vec{F}: \mathbb{R}^4 \to \mathbb{R}^3$, 2 φορές παραγωγίσιμη
Αποδείξτε ότι

$$\frac{d}{dt}\, \vec{F}(t) \times \vec{F}'(t) = \vec{F}(t) \times \left(\frac{d^2}{dt^2} \vec{F}(t)\right)$$

Αποδ. $\frac{d}{dt}\left(\vec{F}(t) \times \left(\frac{d}{dt}\vec{F}(t)\right)\right) =$

$$= \left(\frac{d}{dt}\vec{F}(t)\right) \times \left(\frac{d}{dt}\vec{F}(t)\right) + \vec{F}(t) \times \left(\frac{d^2}{dt^2}\vec{F}(t)\right) = \text{αυτό που θέλουμε}$$

Άσκηση Έστω $\vec{F}(x,\psi,z) = (x\psi z)^m \cdot (x^n \vec{i} + \psi^n \vec{j} + z^n \vec{k})$,

$$\vec{F}: \underset{(0,+\infty)^3}{\cancel{\mathbb{R}^3}} \to \mathbb{R}^3$$

Αν $\operatorname{curl} \vec{F} = \vec{0}$, $\forall x,\psi,z \in (0,+\infty)$, αποδείξτε ότι
είτε $m=0$
είτε $n=-1$

Απ. $\vec{F} = \left((x^m \psi^m z^m)\, x^n,\ (x^m\psi^m z^m)\cdot \psi^n,\ (x^m\psi^m z^m)\cdot z^n \right)$

$$= \left(x^{m+n}\psi^m z^m,\ x^m \psi^{m+n} z^m,\ x^m \psi^m z^{m+n}\right)$$

Οπότε $\operatorname{curl}\vec{F}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial \psi} & \frac{\partial}{\partial z} \\ x^{m+n}\psi^m z^m & x^m\psi^{m+n}z^m & x^m\psi^m z^{m+n} \end{vmatrix} =$$

$$= \left(\frac{\partial}{\partial \psi}\left(x^m \psi^m z^{m+n}\right) - \frac{\partial}{\partial z}\left(x^m \psi^{m+n} z^m\right)\right)\vec{i} -$$
$$\left(\frac{\partial}{\partial x}\left(x^m \psi^m z^{m+n}\right) - \frac{\partial}{\partial z}\left(x^{m+n} \psi^m z^m\right)\right)\vec{j} +$$
$$\left(\frac{\partial}{\partial x}\left(x^m \psi^{m+n} z^m\right) - \frac{\partial}{\partial \psi}\left(x^{m+n} \psi^m z^m\right)\right)\vec{k}$$

$$= \left(m x^m \psi^{m-1} z^{m+n} - m x^m \psi^{m+n} z^{m-1},\ m x^{m+n} \psi^m z^{m-1} - m x^{m-1}\psi^m z^{m+n},\ m x^{m-1}\psi^{m+n} z^m - m x^{m+n}\psi^{m-1} z^m\right)$$

$$= m x^{m-1}\psi^{m-1} z^{m-1}\left(x z^{n+1} - x\psi^{n+1},\ x^{n+1}\psi - \psi z^{n+1},\ \psi^{n+1} z - x^{n+1} z\right)$$

Επειδή $\nabla \times \vec{F} = \vec{0} \iff$

$$m x^{m-1}\psi^{m-1} z^{m-1}\, x\left(z^{n+1} - \psi^{n+1}\right) = 0$$
$$m x^{m-1}\psi^{m-1} z^{m-1}\, \psi\left(x^{n+1} - z^{n+1}\right) = 0$$
$$m x^{m-1}\psi^{m-1} z^{m-1}\, z\left(\psi^{n+1} - x^{n+1}\right) = 0$$

$\iff$ είτε $m=0$

είτε $m \neq 0$
$z^{n+1} = \psi^{n+1}$
$x^{n+1} = z^{n+1}$ $\forall x,\psi,z$
$\psi^{n+1} = x^{n+1}$

Αν $n+1=0 \iff n=-1$ οκ.
Αν $n+1 \neq 0 \Rightarrow x=\psi=z$

<u>Άσκηση</u>

α) Έστω $f,g: \mathbb{R}^2 \to \mathbb{R}$ παραγωγίσιμες ώστε $\nabla f(x,\psi) = \nabla g(x,\psi)$ $\forall x,\psi \in \mathbb{R}$. Αποδείξτε ότι $\exists\, c \in \mathbb{R}$ ώστε $f(x,\psi) = g(x,\psi) + c$, $\forall x,\psi \in \mathbb{R}$

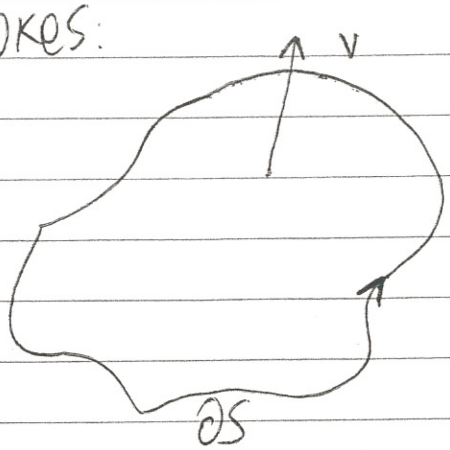
<u>Συνεκτικό χωρίο</u>

Κάθε 2 σημεία ενώνονται με συνεχή καμπύλη μέσα από το χωρίο.

β) Αν $f,g: \Omega \to \mathbb{R}$, Ω ανοικτό συνεκτικό και $\nabla f = \nabla g$ τότε $\exists\, c \in \mathbb{R}: f(x,\psi) = g(x,\psi) + c$, $\forall x,\psi \in \Omega$

29/11/2016 Χρυσοτίνα Ευθυμιάδου

Θ. Stokes:



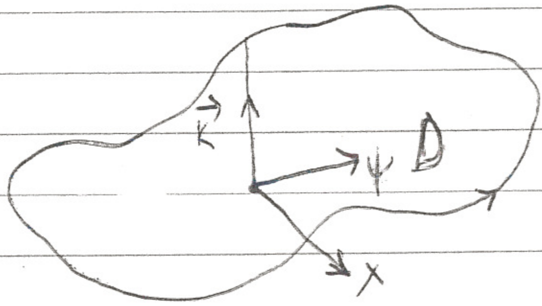
$$\int_{\partial D}(P\,dx+Q\,d\psi)$$

$$\parallel \quad (x,\psi)\in D$$

$$\int_C \vec{F}\cdot d\vec{S} = \iint_S \nabla\times\vec{F}\cdot d\vec{S} = \int_{\partial D}(P,Q,0)(dx,d\psi,dz)$$

$$= \iint_D \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial \psi}\right)\vec{k}\cdot\vec{k}\,\underbrace{dx\,d\psi}_{dS}$$

Θ. Green: $\leftrightarrow$ Θ. Stokes



Green:

$$\int_{\partial D}(P\,dx+Q\,d\psi) = \iint_D \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial \psi}\right)dx\,d\psi$$

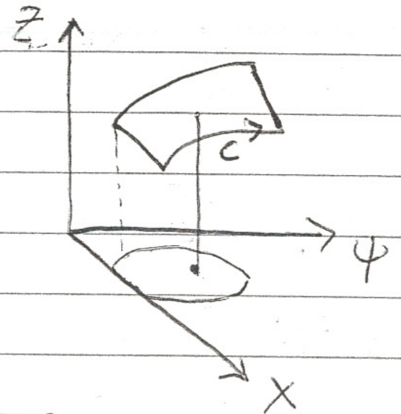
$$\int_{\partial D}(P\,dx+Q\,d\psi+0\,dz) \Longrightarrow \vec{F}=(P,Q,0)$$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial \psi} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \left(-\cancelto{0}{\frac{\partial}{\partial z}Q(x,\psi)}\right)\vec{i} - \left(-\cancelto{0}{\frac{\partial}{\partial z}P(x,\psi)}\right)\vec{j} + \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial \psi}\right)\vec{k}$$

$\Phi(x,\psi) = (x,\psi,0)$, $(x,\psi)\in D$ $\quad \vec{dS} = \Phi_x \times \Phi_\psi \, dx\,d\psi$

$\Phi_x \times \Phi_\psi = \vec{k}$

Αποδ. Θ. stokes όταν η επιφάνεια είναι γράφημα συνάρτησης, δηλ.



$S: \{(x,\psi,z) \mid z = g(x,\psi),\ \forall (x,\psi)\in D\}$.

$\Phi(x,\psi) = (x,\psi, g(x,\psi))$

$\Phi_x(x,\psi) = (1, 0, g_x(x,\psi))$

$\Phi_\psi(x,\psi) = (0, 1, g_\psi(x,\psi))$

$$\Phi_x \times \Phi_\psi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & g_x \\ 0 & 1 & g_\psi \end{vmatrix} = -g_x\vec{i} - g_\psi\vec{j} + \vec{k}$$
$$= (-g_x, -g_\psi, 1)$$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial \psi} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial \psi} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial}{\partial z}P - \frac{\partial}{\partial x}R\right)\vec{j}$$
$$+ \left(\frac{\partial}{\partial x}Q - \frac{\partial}{\partial \psi}P\right)\vec{k}$$

(υποθέτω $\vec{F} = (P,Q,R)$)

Οπότε: $\iint_S \nabla \times \vec{F} \cdot d\vec{S} =$

$$= \iint_D \left(\frac{\partial R}{\partial \psi} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial \psi} \right) \cdot \left(-\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial \psi}, 1 \right) dx\,d\psi$$

$$\iint_D \left[-\left(\frac{\partial R}{\partial \psi} - \frac{\partial Q}{\partial z} \right) \frac{\partial g}{\partial x} - \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \frac{\partial g}{\partial \psi} + \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial \psi} \right] dx\,d\psi$$

$C: \vec{\sigma}(t) = (x(t), \psi(t), g(x(t), \psi(t))), \ t \in [0,1]$

$\vec{\sigma}'(t) = (x'(t), \psi'(t), g_x(x(t), \psi(t)) x'(t) + g_\psi(x(t), \psi(t)) \psi'(t))$

$$\frac{d}{dt}(g(x(t), \psi(t))) = \frac{\partial g}{\partial x}(x(t), \psi(t)) \cdot x'(t) + \frac{\partial g}{\partial \psi}(x(t), \psi(t)) \psi'(t)$$

$$\frac{d}{dt}(f(g(t))) = f'(g(t)) \cdot g'(t)$$

$$\int_C \vec{F} \cdot d\vec{S} = \int_0^1 (P, Q, R) \cdot (x', \psi', g_x x' + g_\psi \psi') dt$$

$$= \int_0^1 [P x' + Q \psi' + R(g_x x' + g_\psi \psi')] dt$$

$$= \int_0^1 [(P + R g_x) x' + (Q + R g_\psi) \psi'] dt$$

$$= \int_{\partial D} [(P + R g_x) dx + (Q + R g_\psi) d\psi] =$$

(Θ. Green: $\int_{\partial D} (P\,dx + Q\,d\psi) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial \psi} \right) dx\,d\psi$

θ. Green

$$= \iint_D \left[\frac{\partial}{\partial x}(Q + R g_\psi) - \frac{\partial}{\partial \psi}(P + R g_x)\right]$$

$$\frac{\partial}{\partial x}\Big(Q(x, \psi, g(x,\psi))\Big) = \frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\cdot\frac{\partial g}{\partial x}$$

$$\frac{\partial}{\partial x}\big(R(x, \psi, g(x,\psi))\big) = \frac{\partial R}{\partial x} + \frac{\partial R}{\partial z}\cdot\frac{\partial g}{\partial x}$$

$$\frac{\partial}{\partial \psi}\Big(P(x, \psi, g(x,\psi))\Big) = \frac{\partial P}{\partial \psi} + \frac{\partial P}{\partial z}\cdot\frac{\partial g}{\partial \psi}$$

$$\frac{\partial}{\partial \psi}\big(R(x, \psi, g(x,\psi))\big) = \frac{\partial R}{\partial \psi} + \frac{\partial R}{\partial z}\cdot\frac{\partial g}{\partial \psi}$$

ΘΕΩΡΗΜΑ Έστω $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ ($\Omega \subseteq \mathbb{R}^3$, Ω απλά συνεκτικό).

Τότε τα ακόλουθα είναι ισοδύναμα

i) $\forall$ C κλειστή καμπύλη, έχουμε

$$\int_C \vec{F}\cdot d\vec{S} = 0$$

ii) $\forall$ C_1, C_2 καμπύλες με την ίδια αρχή και τέλος ισχύει

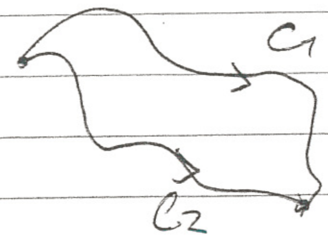
$$\int_{C_1} \vec{F}\cdot d\vec{S} = \int_{C_2} \vec{F}\cdot d\vec{S}$$

iii) Υπάρχει $f: \mathbb{R}^3 \to \mathbb{R}$, ώστε $\vec{F} = \nabla f$

iv) $\nabla \times \vec{F} = 0$

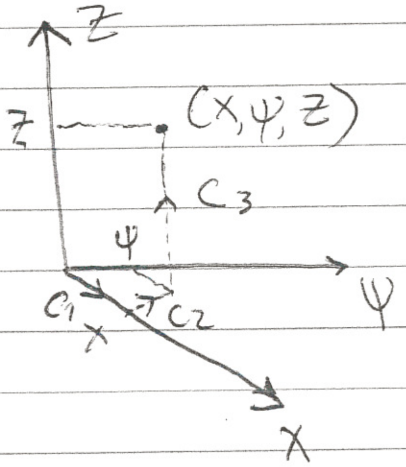
*αστρόβιλα πεδία

<u>Απ</u> $i) \Rightarrow ii) \Rightarrow iii) \Rightarrow iv) \Rightarrow i)$



$C = C_1 \cup (-C_2)$ είναι κλειστό

$$\int_C \vec{F} \cdot d\vec{S} = 0 = \int_{C_1} \vec{F} \cdot d\vec{S} + \int_{C_2} \vec{F} \cdot d\vec{S} = \int_{C_1} \vec{F} d\vec{S} - \int_{C_2} \vec{F} d\vec{S}$$



$\vec{F} = (P, Q, R)$

$$\int_C \vec{F} \cdot d\vec{S} = \int_{C1} \vec{F} \cdot ds + \int_{C2} \vec{F} \cdot dS + \int_{C3} \vec{F} \cdot d\vec{S}$$

$$= \int_0^x P(t, 0, 0)\, dt + \int_0^{\psi} Q(x, t, 0)\, dt + \int_0^z R(x, \psi, t)\, dt = f(x, \psi, z)$$

C_1: $\vec{\sigma}(t) = (t, 0, 0)$

$t = 0$

$t = x$

$$\int_{C_1} \vec{F} d\vec{S} = \int_0^x P(t, 0, 0)\, dt$$

$C_2: \vec{\sigma}(t) = (x, t, 0) \quad 0 \leq t \leq \psi$

$$\int_{C_2} \vec{F} \cdot d\vec{S} = \int_0^{\psi} (P(x,t,0), Q(x,t,0), R(x,t,0)) \cdot e_2 \, dt$$

$$= \int_0^{\psi} Q(x,t,0) dt$$

$C_3: \vec{\sigma}(t) = (x, \psi, t) \quad t [0, z]$

$$\int_{C_3} \vec{F} \cdot d\vec{S} = \int_0^{z} R(x, \psi, t) dt$$

Παράδειγμα 1: σπίτι

Έστω $\vec{F} = (2x\psi z + \sin x, x^2 z, x^2 \psi)$. Βρείτε όλες τις $f: \mathbb{R}^3 \to \mathbb{R}$, ώστε $\vec{F} = \nabla f$

Υπολογίστε $\int_{C_1} \vec{F} \cdot d\vec{S}$ όταν η καμπύλη ξεκινά από $(\pi, 1, 1)$ και καταλήγει στο $(4\pi, 2, -1)$.

Παράδειγμα 2: Βρείτε το επιφανειακό ολοκλήρωμα

$$\iint_S \vec{F} \cdot d\vec{S} \quad * z \geq 0$$

όπου S, είναι η επιφάνεια της μοναδιαίας σφαίρας με κέντρο το 0^* και ο προσανατολισμός είναι προς τα έξω και $\vec{F} = (2x + e^{\psi^{2016}}, \psi^2 + e^{z^{2016}}, z^2 + x^2 e^{\psi^{2016} + x^{2016}})$

7/12/2016

Θ. Stokes:

$$\int_S \nabla \times \vec{F}\, d\vec{S} = \int_C \vec{F} \cdot d\vec{S}$$

Παράδειγμα ①:

Έστω $\vec{F} = (2x\psi z + \sin x,\ x^2 z,\ x^2 \psi)$. Βρείτε όλες τις $f : \mathbb{R}^3 \to \mathbb{R}$, και $\vec{F} = \nabla f$. Υπολογίσετε $\int_{C_1} \vec{F}\, d\vec{S}$, όπου η C_1 ενώνει το $(\pi, 1, 1)$ με το $(4\pi, 2, -1)$.

1ος τρόπος: Έστω f μια βαθμωτή, θα πρέπει $\nabla f = \vec{F} \Leftrightarrow$

$$\frac{\partial f}{\partial x}(x,\psi,z) = 2x\psi z + \sin x$$

$$\frac{\partial f}{\partial \psi}(x,\psi,z) = x^2 z$$

$$\frac{\partial f}{\partial z}(x,\psi,z) = x^2 \psi$$

$$\frac{\partial f}{\partial z}(x,\psi,z) = x^2\psi = \frac{\partial}{\partial z}(x^2\psi z) \Leftrightarrow \frac{\partial}{\partial z}\left(f(x,\psi,z) - x^2\psi z\right)$$

$\Rightarrow \exists\, g = g(x,\psi)$ ώστε $f(x,\psi,z) - x^2\psi z = g(x,\psi)$

$$f(x,\psi,z) = x^2\psi z + g(x,\psi)$$

Επειδή:

$$\frac{\partial}{\partial \psi} f(x,\psi,z) = x^2 z \Leftrightarrow \frac{\partial}{\partial \psi}\left(x^2\psi z + g(x,\psi)\right) = x^2 z$$

$$\Leftrightarrow \cancel{x^2 z} + \frac{\partial}{\partial \psi} g(x,\psi) = \cancel{x^2 z} \Leftrightarrow \frac{\partial g}{\partial \psi}(x,\psi) = 0$$

Οπότε $\exists\, h = h(x)$ ώστε:

$$g(x,\psi) = h(x)$$

Επομένως $f(x,\psi,z)=x^2\psi z+h(x)$.

όμως $\frac{\partial f}{\partial x}(x,\psi,z)=2x\psi z+\sin x \Leftrightarrow \frac{\partial}{\partial x}\left(x^2\psi z+h(x)\right)=2x\psi z+\sin x$

$\Leftrightarrow \cancel{2x\psi z}+h'(x)=\cancel{2x\psi z}+\sin x \Leftrightarrow h'(x)=\sin x=(-\cos x)'$

$\Leftrightarrow (h(x)+\cos x)'=0$

$\Rightarrow \exists\, c\in\mathbb{R}$ ώστε

$h(x)+\cos x=c \Rightarrow h(x)=c-\cos x$

$\Rightarrow f(x,\psi,z)=c+x^2\psi z-\cos x$

2ος τρόπος: Το διανυσματικό πεδίο $\vec{F}$ είναι αστρόβιλο, $\nabla\times\vec{F}=\vec{0}$

$\Rightarrow \exists\, f:\mathbb{R}^3\to\mathbb{R}$, ώστε $\vec{F}=\nabla f$

$$\int_c \vec{F}\cdot d\vec{s}=\int_c \nabla f\cdot d\vec{s}=f(\vec{\sigma}(1))-f(\vec{\sigma}(0))$$

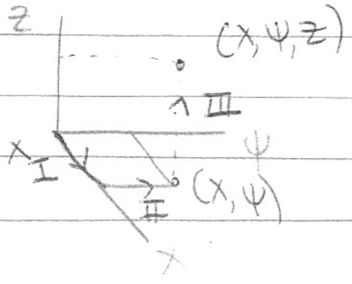
Έστω $c: \vec{\sigma}(t),\ t\in[0,1]$

$$\int_c \nabla f\cdot d\vec{s}=\int_0^1 \underbrace{\nabla f(\vec{\sigma}(t))\cdot\vec{\sigma}'(t)}_{\frac{d}{dt}\left(f(\vec{\sigma}(t))\right)}\,dt$$

$$=\int_0^1 \frac{d}{dt}\left(f(\vec{\sigma}(t))\right)dt=f(\vec{\sigma}(1))-f(\vec{\sigma}(0))$$

Αν $\vec{\sigma}(1)=(x,\psi,z)$

Τότε $f(x,\psi,z)=\int_c \vec{F}\cdot d\vec{s}+c$



Τότε $\int_C \vec{F}\cdot d\vec{s} = \int_{I} + \int_{II} + \int_{III}$

$\int_{I} \vec{F}\cdot d\vec{s}$ $\quad$ I: $\vec{\sigma}(t) = (t,0,0) \quad 0 \le t \le x$

$\vec{\sigma}_1{}'(t) = (1,0,0)$

$= \int_0^x \vec{F}(\vec{\sigma}_1(t))\cdot \vec{\sigma}_1{}'(t)\,dt = \int_0^x (\sin t, 0, 0)\cdot(1,0,0)\,dt =$

$= (-\cos t)\Big|_0^x = -\cos x + 1$

$\int_{II} \vec{F}\cdot d\vec{s}$ $\quad$ II: $\vec{\sigma}_2(t) = (x, t, 0), \quad 0 \le t \le \psi$

$\vec{\sigma}_2{}'(t) = (0,1,0)$

$= \int_0^\psi \vec{F}(\vec{\sigma}_2(t))\cdot \vec{\sigma}_2{}'(t)\,dt =$

$\int_0^\psi (\sin x, 0, x^2 t)\cdot(0,1,0)\,dt = 0$

$\int_{III} \vec{F}\cdot d\vec{s}$ $\quad$ III: $\vec{\sigma}_3(t) = (x, \psi, t), \quad t \in [0, z]$

$\vec{\sigma}_3{}'(t) = (0,0,1)$

$= \int_0^z (2x\psi t + \sin x, x^2 t, x^2\psi)(0,0,1)\,dt =$

$= \int_0^z x^2\psi\,dt = x^2\psi z$

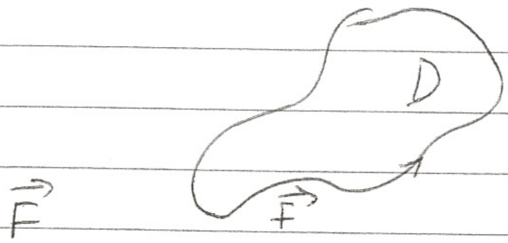
Επομένως $f(x,\psi,z) = x^2\psi z - \cos x + \overbrace{1 + c}^{c}$

τιμή ενός άκρου − τιμή άλλου άκρου

⋮

Θεώρημα Gauss (Θεώρημα απόκλισης)

$\Rightarrow$ Θ. Green



$$\int_C (-P dx + Q d\psi) = \iint_D \left(\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial \psi} \right) dx$$

$$\iint_D \operatorname{div}(\overbrace{Q, P}^{\vec{F}})\, dx\, d\psi = \int_C (Q, P) \cdot d\vec{s}$$

$$= \int_C (Q, P)\ \nu\, ds = \int_C \vec{F} \cdot \nu \cdot ds$$

Θ. απόκλισης Green

$$\iint_D \nabla \cdot (\overbrace{Q, P}^{\vec{F}})\, dx\, d\psi = \int_C \vec{F} \cdot \vec{\nu}\, ds = \int_C (Q, P) \cdot (\nu_1 , \nu_2)\, ds$$

Θ Gauss:

$\vec{F} : \Omega \to \mathbb{R}^3$, Ω στερεό $\subseteq \mathbb{R}^3$.

$$\iiint_\Omega \nabla \cdot \vec{F}\, dx\, d\psi\, dz = \iint_S \vec{F} \cdot \vec{\nu}\, dS = \int_C (Q, P) \cdot (\nu_1, \nu_2)\, dS$$

$S = \partial \Omega$, $\vec{\nu}$ προς τα έξω του στερεού

$$\vec{F} = (P, Q, R) \quad \iiint_\Omega \nabla \cdot \vec{F}\, dx\, d\psi\, dz = \iiint_\Omega \left[\frac{\partial}{\partial x} P + \frac{\partial}{\partial \psi} Q + \frac{\partial}{\partial z} R \right] dx\, d\psi\, dz$$

$$= \iint_S (P, Q, R) \cdot \vec{\nu}\, dS = \iint_S (P\nu_1 + Q\nu_2 + R\nu_3)\, dS$$

$\vec{\nu} = (\nu_1, \nu_2, \nu_3)$

<u>Πρόβλημα Poisson</u>

$$\left.\begin{aligned} \Delta f(x) &= g(x), \quad x\in\Omega \\ \nabla f(x)\cdot n &= h(x), \quad x\in\partial\Omega \end{aligned}\right\} \ *$$

Έχει λύση όταν $\int_{\Omega} g(x)dx = \int_{\partial\Omega} h(x)ds$

$$\nabla\times\vec{G} = 0 \Rightarrow \vec{G} = \nabla f(x)$$

τότε $\vec{F}(x) = \nabla f(x) + \vec{Q}$

$$\Rightarrow \nabla\cdot\vec{F}(x) = \Delta f(x) + 0$$

$$\Rightarrow \Delta f(x) = \nabla\cdot\vec{F}(x), \quad x\in\Omega$$

$$\underset{(\vec{G}\cdot\vec{n})}{\nabla f(x)\cdot\vec{n}} = \underbrace{\vec{F}(x)\cdot\vec{n}}_{h}, \quad x\in\partial\Omega$$

[illegible] το Θ. Gauss

$$\int_{\partial\Omega}\vec{F}(x)\cdot\vec{n}\,dS = \int_{\Omega}\nabla\cdot\vec{F}(x)\,dx \ \checkmark$$

Έχουμε βρει μια f, τότε

$$\vec{F}(x) = \nabla f(x) + \vec{Q}(x) \Rightarrow$$

$$\vec{Q}(x) = \vec{F}(x) - \nabla f(x), \quad x\in\partial\Omega$$

τότε $\nabla\cdot\vec{Q}(x) = \nabla\cdot(\vec{F}(x) - \nabla f(x))$

$$= \nabla\cdot\vec{F}(x) - \nabla^2 f(x)$$

$$= \nabla\cdot\vec{F}(x) - \Delta f(x)$$

$$= 0 \qquad \blacksquare$$

Για τις μονοσημαντες. Έστω: $\vec{G_1}, \vec{G_2}, \vec{Q_1}, \vec{Q_2}$

$(\vec{G_1}, \vec{Q_1}), (\vec{G_2}, \vec{Q_2})$

δύο διαμερισμένες διαστάσεις ώστε

$$\vec{F} = \vec{G_1} + \vec{Q_1} = \vec{G_2} + \vec{Q_2} \Leftrightarrow \vec{G_1} - \vec{G_2} = -(\vec{Q_1} - \vec{Q_2})$$

$$\nabla\times\vec{G_1} = \nabla\cdot\vec{G_2} = \vec{0}$$

$$\nabla\cdot\vec{Q_1} = \nabla\cdot\vec{Q_2} = 0$$

$\vec{G_1}\cdot\vec{n} = \vec{F}\cdot\vec{n} = \vec{G_2}\cdot\vec{n} \quad \partial\Omega$

Οπότε αν θέσουμε

$$\vec{G} = \vec{G_1} - \vec{G_2}$$
$$\vec{Q} = \vec{Q_1} - \vec{Q_2}$$

Τότε έχουμε:

$$\nabla\times\vec{G} = \nabla\times\vec{G_1} - \nabla\times\vec{G_2} = \vec{0}$$
$$\nabla\cdot\vec{Q} = \nabla\cdot\vec{Q_1} - \nabla\cdot\vec{Q_2} = 0$$
$$\vec{G}\cdot\vec{n} = \vec{G_1}\cdot\vec{n} - \vec{G_2}\cdot\vec{n} = \vec{F}(x)\cdot\vec{n} - \vec{F}(x)\cdot\vec{n} = 0$$

$\nabla\times\vec{G} = 0 \quad \Omega \quad \Rightarrow \exists f : \vec{G} = \nabla f \quad \Omega$

$\vec{G}\cdot\vec{n} = 0 \quad \partial\Omega \qquad \nabla f\cdot\vec{n} = 0 \quad \partial\Omega$

$\vec{F}(x) = -\vec{Q}(x) \quad x\in\Omega$

$\vec{G} = (g_1, g_2, g_3)$

και επειδή $\nabla\cdot\vec{Q} = 0 \Rightarrow \nabla\cdot\vec{G}(x) = 0,\ x\in\Omega$

$$\boxed{\begin{array}{l} \nabla f\cdot\vec{n} = 0 \quad \partial\Omega \\ \Delta f(x) = 0 \quad \Omega \end{array}}$$

Έστω g

$$g\,\Delta f(x) = 0$$
$$\Downarrow$$
$$\int_\Omega g\,\Delta f(x)\,dx = 0$$
$$\|$$

1η ταυτότητα Green

$$-\int_\Omega \nabla g\cdot\nabla f\,dx + \int_{\partial\Omega} g\,\overbrace{\nabla f\cdot\vec{n}}^{0}\,dS$$

$$\Rightarrow \int_\Omega \nabla g\cdot\nabla f\,dx = 0$$

Επιλέγουμε $g=f$

$$\int_\Omega |\nabla f|^2 dx = 0$$

$$\Rightarrow \nabla f(x) = \vec{0}, \ x\in\Omega \Rightarrow \vec{G}(x)=0, \ x\in\Omega$$
$$\Rightarrow \vec{Q}(x)=0, \ x\in\Omega$$

$$\Downarrow$$
$$\vec{G_1}=\vec{G_2}$$
$$\vec{Q_1}=\vec{Q_2}$$

$\blacksquare$

<u>Άσκηση</u>: Έστω $f: B_1 \to \mathbb{R}$, ομαλή συνάρτηση που ικανοποιεί:
$B_1 = \{(x,\psi,z) \mid x^2+\psi^2+z^2 \le 1\}$

$$\Delta f(x) = 2, \quad x\in B_1 \qquad x=(x,\psi,z)$$
$$f(x) = \frac{1}{3}, \quad x\in\partial B_1$$

Αποδείξτε ότι $f(x) = \frac{1}{3}(x^2+\psi^2+z^2)$, $x=(x,\psi,z)\in B_1$.

Έστω f τυχαία λύση

$$\Delta f(x)=2, \quad x\in B$$
$$f(x)=\frac{1}{3}, \quad x\in\partial B,$$

φτιάχνουμε $g(x) = f(x) - \frac{1}{3}(x^2+\psi^2+z^2)$

Τότε η g ικανοποιεί

$$\Delta g(x) = \Delta f(x) - \frac{1}{3}\Delta(x^2+\psi^2+z^2)$$
$$= 2 - \frac{1}{3}\left(\frac{\partial^2}{\partial x^2}(\ ) + \frac{\partial^2}{\partial \psi^2}(\ ) + \frac{\partial^2}{\partial z^2}(\ )\right)$$
$$= 2 - \frac{1}{3}(2+2+2) = 0$$

$x \in \partial B_2, \quad g(x) = f(x) - \frac{1}{3}(x^2+\psi^2+z^2) = \frac{1}{3} - \frac{1}{3} = 0,$

$$\int_{B_1} f \Delta g \, dx = \int_{B_1} 0 \, dx = 0$$

Green ταυτότητα

$$-\int_{B_1} \nabla f \cdot \nabla g \, dx + \int_{\partial B_1} f \nabla g(x) \cdot \vec{n} \, dS = 0 \quad \forall f$$

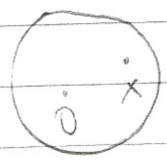
Επιλέγω $f = g$

Τότε έχω:

$$-\int_{B_1} \nabla g \cdot \nabla g \, dx + \int_{\partial B_1} g \nabla g \cdot n \, dS = 0$$

$$\int_{B_1} |\nabla g|^2 dx = 0 \Rightarrow \nabla g(x) = 0 \Longrightarrow \exists c \in \mathbb{R} \quad g(x) = c, \ \forall x \in B_1$$

$$g_x + g_\psi + g^2_z = 0$$


$\Rightarrow g(x) = g(0) \quad \forall x \in B_1$

$$h(t) = g(tx)$$

ΘΜΤ h

$$h(1) - h(0) = (1-0) \cdot h'(s)$$

$$h(t) = g(tx, t\psi, tz)$$

$$h'(t) = \frac{\partial g}{\partial x}(tx, t\psi, tz)x + \frac{\partial g}{\partial \psi}(tx, t\psi, tz) + \frac{\partial}{\partial z} g(tx, t\psi, tz) \frac{d}{dt}(tz)$$

$$g(x)-g(0)=\frac{\partial g}{\partial x}(s_x,s_\psi,s_z)\cdot x+\frac{\partial g}{\partial \psi}(s_x,s_\psi,s_z)\psi+$$

$$\frac{\partial g}{\partial z}(s_x,s_\psi,s_z)\cdot z$$

$$\Rightarrow \exists c\in\mathbb{R},\quad g(x)=c \qquad \forall x\in B_1$$

$$x\in\partial B_1 \quad g(x)=0$$

$$\Rightarrow c=0$$

$$\Rightarrow g(x)\equiv 0$$

$$\lim_{n\to\infty} a_n=0 \Leftrightarrow \lim_{n\to\infty}|a_n|=0$$

<u>Άσκηση</u> Αποδείξτε ότι: $\qquad X=(x,\psi,z)$

$$\underbrace{\iint_{\partial\Omega}\sqrt{x^2+\psi^2+z^2}\cos(X,\nu)}\,dS=3\iiint_{\Omega}dx\,d\psi\,dz$$

$$\vec{F}(x)\cdot\vec{\nu}(x)=|\vec{F}(x)|\cdot\cos(\vec{F}\cdot\vec{\nu})$$

$$|\vec{F}(x)|=\sqrt{x^2+\psi^2+z^2}$$

$$\vec{F}(x)=\lambda(x)\cdot x\Rightarrow|\vec{F}(x)|=|\lambda(x)|\cdot|x|=|X|$$

$$\vec{F}(x)=(x,\psi,z)=X$$

$$\iiint_{\Omega}\underset{3}{\underset{\parallel}{D}}\cdot\vec{F}(x)\,dx\,d\psi\,dz=\iint_{\partial\Omega}\vec{F}(x)\cdot\vec{n}\,ds$$

Άσκηση Με Θ. Gauss να υπολογίσετε:

$$\iint \vec{F}\cdot\vec{n}\, dS$$

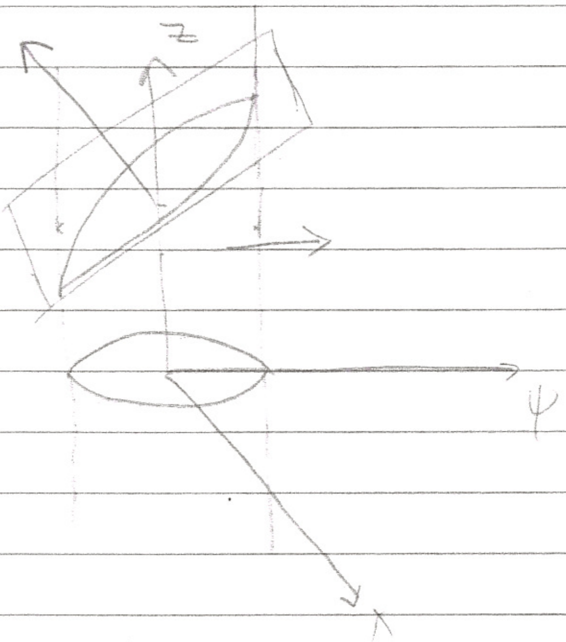
$$\vec{F} = (x^4, -x^3 z^2, 4x\psi^3 z)$$

όπου S επιφάνεια του στερεού

$$x^2+\psi^2 \le 1$$

$$0 \le z \le x+2$$

, $\vec{n}$ προς τα έξω στερεού



$$\iiint \underbrace{\nabla\cdot\vec{F}}_{4x^3+4x\psi^3} = \iint_{x^2+\psi^2\le 1}\left(\int_0^{x+2} dz\right)dx\,d\psi$$

$$\iiint_\Omega \frac{\partial P}{\partial x}\,dx\,d\psi\,dz = \iint_S P\,\nu_1\,dS$$

$$\iiint_\Omega \frac{\partial Q}{\partial \psi}\,dx\,d\psi\,dz = \iint_S Q\cdot \nu_2\,dS$$

<u>Ερώτημα</u>: Έστω $f, g : \Omega \to \mathbb{R}$ και $\iiint_\Omega \frac{\partial f}{\partial x} g\,dx\,d\psi\,dz \overset{*}{=}$

$$\int_0^1 f'(t)g(t)\,dt = -\int_0^1 f(t)g'(t)\,dt + (fg)\Big|_0^1$$

$$(fg)\Big|_0^1 = f(1)g(1) - f(0)g(0)$$

$$\overset{*}{=} -\iiint_\Omega f\frac{\partial g}{\partial x}\,dx\,d\psi\,dz + \iint_S fg\,\nu_1\,dS$$

<u>Παράδειγμα ①</u>: Βρείτε το επιφανειακό ολοκλήρωμα $\iint_S \vec{F}\cdot\vec{\nu}\,dS$ όπου η επιφάνεια S είναι η επιφάνεια της μοναδιαίας σφαίρας με κέντρο το O και προσανατολισμό του μοναδιαίου προς τα έξω και $\vec{F} = (2x + e^{\psi^{2016}}, \psi^2 + e^{z^{2016}}, z + x^2 e^{\psi^{2016} + x}$

$$\text{Gauss} = \iiint_\Omega \nabla\cdot\vec{F}\,dx\,d\psi\,dz$$

$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(2x + e^{\psi^{2016}}) + \frac{\partial}{\partial \psi}(\psi^2 + e^{z^{2016}}) + \frac{\partial}{\partial z}(z^2 + x^2 e^{\psi^{2016} + x^{2016}})$$

$$= 2 + 2\psi + 2z$$

Σφαιρικές συντεταγμένες

$$x = \rho \sin\varphi \cos\vartheta \qquad 0 \le \vartheta < 2\pi$$
$$\psi = \rho \sin\varphi \sin\vartheta \qquad 0 \le \varphi \le \pi$$
$$z = \rho \cos\varphi \qquad 0 \le \rho \le 1$$

$$\left|\frac{\partial(x,\psi,z)}{\partial(\rho,\varphi,\vartheta)}\right| = \rho^2 \sin\varphi$$

$$= \int_0^1\int_0^{2\pi}\int_0^{\pi} (2 + 2\rho\sin\varphi\sin\vartheta + 2\rho\cos\varphi) \left|\frac{\partial(x,\psi,z)}{\partial(\rho,\varphi,\vartheta)}\right| d\varphi\, d\vartheta\, d\rho$$

$$= \int_0^1\int_0^{2\pi}\int_0^{\pi} (2 + 2\rho\sin\varphi\sin\vartheta + 2\rho\cos\varphi)\rho^2 \sin\varphi \, d\varphi\, d\vartheta\, d\rho$$

$$= \int_0^1\int_0^{2\pi}\int_0^{\pi} 2\rho^2 \sin\varphi \, d\varphi\, d\vartheta\, d\rho + 2\iiint \rho^3 \sin^2\varphi \sin\vartheta \, d\varphi\, d\vartheta\, d\rho +$$
$$+ 2\iiint \rho^3 \sin\varphi\cos\varphi \, d\varphi\, d\vartheta\, d\rho$$

$$= 2 \cdot 2\pi \cdot \frac{2}{3} = \frac{8\pi}{3}$$

__Παράδειγμα__: Βρείτε το επιφανειακό ολοκλήρωμα
$$\iint_S (2x + (\psi^2 + z^2)e^{\psi^3})\, x\, dS \quad *$$

όπου S η επιφάνεια της μοναδιαίας σφαίρας με κέντρο το O.

Απ. Με θ. Gauss:
$$\iiint \nabla \cdot \vec{F}\, dx\, d\psi\, dz = \iint_S \vec{F} \cdot \vec{\nu}\, dS$$

$$* = \iint_S (2x + (\psi^2 + z^2)e^{\psi^3}) \cdot \nu_1 \, dS$$

$\vec{F} = (f, 0, 0)$

$$\iiint_{B_1} \frac{\partial f}{\partial x}\, dx\, d\psi\, dz = \iint_S f \cdot \nu_1\, dS$$

Επιλέγω $f(x,\psi,z) = 2x + (\psi^2+z^2)e^{\psi^3}$

$$\Rightarrow \frac{\partial f}{\partial x}(x,\psi,z) = 2$$

Επομένως $\iint_S (2x+(\psi^2+z^2)e^{\psi^3}) x\, dS =$

$$= 2\iiint_{B_1} dx\, d\psi\, dz = \frac{8\pi}{3}$$

//

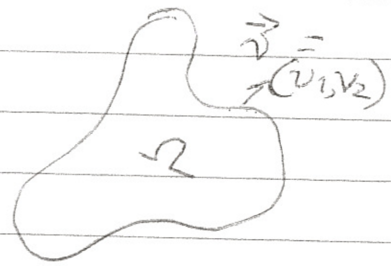
6/12/2016

Θ. Gauss $\quad \vec{F} = (fg, 0, 0)$

$$\iiint_\Omega \nabla \cdot \vec{F}\, dx\, d\psi\, dz = \iint_{\partial\Omega} \vec{F}\cdot\vec{\nu}\, dS$$

ολοκλήρωση κατά μέρη στις πολλές μεταβλητές.

$$\iiint_\Omega \frac{\partial f}{\partial x}\, dx\, d\psi\, dz = \iint_{\partial\Omega} f \nu_1\, dS$$



$$\iiint_\Omega \frac{\partial f}{\partial x} g\, dx\, d\psi\, dz = -\iiint_\Omega f \frac{\partial g}{\partial x}\, dx\, d\psi\, dz + \iint_{\partial\Omega} fg\,\nu_1\, dS$$

$$\iiint_\Omega \frac{\partial f}{\partial \psi} g\, dx\, d\psi\, dz = -\iiint_\Omega f \frac{\partial g}{\partial \psi}\, dx\, d\psi\, dz + \iint_{\partial\Omega} fg\,\nu_2\, dS$$

$$\iiint_\Omega \frac{\partial f}{\partial z} g\, dx\, d\psi\, dz = -\iiint_\Omega f \frac{\partial g}{\partial z}\, dx\, d\psi\, dz + \iint_{\partial\Omega} fg\,\nu_3\, dS$$

Ταυτότητες Green

1_η// $\iiint_\Omega f\,\Delta g\,dx\,d\psi\,dz = -\iiint_\Omega \nabla f\cdot\nabla g\,dx\,d\psi\,dz + \iint_{\partial\Omega} f\,\nabla g\cdot\vec{\nu}\,dS$

$$\Delta g(x,\psi,z) = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial \psi^2} + \frac{\partial^2 g}{\partial z^2}$$

$$H(g) = \begin{pmatrix} g_{xx} & g_{x\psi} & g_{xz} \\ g_{\psi x} & g_{\psi\psi} & g_{\psi z} \\ g_{zx} & g_{z\psi} & g_{zz} \end{pmatrix} \quad \Delta g(x,\psi,z) = \operatorname{tr} H(g)$$

$$\int_\Omega f(x,\psi,z)\,g_{xx}(x,\psi,z)\,dx\,d\psi\,dz = -\int_\Omega f_x g_x\,dx\,d\psi\,dz + \iint_{\partial\Omega} f\cdot g_x \cdot \nu_1\,dS$$

$$\int_\Omega f\,g_{\psi\psi}\,dx\,d\psi\,dz = -\int_\Omega f_\psi\,g_\psi\,dx\,d\psi\,dz + \iint_{\partial\Omega} f\cdot g_\psi\,\nu_2\,dS$$

$$\int_\Omega f\,g_{zz}\,dx\,d\psi\,dz = -\int_\Omega f_z\,g_z\,dx\,d\psi\,dz + \iint_{\partial\Omega} f\,g_z\,\nu_3\,dS$$

2_η// $\iiint (f\Delta g - g\Delta f)\,dx\,d\psi\,dz = \iint_{\partial\Omega} [f\nabla g - g\nabla f]\cdot\vec{\nu}\,dS$

<u>Άσκηση</u> Έστω Ω φραγμένο στερεό στον $\mathbb{R}^3$ ($x=(x,\psi,z)$) και $f:\Omega\to\mathbb{R}$, $\vec{G}:\Omega\to\mathbb{R}^3$, $g:\partial\Omega\to\mathbb{R}$ ομαλές συναρτήσεις ώστε να υπάρχει διανυσματικό πεδίο $\vec{F}:\Omega\to\mathbb{R}^3$ ώστε:

$$\nabla\cdot\vec{F}(x) = f(x), \quad x\in\Omega$$
$$\nabla\cdot\vec{F}(x) = \vec{G}(x), \quad x\in\Omega$$
$$\vec{F}(x)\cdot\vec{\nu}(x) = g(x), \quad x\in\partial\Omega$$

Αποδείξτε ότι το πρόβλημα έχει ακριβώς μια λύση.

Έστω $\vec{F_1}, \vec{F_2}$ δύο διακεκριμένες λύσεις του προβλήματος δηλαδή,

$$\nabla \cdot \vec{F_1}(x) = f(x) = \nabla \cdot \vec{F_2}(x), \quad x \in \Omega$$
$$\nabla \times \vec{F_1}(x) = \vec{G}(x) = \nabla \times \vec{F_2}(x), \quad x \in \Omega$$
$$\vec{F_1} \cdot \vec{\nu} = g = \vec{F_2} \cdot \vec{\nu} \quad \partial\Omega$$

Αν $\vec{F} = \vec{F_1} - \vec{F_2}$

τότε έχουμε $\nabla \cdot \vec{F}(x) = 0 \quad x \in \Omega$

$$\nabla \times \vec{F}(x) = 0, \quad x \in \Omega$$
$$\vec{F}(x) \cdot \vec{\nu}(x) = 0, \quad x \in \partial\Omega$$

Επειδή $\nabla \times \vec{F}(x) = 0, \ x \in \Omega \Rightarrow \exists f : \Omega \to \mathbb{R}$ ώστε $\vec{F}(x) = \nabla f(x), \ x \in \Omega$

οπότε από $\nabla \cdot \vec{F}(x) = 0, \ x \in \Omega$

$$\nabla \cdot (\nabla f(x)) = 0, \quad x \in \Omega$$
$$\left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) = 0\right.$$

$$\Delta f(x) = 0 \quad x \in \Omega$$
$$\nabla f(x) \cdot \vec{\nu}(x) = 0, \quad x \in \partial\Omega$$

<u>1η ταυτότητα:</u>

$$\int_\Omega g \overset{0}{\cancel{\Delta f}} \, dx = -\int \nabla g \cdot \nabla f \, dx + \int_{\partial\Omega} g \, \overset{0}{\cancel{\nabla f \, \vec{\nu}}} \, ds$$

Επιλέγουμε $h = f$

οπότε $\int_\Omega \nabla g \cdot \nabla f \, dx = 0, \ \forall g \in C^1(\overline{\Omega})$

Σφαιρικές συντεταγμένες

$$x = \rho \sin\varphi \cos\vartheta \qquad 0 \le \vartheta < 2\pi$$
$$\psi = \rho \sin\varphi \sin\vartheta \qquad 0 \le \varphi \le \pi$$
$$z = \rho\cos\varphi \qquad 0 \le \rho \le 1$$

$$\left|\frac{\partial(x,\psi,z)}{\partial(\rho,\varphi,\vartheta)}\right| = \rho^2 \sin\varphi$$

$$= \int_0^1\int_0^{2\pi}\int_0^{\pi} (2 + 2\rho\sin\varphi\sin\vartheta + 2\rho\cos\varphi)\left|\frac{\partial(x\psi z)}{\partial(\rho,\varphi,\vartheta)}\right| d\varphi\, d\vartheta\, d\rho$$

$$= \int_0^1\int_0^{2\pi}\int_0^{\pi} (2 + 2\rho\sin\varphi\sin\vartheta + 2\rho\cos\varphi)\, \rho^2 \sin\varphi\, d\varphi\, d\vartheta\, d\rho.$$

$$= \int_0^1\int_0^{2\pi}\int_0^{\pi} 2\rho^2 \sin\varphi\, d\varphi\, d\vartheta\, d\rho + 2\iiint \rho^3 \sin^2\varphi \sin\vartheta\, d\varphi\, d\vartheta\, d\rho +$$
$$+ 2\iiint \rho^3 \sin\varphi\cos\varphi\, d\varphi\, d\vartheta\, d\rho$$

$$= 2\cdot 2\pi \cdot \frac{2}{3} = \frac{8\pi}{3}$$

<u>Παράδειγμα</u>: Βρείτε το επιφανειακό ολοκλήρωμα
$$\iint_S \left(2x + (\psi^2+z^2)e^{\psi^3}\right) x\, dS \quad *$$

όπου S η επιφάνεια της μοναδιαίας σφαίρας με κέντρο το O.

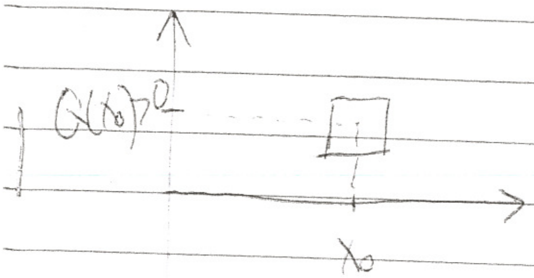
Απ. Με Θ. Gauss:
$$\iiint \nabla\cdot \vec{F}\, dx\, d\psi\, dz = \iint_S \vec{F}\cdot\vec{\nu}\, dS$$

$$* = \iint_S \left(2x + (\psi^2+z^2)e^{\psi^3}\right)\cdot \nu_1\, dS$$

Πότε $Q \in C(\bar{\Omega})$

$\int_\Omega Q(x)\,dx = 0 \Rightarrow Q = 0$

$Q \geq 0$



$\exists\, \delta > 0 : \quad Q(x) \geq \frac{1}{2} Q(x_0)$

$|x - x_0| < \delta$

$$\int_\Omega Q \geq \int_{|x-x_0|<\delta} Q \geq \frac{1}{2} Q(x_0)\,\delta^3 > 0$$

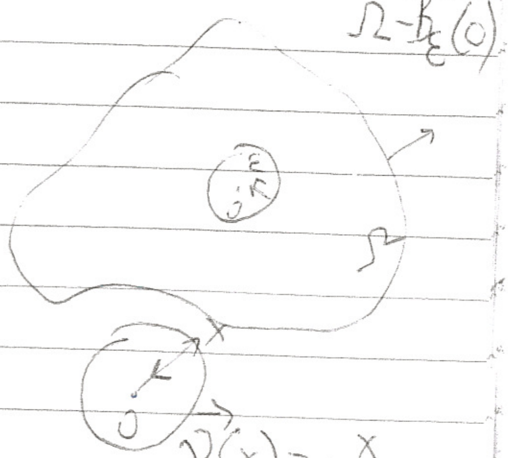
Επιλέγουμε $g = Q(f) \Rightarrow \nabla g = Q(f)\nabla F$

$\Rightarrow \int Q'(f)\,|\nabla f|^2\,dx = 0$

$\Rightarrow \underset{\vec{F}(x)}{\underset{\parallel}{\nabla f(x)}} = \vec{0} \quad \forall x \in \Omega$ άτοπο

Προβλ. Έστω $0 \in \Omega$, φραγμένο στερεό. Αποδείξτε ότι:

$$\iiint_\Omega \frac{dx\,d\psi\,dz}{x^2+\psi^2+z^2} = \iint_{\partial\Omega} \frac{\vec{x \cdot \nu}}{x^2+\psi^2+z^2}\,dS$$



$$\iiint_{\Omega - B_\varepsilon} \nabla \cdot \vec{F}\,dX = \iint_{\partial(\Omega - B_\varepsilon)} \vec{F} \cdot \vec{\nu}\,dS$$

$\vec{\nu}(x) = -\frac{x}{\varepsilon}$

$$\vec{F}(x) = \frac{1}{x^2+\psi^2+z^2} \cdot x = \frac{1}{x^2+\psi^2+z^2}(x,\psi,z) = \left(\frac{x}{x^2+\psi^2+z^2}, \frac{\psi}{x^2+\psi^2+}\right.$$

$$\left.\frac{z}{x^2+\psi^2+z^2}\right)$$

$$\Rightarrow \underbrace{\nabla\cdot\vec{F}}_{\operatorname{div}\vec{F}} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+\psi^2+z^2}\right) + \frac{\partial}{\partial \psi}\left(\frac{\psi}{x^2+\psi^2+z^2}\right) + \frac{\partial}{\partial z}\left(\frac{z}{x^2+\psi^2+\right.$$

$$= \frac{1}{x^2+\psi^2+z^2} - \frac{2x\cdot x}{(x^2+\psi^2+z^2)^2} + \frac{1}{x^2+\psi^2+z^2} - \frac{2\psi\cdot\psi}{(x^2+\psi^2+z^2)^2}$$

$$\frac{1}{x^2+\psi^2+z^2} - \frac{2z\cdot z}{(x^2+\psi^2+z^2)^2} = \frac{3}{x^2+\psi^2+z^2} - \frac{2\cancel{(x^2+\psi^2+z^2)}}{(x^2+\psi^2+z^2)^{\cancel{2}}}$$

$$= \frac{1}{x^2+\psi^2+z^2}, \qquad x \in \Omega - B_\varepsilon$$

$$\iiint_{\Omega - B_\varepsilon} \frac{dx\,d\psi\,dz}{x^2+\psi^2+z^2} = \iint_{\partial(\Omega - B_\partial)} \left(\frac{x}{x^2+\psi^2+z^2}, \frac{\psi}{x^2+\psi^2+z^2}, \frac{z}{x^2+\psi^2}\right.$$

$$= \iint_{\partial\Omega} \left(\frac{x}{x^2+\psi^2+z^2}, \frac{\psi}{x^2+\psi^2+z^2}, \frac{z}{x^2+\psi^2+z^2}\right)\cdot \vec{\nu}\, dS \quad +$$

$$\iint_{\partial B_\varepsilon} \left(\frac{x}{x^2+\psi^2+z^2}, \frac{\psi}{x^2+\psi^2+z^2}, \frac{z}{x^2+\psi^2+z^2}\right)\cdot \vec{\nu}\, dS$$

$$\vec{\nu} = -\frac{1}{\varepsilon}(x,\psi,z)$$

όμως $-\frac{1}{\varepsilon}\iint_{\partial B_\varepsilon} \left(\frac{x}{x^2+\psi^2+z^2}, \frac{\psi}{x^2+\psi^2+z^2}, \frac{z}{x^2+\psi^2+z^2}\right)\cdot(x,\psi,z)\, dS$

$$= -\frac{1}{\varepsilon}\iint_{\partial B_\varepsilon} \frac{x^2+\psi^2+z^2}{x^2+\psi^2+z^2}\, dS = -\frac{1}{\varepsilon}\iint_{\partial B_\varepsilon} dS = -\frac{1}{\varepsilon}\, 4\pi\varepsilon^2 = -4\pi\varepsilon.$$

$$\iiint_{\Omega - B_\varepsilon} \frac{dX}{x^2+\psi^2+z^2} = \iint_{\partial\Omega} \left(\frac{x}{x^2+\psi^2+z^2}, \frac{\psi}{x^2+\psi^2+z^2}, \frac{z}{x^2+\psi^2+z^2}\right)\cdot \vec{\nu}\, dS$$

οπότε

$$\lim_{\varepsilon\to 0^+} \iiint_{\Omega - B_\varepsilon} \frac{dX}{x^2+\psi^2+z^2} = \iint_{\partial\Omega} \left(\frac{x}{|X|^2}, \frac{\psi}{|X|^2}, \frac{z}{|X|^2}\right)\cdot \vec{\nu}\, dS - \lim_{\varepsilon\to 0^+}(4\pi\varepsilon$$

$$\iiint_{\Omega} \frac{dX}{x^2+\psi^2+z^2} = \iint_{\partial\Omega} \left(\frac{x}{|X|^2}, \frac{\psi}{|X|^2}, \frac{z}{|X|^2}\right) \vec{\nu}\, dS \qquad //$$

Helmholtz (Ανάλυση)

$\Omega \subseteq \mathbb{R}^3$, $\vec{F}$

<u>ΘΕΩΡΗΜΑ</u> Έστω Ω ανοικτό, φραγμένο (απλά συνεκτικό) Τότε $\forall \vec{F}$ ομαλό, <u>υπάρχουν μοναδικά</u> διανυσματικά πεδία $\vec{G}, \vec{Q}$ τ.ώ. να ισχύουν:

$$\vec{F}(x) = \vec{G}(x) + \vec{Q}(x), \quad x\in\Omega$$
$$\nabla\times\vec{G}(x) = 0 \quad x\in\Omega$$
$$\nabla\cdot\vec{Q}(x) = 0 \quad x\in\Omega$$
$$\vec{G}(x)\cdot\vec{\nu}(x) = \vec{F}(x)\cdot\vec{\nu}(x), \quad x\in\partial\Omega$$

$$\nabla\times\vec{G} = 0 \Rightarrow \vec{G} = \nabla f$$
$$\vec{F} = \nabla f + \vec{Q}$$
$$\Downarrow$$
$$\nabla\cdot\vec{F} = \nabla\cdot(\nabla f + \vec{Q})$$
$$= \Delta f + 0$$

Οπότε η f

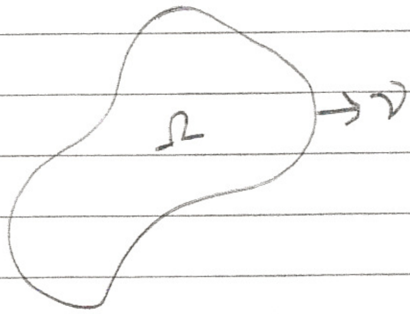
$$\Delta f(x) = \nabla \cdot \vec{F} \quad (\text{Poisson})$$

$$\nabla f(x)\, \vec{\nu}(x) = \vec{F}(x) \cdot \vec{\nu} \quad , x \in \partial\Omega$$

8/12/2016

Ταυτότητες Green

$$\int_\Omega \frac{\partial f}{\partial x} g\, dx = -\int_\Omega f \frac{\partial g}{\partial x} dx + \int_{\partial\Omega} f g \nu_1 ds$$



1η: $$\int_\Omega f \Delta g\, dx = -\int_\Omega \nabla f \cdot \nabla g\, dx + \int_{\partial\Omega} f \nabla g \cdot n\, ds$$

2η: $$\int_\Omega (f\Delta g - g \Delta f)\, dx = \int_{\partial\Omega} (f \nabla g - g \nabla f) \cdot n\, ds$$

$$\frac{\partial g}{\partial \vec{\nu}} =$$

Helmholtz

Έστω $F: \Omega \to \mathbb{R}^3$ (ομαλό)

τότε $\exists\ \vec{G}, \vec{Q}$, $\vec{F} = \vec{G} + \vec{Q}$, τ.ω.

$\nabla \times \vec{G}(x) = \vec{0}$, $x \in \Omega$

$\nabla \cdot \vec{Q}(x) = 0$, $x \in \Omega$

$\vec{F}(x) \cdot \vec{n} = \vec{G}(x) \cdot \vec{n}$, $x \in \partial\Omega$