On a question of Brezis and Marcus

S. Filippas,

Department of Applied Mathematics, University of Crete, 71409 Heraklion, Greece
and Institute of Applied and Computational Mathematics, FORTH, 71110 Heraklion, Greece

V. Maz'ya,

Department of Mathematical Sciences, The University of Liverpool,
M&O Building, Peach Street, Liverpool, L69 72L UK
and Department of Mathematics, Ohio State University, Columbus,OH 43210 USA

& A. Tertikas

Department of Mathematics, University of Crete, 71409 Heraklion, Greece
and Institute of Applied and Computational Mathematics, FORTH, 71110 Heraklion, Greece

December 21, 2004

Abstract

Motivated by a question of Brezis and Marcus, we show that the L^p–Hardy inequality involving the distance to the boundary of a convex domain, can be improved by adding an L^q norm $q \geq p$, with a constant depending on the interior diameter of Ω.

1 Introduction

Recently a lot of attention has been paid to the so called improved Hardy inequalities; see e.g. [BV], [BM], [BFT], [DD], [HHL], [FT],[M], [VZ], [T], and references therein. By "improved" it is meant that one considers a classical Hardy inequality with best constant and adds a positive term in the right hand side, as for instance in (1.1) or (1.2) below. These inequalities play an essential role in the theory of partial differential equations and nonlinear analysis. The are used, for instance, in the study of the stability of solutions of elliptic and parabolic equations (cf [BV], [PV]), as well as in the study of existence and asymptotic behavior of solutions of heat equations with singular potentials; see e.g. [CM], [DD], [GP], [VZ].

Multidimensional inequalities of this kind first appeared in [M, sec. 2.1.6] where functions defined in the whole space $I\!R^n$ were considered. More recently, Brezis and Marcus [BM] showed that if Ω is a bounded convex domain in $I\!R^n$ and $d(x) = \text{dist}(x, \partial\Omega)$ then for $u \in C_0^\infty(\Omega)$,

$$\int_\Omega |\nabla u|^2 dx - \frac{1}{4}\int_\Omega \frac{u^2}{d^2} dx \geq \frac{1}{4\text{diam}^2(\Omega)}\int_\Omega u^2 dx, \tag{1.1}$$

and they asked whether the constant in the right hand side can be replaced by one depending only on the volume of Ω. This question was answered in affirmative in [HHL] for $p = 2$ and later in [T] for any $p > 1$. The constant obtained in these two papers has the form $C = c(p, n)\,(vol(\Omega))^{-p/n}$, where $c(p, n)$ is an explicitly given constant independent of the domain Ω.

The main goal of the present work is to study the dependence of the best constant $C(\Omega)$ in the inequality

$$\int_\Omega |\nabla u|^p dx - \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u|^p}{d^p} dx \geq C \left(\int_\Omega |u|^q d^\alpha dx\right)^{\frac{p}{q}}, \tag{1.2}$$

$u \in C_0^\infty(\Omega)$, on the domain Ω. We establish that $C(\Omega)$ depends on Ω through its interior diameter $\mathrm{D}_{int} := 2\sup_{x\in\Omega} d(x)$.

In case $\alpha = 0$, we have

Theorem 1.1 *Suppose $\Omega \subset \mathbb{R}^n$ is a convex domain with D_{int} finite. For $1 < p < n$ and $p \leq q < \frac{np}{n-p}$, let $C(\Omega)$ be the best constant in the inequality*

$$\int_\Omega |\nabla u|^p dx - \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u|^p}{d^p} dx \geq C(\Omega) \left(\int_\Omega |u|^q dx\right)^{\frac{p}{q}}, \qquad u \in W_0^{1,p}(\Omega). \tag{1.3}$$

Then, there exist positive constants $c_i = c_i(p,q,n),\ i = 1,2$ independent of Ω, such that

$$c_1(p,q,n)\mathrm{D}_{int}^{n-p-\frac{np}{q}} \geq C(\Omega) \ \geq\ c_2(p,q,n)\mathrm{D}_{int}^{n-p-\frac{np}{q}}. \tag{1.4}$$

This kind of dependence of the best constant appears for example when estimating the first eigenvalue $\lambda_1(\Omega)$ of the *p–Laplacian* under the Dirichlet boundary conditions,

$$\int_\Omega |\nabla u|^p dx \ \geq\ \lambda_1(\Omega) \int_\Omega |u|^p dx\ . \tag{1.5}$$

In particular if Ω is convex with D_{int} finite, then (see [PS] section 5.11 for $p = 2$ and in [M] Theorem 11.4.1 on page 434 for the general case) there are positive constants $c_i(p,n),\ i = 1,2$ independent of Ω, such that

$$c_1(p,n)\mathrm{D}_{int}^{-p} \geq \lambda_1(\Omega) \ \geq\ c_2(p,n)\mathrm{D}_{int}^{-p}. \tag{1.6}$$

For $p = q = 2$ our lower bound for $C(\Omega)$ in (1.4) is $3D_{int}^{-2}$. Needless to say, it is better than the bound $4^{-1}(\mathrm{diam}(\Omega))^{-2}$ in (1.1) of Brezis and Marcus [BM]. Moreover, since $3D_{int}^{-2} \geq 3/4(v_n/vol(\Omega))^{2/n}$, where v_n is the volume of the unit ball, our lower bound on $C(\Omega)$ implies a bound in terms of of $vol(\Omega)$ as in [HHL]. In particular our estimate is stronger than that in [HHL] in the two or three dimensional case. However these estimates do not imply each other for $n > 3$.

The Sobolev exponent $q = \frac{np}{n-p}$ is not allowed in (1.3) since our proof fails. For some results in this case we refer to [FMT1,2].

We actually establish lower bounds of the best constant in inequality (1.2) for a suitable range of the parameters p, q, α. These results are formulated in Theorems 3.1 and 3.2 of Section 3. Theorem 3.1 deals with the special case $p = q = 2$, which is particularly simple and allows for the calculation of an explicit lower bound of the best constant in (1.2). In Theorem 3.2 then we consider the general case. In Section 3 we also have the proof of Theorem 1.1. An auxiliary estimate is presented in Section 2.

2 Preliminaries

Here we will present an auxiliary estimate. Let $X(t) = (1 - \log t)^{-1}$, for $t \in [0,1]$. $X(t)$ is an increasing function with $X(0) = 0$ and $X(1) = 1$. In the sequel we will write X instead of $X(\frac{d(x)}{\mathrm{R}_{int}})$, where $\mathrm{R}_{int} = \sup_{x\in\Omega} d(x)$ is the interior radius of Ω.

The Proposition that follows has been proved in [BFT], but we include its proof here for completeness. The proof we present is slightly simpler than in [BFT].

Proposition 2.1 *Let $\Omega \subset I\!\!R^n$ be a convex domain. For $u \in C_0^\infty(\Omega)$ we set $u(x) = v(x)d^{\frac{p-1}{p}}$.*

(i) If $1 < p < 2$ then

$$\int_\Omega |\nabla u|^p dx - \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u|^p}{d^p} dx \geq$$
$$\geq c(p,n) \left[\int_\Omega d^{p-1} X^{2-p} |\nabla v|^p \, dx + \int_\Omega (-\Delta d)|v|^p \, dx \right]. \tag{2.1}$$

(ii) If $p \geq 2$ then

$$\int_\Omega |\nabla u|^p dx - \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u|^p}{d^p} dx \geq c(p,n) \left[\int_\Omega d^{p-1} |\nabla v|^p dx \int_\Omega (-\Delta d)|v|^p dx \right]. \tag{2.2}$$

Proof: We first consider the case $p \geq 2$ which is easier. For $p \geq 2$ we will use the following pointwise inequality valid for any a, $b \in I\!\!R^n$,

$$|a+b|^p - |a|^p \geq c(p,n)|b|^p + p|a|^{p-2} a \cdot b \, . \tag{2.3}$$

We have that

$$\nabla u = \frac{p-1}{p} d^{-\frac{1}{p}} v \nabla d + d^{\frac{p-1}{p}} \nabla v =: a + b, \qquad \frac{p-1}{p}\frac{|u|}{d} = \frac{p-1}{p}|v| d^{-\frac{1}{p}} = |a|. \tag{2.4}$$

Using (2.3) we obtain

$$\int_\Omega |\nabla u|^p dx - \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u|^p}{d^p} dx \geq c(p,n) \left[\int_\Omega d^{p-1} |\nabla v|^p dx + \int_\Omega \nabla d \cdot \nabla |v|^p dx \right]. \tag{2.5}$$

For any domain Ω, the distance function is a Lipschitz continuous function and therefore differentiable a.e.. Moreover, if Ω is convex, then $-d(x)$, $x \in \Omega$ is a convex function. It then follows that $-\Delta d(x)$ is a nonnegative Radon measure, see e.g., [EG, sec. 6.3]. That is

$$\int_\Omega \nabla d \cdot \nabla \phi dx = \int_\Omega \phi d\mu, \qquad \phi \in C_0^1(\Omega), \tag{2.6}$$

with $d\mu \geq 0$. For convenience we will write $(-\Delta d(x))\, dx$ in the place of $d\mu$, and in this sense, integration by parts is permissible in the left hand side of (2.6).

Integrating by parts the last term in (2.5) we obtain (2.2).

Next we consider the case $1 < p < 2$. In this case, the following pointwise inequality is true for a, $b \in I\!\!R^n$,

$$|a+b|^p - |a|^p \geq c(p,n) \frac{|b|^2}{(|a|+|b|)^{2-p}} + p|a|^{p-2} a \cdot b. \tag{2.7}$$

In view of (2.4) we have that

$$\int_\Omega |\nabla u|^p dx - \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u|^p}{d^p} dx \geq$$
$$c(p,n) \left[\int_\Omega \frac{d|\nabla v|^2}{(|v| + |d\nabla v|)^{2-p}} dx + \int_\Omega (-\Delta d)|v|^p dx \right]. \tag{2.8}$$

To simplify the subsequent calculations we set

$$A_1 := \int_\Omega \frac{d|\nabla v|^2}{\left(|v| + |d\nabla v|\right)^{2-p}} dx, \quad A_2 := \int_\Omega d^{-1} X^2 |v|^p dx,$$

$$A_3 := \int_\Omega d^{p-1} X^{2-p} |\nabla v|^p dx, \quad A_4 := \int_\Omega (-\Delta d)|v|^p dx.$$

Taking into account (2.8) we need to show that for some constant c depending only on p, n there holds

$$(A_3 + A_4) \leq c(A_1 + A_4). \tag{2.9}$$

To this end, using elementary inequalities we have

$$\begin{aligned} A_3 &= \int_\Omega \frac{d^{\frac{p}{2}} |\nabla v|^p}{(|v| + |d\nabla v|)^{p(2-p)/2}} \cdot (|v| + |d\nabla v|)^{p(2-p)/2} \, d^{\frac{p-2}{2}} X^{2-p} dx \\ &\leq A_1^{p/2} \left(\int_\Omega d^{-1} X^2 \, (|v| + |d\nabla v|)^p \, dx \right)^{(2-p)/2} \\ &\leq c\, A_1^{p/2} \left(\int_\Omega d^{-1} X^2 |v|^p dx + \int_\Omega d^{p-1} X^{2-p} |\nabla v|^p dx \right)^{(2-p)/2} \\ &\leq c\, A_1^{p/2} A_2^{(2-p)/2} + c\, A_1^{p/2} A_3^{(2-p)/2} \\ &\leq c_\varepsilon A_1 + \varepsilon A_2 + \varepsilon A_3 + c_\varepsilon A_1, \end{aligned}$$

where ε is small and the constant c_ε depends only on ε and p. Hence,

$$(1 - \varepsilon) A_3 \leq 2 c_\varepsilon A_1 + \varepsilon A_2. \tag{2.10}$$

We will also use the estimate

$$A_2 \leq c_0(p, n)(A_3 + A_4). \tag{2.11}$$

If we accept this we get from (2.10) that

$$(1 - \varepsilon - \varepsilon c_0) A_3 \leq 2 c_\varepsilon A_1 + \varepsilon c_0 A_4,$$

from which (2.9) follows.

It remains to prove (2.11). Using the fact that $\nabla d \cdot \nabla d = 1$ a.e. and noticing that $\nabla d \cdot \nabla X = d^{-1} X^2 \nabla d \cdot \nabla d = d^{-1} X^2$ we integrate by parts to get

$$\begin{aligned} A_2 &= -\int_\Omega X \operatorname{div}(\nabla d |v|^p) dx \\ &\leq p \int_\Omega X |v|^{p-1} |\nabla v| dx + \int_\Omega (-\Delta d) X |v|^p \\ &\leq p \left(\int_\Omega d^{-1} X^2 |v|^p dx \right)^{\frac{p-1}{p}} \left(\int_\Omega d^{-1} X^2 |\nabla v|^p dx \right)^{\frac{1}{p}} + \int_\Omega (-\Delta d) X |v|^p \\ &\leq p A_2^{\frac{p-1}{p}} A_3^{\frac{1}{p}} + A_4 \\ &\leq p \varepsilon A_2 + p C_\varepsilon A_3 + A_4, \end{aligned}$$

form which (2.11) follows. The proof of the Proposition is now complete. □

3 Main Theorems and proofs

We first consider the special case $p = q = 2$. We have

Theorem 3.1 *If $\Omega \subset I\!R^n$ is a convex domain then for any $\alpha > -2$ and all $u \in H_0^1(\Omega)$,*

$$\int_\Omega |\nabla u|^2 dx - \frac{1}{4}\int_\Omega \frac{u^2}{d^2} dx \geq C_\alpha \, \mathrm{D}_{int}^{-(\alpha+2)} \int_\Omega u^2 d^\alpha dx, \tag{3.1}$$

with $C_\alpha = 2^\alpha(\alpha+2)^2$ if $-2 < \alpha < -1$ and $C_\alpha = 2^\alpha(2\alpha+3)$ if $\alpha \geq -1$.

Proof: We will prove the result for $u \in C_0^\infty(\Omega)$, the general case following by a density argument.

Using the change of variables $u(x) = d^{\frac{1}{2}}(x)v(x)$ we have

$$\int_\Omega u^2 d^\alpha dx = \int_\Omega v^2 d^{\alpha+1} dx, \tag{3.2}$$

and

$$\int_\Omega |\nabla u|^2 dx - \frac{1}{4}\int_\Omega \frac{u^2}{d^2} dx = \int_\Omega d|\nabla v|^2 dx + \frac{1}{2}\int_\Omega (-\Delta d)|v|^2 dx. \tag{3.3}$$

Using the fact that $|\nabla d(x)| = 1$ a.e and integrating by parts we get

$$\begin{aligned}
\int_\Omega d^{\alpha+1} v^2 dx &= \frac{1}{\alpha+2}\int_\Omega \nabla d^{\alpha+2}\cdot \nabla d \; v^2 dx = -\frac{1}{\alpha+2}\int_\Omega d^{\alpha+2}\mathrm{div}(\nabla d \; v^2) dx \\
&= -\frac{2}{\alpha+2}\int_\Omega d^{\alpha+2}|v|\nabla d\cdot\nabla v dx + \frac{1}{\alpha+2}\int_\Omega d^{\alpha+2}(-\Delta d)v^2 dx.
\end{aligned}$$

Using elementary inequalities we have

$$\begin{aligned}
(\alpha+2)\int_\Omega d^{\alpha+1}v^2 dx &\leq 2\left(\int_\Omega d^{\alpha+1}v^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega d^{\alpha+3}|\nabla v|^2\right)^{\frac{1}{2}} + \mathrm{R}_{int}^{\alpha+2}\int_\Omega(-\Delta d)v^2 dx \\
&\leq \delta\int_\Omega d^{\alpha+1}v^2 dx + \delta^{-1}\int_\Omega d^{\alpha+3}|\nabla v|^2 + \mathrm{R}_{int}^{\alpha+2}\int_\Omega(-\Delta d)v^2 dx \\
&\leq \delta\int_\Omega d^{\alpha+1}v^2 dx + 2\mathrm{R}_{int}^{\alpha+2}\left(\frac{1}{2\delta}\int_\Omega d|\nabla v|^2 + \frac{1}{2}\int_\Omega(-\Delta d)v^2 dx\right).
\end{aligned}$$

Hence, we have

$$(\alpha+2-\delta)\int_\Omega d^{\alpha+1}v^2 dx \leq 2\mathrm{R}_{int}^{\alpha+2}\left(\frac{1}{2\delta}\int_\Omega d|\nabla v|^2 + \frac{1}{2}\int_\Omega(-\Delta d)v^2 dx\right). \tag{3.4}$$

We next choose $\delta = \min\{\frac{1}{2}, \frac{\alpha+2}{2}\}$ and recall that $\mathrm{D}_{int} = 2\mathrm{R}_{int}$. The result then follows taking into account (3.2) and (3.3). □

We next consider the general case.

Theorem 3.2 *Let $\Omega \subset I\!R^n$ be a convex domain. Then for any $u \in W_0^{1,p}(\Omega)$ we have*

$$\int_\Omega |\nabla u|^p dx - \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u|^p}{d^p} dx \geq c\,\mathrm{D}_{int}^{-\left(\frac{p(\alpha+n)}{q}-n+p\right)}\left(\int_\Omega d^\alpha |u|^q dx\right)^{\frac{p}{q}}, \tag{3.5}$$

with $c = c(p,q,n,\alpha) > 0$ a constant independent of Ω and

$$1 < p \leq q \leq \frac{np}{n-p}, \qquad n > p, \qquad \alpha > \frac{q}{p}(n-p) - n. \tag{3.6}$$

If $p = q$ then $n = p$ is allowed.

Proof: By standard density arguments it is enough to prove (3.5) for $u \in C_0^\infty(\Omega)$.

We first consider the case $1 < p < 2$. Using the change of variables $u(x) = v(x)d^{\frac{p-1}{p}}$ we have by Proposition 2.1 that

$$\int_\Omega |\nabla u|^p dx - \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u|^p}{d^p} dx \geq \tag{3.7}$$
$$\geq c(p,n)\left(\int_\Omega d^{p-1}X^{2-p}|\nabla v|^p dx + \int_\Omega (-\Delta d)|v|^p dx\right).$$

On the other hand

$$\int_\Omega d^\alpha |u|^q dx = \int_\Omega d^{\alpha+q-\frac{q}{p}}|v|^q dx. \tag{3.8}$$

For simplicity we set

$$A = \alpha + q - \frac{q}{p}. \tag{3.9}$$

Let $\{Q_m\}$, $m = 1, 2, \ldots$, be a covering of Ω by Whitney cubes (see [S, chapter VI, sec. 1]). In particular each side of the cube Q_m has length d_m such that $c_0 d_m \leq d(x) \leq c_1 d_m$ and

$$c_0' X\left(\frac{d_m}{\mathrm{R}_{in}}\right) \leq X\left(\frac{d(x)}{\mathrm{R}_{in}}\right) \leq c_1' X\left(\frac{d_m}{\mathrm{R}_{in}}\right),$$

for any $x \in Q_m$ and any $m = 1, 2\ldots$, where c_0, c_1, c_0', c_1' are universal constants. Then, for a universal constant c we have

$$\left(\int_\Omega d^A|v|^q dx\right)^{\frac{p}{q}} \leq c\left(\sum_m \int_{Q_m} d_m^A |v|^q dx\right)^{\frac{p}{q}} \leq c\sum_m d_m^{\frac{pA}{q}}\left(\int_{Q_m}|v|^q dx\right)^{\frac{p}{q}}. \tag{3.10}$$

From now on we denote by c a positive constant, not necessarily the same in each occurrence, that may depend only on n, p, q or α. Using Sobolev's inequality in Q_m for functions defined in $W^{1,p}(Q_m)$, we have

$$c\left(\int_{Q_m}|v|^q dx\right)^{\frac{p}{q}} \leq d_m^{\frac{np}{q}-n+p}\int_{Q_m}|\nabla v|^p dx + d_m^{\frac{np}{q}-n}\int_{Q_m}|v|^p dx.$$

Then it follows

$$\begin{aligned} c\left(\int_{Q_m}|v|^q dx\right)^{\frac{p}{q}} &\leq d_m^{\frac{np}{q}-n+1}X^{p-2}\left(\frac{d_m}{\mathrm{R}_{int}}\right)\int_{Q_m} d^{p-1}X^{2-p}|\nabla v|^p dx \\ &\quad + d_m^{\frac{np}{q}-n}X^{p-2}\left(\frac{d_m}{\mathrm{R}_{int}}\right)\int_{Q_m} X^{2-p}|v|^p dx. \end{aligned}$$

Combining this with (3.10) we get

$$\begin{aligned} \left(\int_\Omega d^A|v|^q dx\right)^{\frac{p}{q}} &\leq c\sum_m \Bigg[d_m^{\frac{(A+n)p}{q}-n+1}X^{p-2}\left(\frac{d_m}{\mathrm{R}_{int}}\right)\int_{Q_m} d^{p-1}X^{2-p}|\nabla v|^p dx \\ &\quad + d_m^{\frac{(A+n)p}{q}-n}X^{p-2}\left(\frac{d_m}{\mathrm{R}_{int}}\right)\int_{Q_m} X^{2-p}|v|^p dx\Bigg]. \end{aligned} \tag{3.11}$$

For the first term in the bracket in (3.11), noting that

$$\frac{(A+n)p}{q} - n + 1 = \frac{(\alpha+n)p}{q} - n + p > 0, \tag{3.12}$$

we use the estimate

$$
\begin{aligned}
d_m^{\frac{(A+n)p}{q}-n+1} X^{p-2}\left(\frac{d_m}{\mathrm{R}_{int}}\right) &\leq \max_{0\leq t\leq 1}\{t^{\frac{(A+n)p}{q}-n+1} X^{p-2}(t)\}\ \mathrm{R}_{int}^{\frac{(A+n)p}{q}-n+1} \\
&= c\ \mathrm{R}_{int}^{\frac{(A+n)p}{q}-n+1}. \qquad (3.13)
\end{aligned}
$$

To estimate the second term in the brackets in (3.11) we notice that, by (3.12) there exists an $\varepsilon = \varepsilon(p,q,n,\alpha) > 0$ such that $\frac{(A+n)p}{q} - n - \varepsilon > -1$. Then we have

$$
\begin{aligned}
d_m^{\frac{(A+n)p}{q}-n} X^{p-2}\left(\frac{d_m}{\mathrm{R}_{int}}\right)\int_{Q_m} X^{2-p}|v|^p dx \ &\leq \\
&\leq c d_m^{\varepsilon} X^{p-2}\left(\frac{d_m}{\mathrm{R}_{int}}\right)\int_{Q_m} d^{\frac{(A+n)p}{q}-n-\varepsilon} X^{2-p}|v|^p dx \\
&\leq c\ \mathrm{R}_{int}^{\varepsilon}\int_{Q_m} d^{\frac{(A+n)p}{q}-n-\varepsilon} X^{2-p}|v|^p dx. \qquad (3.14)
\end{aligned}
$$

Combining (3.11), (3.13) and (3.14) we get

$$
\begin{aligned}
\left(\int_\Omega d^A|v|^q dx\right)^{\frac{p}{q}} &\leq c\mathrm{R}_{int}^{\frac{(A+n)p}{q}-n+1}\int_\Omega d^{p-1}X^{2-p}|\nabla v|^p dx \\
&\quad + c\ \mathrm{R}_{int}^{\varepsilon}\int_\Omega d^{\frac{(A+n)p}{q}-n-\varepsilon} X^{2-p}|v|^p dx. \qquad (3.15)
\end{aligned}
$$

To continue we will estimate the last term in (3.15). For simplicity we set

$$
\theta := \frac{(A+n)p}{q} - n - \varepsilon + 1 > 0. \qquad (3.16)
$$

Using the fact that $\nabla d \cdot \nabla d = 1$ a.e. and integrating by parts we have

$$
\begin{aligned}
\int_\Omega d^{\theta-1}X^{2-p}|v|^p dx &= \theta^{-1}\int_\Omega \nabla d^\theta \cdot \nabla d\, X^{2-p}|v|^p dx \\
&= -\theta^{-1}\int_\Omega d^\theta \operatorname{div}(\nabla d\, X^{2-p}\,|v|^p)dx \\
&= \theta^{-1}\int_\Omega d^\theta(-\Delta d)X^{2-p}|v|^p dx - (2-p)\theta^{-1}\int_\Omega d^{\theta-1}X^{3-p}|v|^p dx \\
&\quad -p\theta^{-1}\int_\Omega d^\theta X^{2-p}|v|^{p-1}\nabla d\cdot\nabla|v| dx \\
&\leq \theta^{-1}\int_\Omega d^\theta(-\Delta d)X^{2-p}|v|^p dx + p\theta^{-1}\int_\Omega d^\theta X^{2-p}|v|^{p-1}|\nabla v| dx. \qquad (3.17)
\end{aligned}
$$

Using Hölder's inequality for the last term in (3.17) we have

$$
\begin{aligned}
\int_\Omega d^\theta X^{2-p}|v|^{p-1}|\nabla v| dx &\leq \left(\int_\Omega d^{\theta-1}X^{2-p}|v|^p dx\right)^{\frac{p-1}{p}}\left(\int_\Omega d^{\theta-1+p}X^{2-p}|\nabla v|^p dx\right)^{\frac{1}{p}} \\
&\leq \delta\int_\Omega d^{\theta-1}X^{2-p}|v|^p dx + c_\delta\int_\Omega d^{\theta-1+p}X^{2-p}|\nabla v|^p dx.
\end{aligned}
$$

Combining this with (3.17) we easily arrive at

$$\begin{aligned} c\int_\Omega d^{\theta-1}X^{2-p}|v|^p dx &\leq \int_\Omega d^{\theta-1+p}X^{2-p}|\nabla v|^p dx + \int_\Omega d^\theta(-\Delta d)X^{2-p}|v|^p dx \\ &\leq \mathrm{R}_{int}^\theta\left[\int_\Omega d^{p-1}X^{2-p}|\nabla v|^p dx + \int_\Omega(-\Delta d)|v|^p dx\right]. \end{aligned}$$

This is the sought for estimate for the last term in (3.15); see (3.16) for the value of θ. Using this estimate in (3.15) we conclude

$$\left(\int_\Omega d^A|v|^q dx\right)^{\frac{p}{q}} \leq c\,\mathrm{R}_{int}^{\frac{(A+n)p}{q}-n+1}\left[\int_\Omega d^{p-1}X^{2-p}|\nabla v|^p dx + \int_\Omega(-\Delta d)|v|^p dx\right]. \tag{3.18}$$

From this and (3.7)–(3.9) the result follows. The case $1 < p < 2$ has been proved.

The case $p \geq 2$ is similar but simpler since no logarithmic corrections are involved in this case. We will therefore sketch it.

For $u(x) = v(x)d^{\frac{p-1}{p}}$ we have by Proposition 2.1 that

$$\begin{aligned} &\int_\Omega |\nabla u|^p dx - \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u|^p}{d^p}dx \geq \\ &\qquad \geq c(p,n)\left(\int_\Omega d^{p-1}|\nabla v|^p dx + \int_\Omega(-\Delta d)|v|^p dx\right). \end{aligned} \tag{3.19}$$

The L^q–integral is again given by (3.8)–(3.9).

By the same covering argument as before and the fact that $d(x) \leq \mathrm{R}_{in}$ we obtain the analogue of (3.15) which is

$$\left(\int_\Omega d^A|v|^q dx\right)^{\frac{p}{q}} \leq c\mathrm{R}_{int}^{\frac{(A+n)p}{q}-n+1}\int_\Omega d^{p-1}|\nabla v|^p dx + c\int_\Omega d^{\frac{(A+n)p}{q}-n}|v|^p dx. \tag{3.20}$$

We note that (3.20) is trivially true in the case $p = q = n$.

As before, we will estimate the last term in (3.20). For simplicity we now set

$$\theta := \frac{(A+n)p}{q} - n + 1 > 0. \tag{3.21}$$

Using the identity $d^{\theta-1} = \theta^{-1}\nabla d^\theta \cdot \nabla d$ and integration by parts we have

$$\begin{aligned} \int_\Omega d^{\theta-1}|v|^p dx &= -\theta^{-1}\int_\Omega d^\theta\,\mathrm{div}(\nabla d\,|v|^p)dx \\ &\leq \theta^{-1}\int_\Omega d^\theta(-\Delta d)|v|^p dx + p\theta^{-1}\int_\Omega d^\theta|v|^{p-1}|\nabla v|dx. \end{aligned} \tag{3.22}$$

The last term above is estimated using Hölder's inequality to get

$$\int_\Omega d^\theta|v|^{p-1}|\nabla v|dx \leq \delta\int_\Omega d^{\theta-1}|v|^p dx + c_\delta\int_\Omega d^{\theta-1+p}|\nabla v|^p dx.$$

Combining this with (3.22) we obtain

$$\int_\Omega d^{\frac{(A+n)p}{q}-n}|v|^p dx \leq c\,\mathrm{R}_{int}^{\frac{(A+n)p}{q}-n+1}\left(\int_\Omega d^{p-1}|\nabla v|^p dx + \int_\Omega(-\Delta d)|v|^p dx\right). \tag{3.23}$$

The result follows by (3.8)–(3.9) and (3.19). □

Proof of Theorem 1.1: The lower bound of $C(\Omega)$ comes from Theorems 3.1, 3.2. The upper bound is a consequence of the corresponding upper bound for the best constant $c_{p,q}(\Omega)$ in

$$\int_\Omega |\nabla u|^p dx \ \geq \ c_{p,q}(\Omega) \left(\int_\Omega |u|^q dx \right)^{\frac{p}{q}}, \tag{3.24}$$

for $u \in W_0^{1,p}(\Omega)$ and $1 < p \leq q < \frac{np}{n-p}$. In particular if B_{int} is the ball of maximum interior diameter, we have that $c_{p,q}(\Omega) \leq c_{p,q}(B_{int})$ and then the result follows.

Acknowledgments This work started when SF visited the University of Liverpool, the hospitality of which is acknowledged. SF and AT acknowledge partial support by the RTN European network Fronts–Singularities, HPRN-CT-2002-00274. SF acknowledges additional support by RTN European network HYKE, HPRN-CT-2002-00282. AT acknowledges the hospitality and support of Rutgers University.

References

[BFT] Barbatis G., Filippas S., and Tertikas A. A unified approach to improved L^p Hardy inequalities with best constants. Trans. Amer. Math. Soc. 356(6) (2004) 2169–2196.

[BM] Brezis H. and Marcus M. Hardy's inequalities revisited. Ann. Scuola Norm. Pisa 25 (1997) 217-237.

[BV] Brezis H. and Vázquez J.-L. Blow-up solutions of some nonlinear elliptic problems. Rev. Mat. Univ. Comp. Madrid **10** (1997) 443-469.

[CM] Cabré X. and Martel Y. Existence versus instantaneous blowup for linear heat equations with singular potentials. C.R. Acad. Sci. Paris Ser. I Math. **329** (1999) 973-978

[DD] Dávila, J. and Dupaigne, L. Hardy-type inequalities. J. Eur. Math. Soc. 6(3) (2004) 335–365.

[EG] Evans L. C. and Gariepy R. F., Measure Theory and Fine Properties of Functions. Studies in Advanced Math., CRC Press, 1992.

[FMT1] Filippas S., Maz'ya V. and Tertikas A. A sharp Hardy Sobolev inequality. C. R. Acad. Paris Ser. I, 339, (2004), 483–486.

[FMT2] Filippas S., Maz'ya V. and Tertikas A. Critical Hardy Sobolev Inequalities. In preparation.

[FT] Filippas S. and Tertikas A. Optimizing Improved Hardy inequalities. J. Funct. Anal. 192 (2002) 186–233.

[GP] Garcia J.P. and Peral I. Hardy inequalities and some critical elliptic and parabolic problems. J. Diff. Equations **144** (1998) 441-476.

[HHL] Hoffmann-Ostenhof M., Hoffmann-Ostenhof T. and Laptev A. A geometrical version of Hardy's inequality. J. Funct. Anal. 189, (2002), 539–548.

[M] Maz'ya V., Sobolev spaces. Springer 1985.

[PS] Pólya, G. and Szegö, G. Isoperimetric Inequalities in Mathematical Physics. Annals of Mathematics Studies, no. 27, Princeton University Press, Princeton, N. J., 1951.

[PV] Peral I. and Vázquez J.L. On the stability or instability of the singular solution of the semilinear heat equation with exponential reaction term. Arch. Rational Mech. Anal. 129 (1995) 201-224.

[S] Stein E. M., Singular Integrals and Differentiability Properties of Functions, Princeton Univ. Press, 1970.

[T] Tidblom J., A geometrical version of Hardy's inequality for $W_0^{1,p}(\Omega)$, Proc. Amer. Math. Soc. 132(8) (2004), 2265–2271.

[VZ] J. L. Vázquez, and E. Zuazua, The Hardy Inequality and the Asymptotic Behaviour of the Heat Equation with an Inverse-Square Potential, J. Funct. Anal. 173 (2000) 103–153.