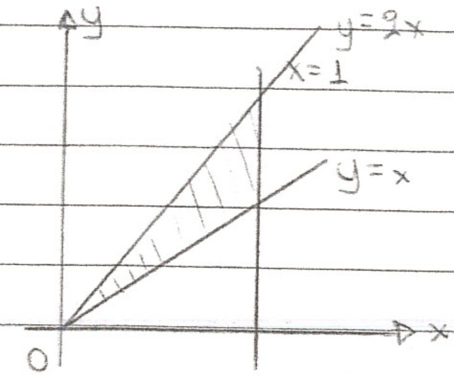
Ασκησεις ΑΠ ΙΙΙ 27/9/16

Ασκηση 1

$$\iint_D (x+y)\,dx\,dy$$

ωλ 1ο



Επιλέγω την x ως πρώτη μεταβλητή.

Άρα $\iint_D f = \int_0^1\left(\int_x^{2x}(x+y)\,dy\right)dx$ ①

D = { f(x,y) | y=x, y=2x, x=1 }

Για να υπολογίσω το $\int_x^{2x}(x+y)\,dy$ θεωρώ το x ως μια σταθερά και ολοκληρώνω ως προς y, άρα

$$\int_x^{2x}(x+y)\,dy = \left[xy+\frac{y^2}{2}\right]_x^{2x} = 2x^2 + \frac{4x^2}{2} - x^2 - \frac{x^2}{2} = \frac{5x^2}{2}$$

Άρα η ① → $\int_0^1 \frac{5x^2}{2}\,dx = \frac{5}{2}\left[\frac{x^3}{3}\right]_0^1 = \frac{5}{2}\cdot\frac{1}{3} = \frac{5}{6}$

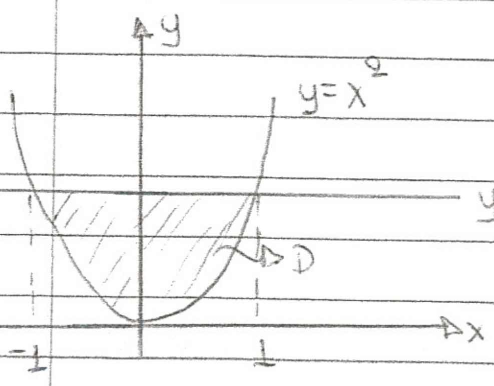
Ασκηση 2

$$\iint_D e^x\,dx\,dy \quad \text{με } D = \{\, y=x^2,\ y=1 \mid f(x,y) \,\}$$

Επιλέγω την x ως πρώτη μεταβλητή

Άρα,

$$\int\left(\int e^x\,dy\right)dx = \int_{-1}^{1}\left(\int_1^{x^2} e^x\,dy\right)dx \ ①$$



$\left.\begin{array}{l} y=1 \\ y=x^2 \end{array}\right\} \Rightarrow 1=x^2 \Leftrightarrow x=\pm 1$

Για να υπολογίσω το $\int^{x^2} e^x\,dy$ θεωρώ το x σταθερά και ολοκληρώνω ως προς y $\int_1^{x^2} e^x \, dy = \left[e^x y\right]_1^{x^2} =$

$= e^x x^2 - e^x = e^x(x^2-1)$

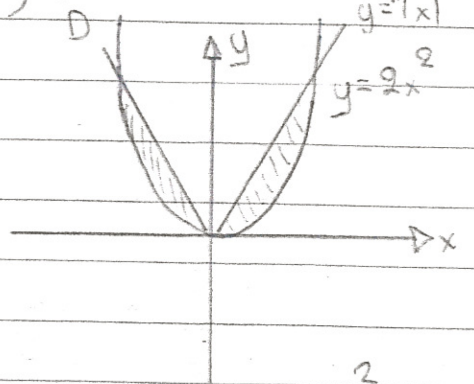
Άρα η ① $\rightarrow \int_{-1}^{1} e^x(x^2-1)\,dx = \cancel{\left[e^x(x^2-1)\right]_{-1}^{1}}^{\,0} - \int_{-1}^{1} e^x 2x =$

$= -\left[e^x 2x\right]_{-1}^{1} - \left(-\int_{-1}^{1} e^x \cdot 2\,dx\right) = \left[-e^x 2x\right]_{-1}^{1} + 2\left[e^x\right]_{-1}^{1} =$

$= -\cancel{2e} + 2e^{-1} + \cancel{2e} + 2e^{-1} = 4e^{-1} = \frac{4}{e}$

Άσκηση 3

$\iint_D (\cos x + \sin x)\,dx\,dy$ με $D = \{P(x,y) \mid y = 2x^2,\ y = \pi|x|\}$



Επιλέγω την x ως πρώτη μεταβλητή

Λύνω το σύστημα: $\left.\begin{array}{l} y = \pi|x| \\ y = 2x^2 \end{array}\right\} \iff$

$2x^2 = \pi|x| \iff x^2 = \frac{\pi|x|}{2} \iff$

$x^2 - \frac{\pi}{2}|x| = 0 \iff \left\{\begin{array}{l} x^2 - \frac{\pi}{2}x = 0 \\ x^2 + \frac{\pi}{2}x = 0 \end{array}\right. \iff$

$\left.\begin{array}{l} x(x-\pi/2) = 0 \\ x(x+\pi/2) = 0 \end{array}\right\} \iff \begin{array}{l} x = 0 \text{ ή } x = \pi/2 \\ x = 0 \text{ ή } x = -\pi/2 \end{array}$

Άρα προκύπτουν 2 ολοκληρώματα

Άρα $\int_0^{\pi/2}\left(\int_{2x^2}^{\pi x} (\cos x + \sin x)\,dy\right) dx$ ①

Για να υπολογίσω το $\int_{2x^2}^{\pi x} (\cos x + \sin x)\,dy$ θεωρώ το x σταθερό και ολοκληρώνω ως προς y.

ολοκληρώνω ως προς y $\int_1^{x^2} e^x \, dy = \left[e^x y\right]_1^{x^2} =$

$= e^x x^2 - e^x = e^x(x^2-1)$

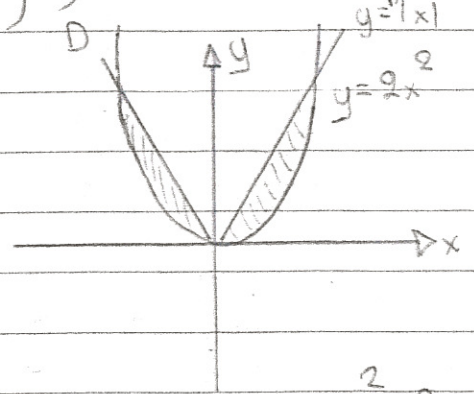
Άρα η ① → $\int_{-1}^{1} e^x(x^2-1)\,dx = \left[e^x \cancelto{0}{(x^2-1)}\right]_{-1}^{1} - \int_{-1}^{1} e^x 2x =$

$= -\left[e^x 2x\right]_{-1}^{1} - \left(-\int_{-1}^{1} e^x \cdot 2\,dx\right) = \left[-e^x 2x\right]_{-1}^{1} + 2\left[e^x\right]_{-1}^{1} =$

$= -\cancel{2e} + 2e^{-1} + \cancel{2e} + 2e^{-1} = 4e^{-1} = \frac{4}{e}$

Άσκηση 3

$\iint_D (\cos x + \sin x)\,dx\,dy$ με $D = \{f(x,y) \mid y = 2x^2,\ y = \pi|x|\}$



Επιλέγω την x ως πρώτη μεταβλητή

Λύνω το σύστημα: $\left.\begin{array}{l} y = \pi|x| \\ y = 2x^2 \end{array}\right\} \Leftrightarrow$

$2x^2 = \pi|x| \Leftrightarrow x^2 = \frac{\pi|x|}{2} \Leftrightarrow$

$x^2 - \frac{\pi}{2}|x| = 0 \Leftrightarrow \left\{\begin{array}{l} x^2 - \frac{\pi}{2}x = 0 \\ x^2 + \frac{\pi}{2}x = 0 \end{array}\right. \Leftrightarrow$

$\left.\begin{array}{l} x(x-\pi/2) = 0 \\ x(x+\pi/2) = 0 \end{array}\right\} \Leftrightarrow$ $x=0$ ή $x=\pi/2$

$x=0$ ή $x=-\pi/2$

Άρα προκύπτουν 2 ολοκληρώματα

Άρα $\int_0^{\pi/2}\left(\int_{2x^2}^{\pi x}(\cos x + \sin x)\,dy\right)dx$ ①

Για να υπολογίσω το $\int_{2x^2}^{\pi x}(\cos x + \sin x)\,dy$ θεωρώ το x σταθερό και ολοκληρώνω ως προς y.

Άρα έχω: $\int_{-\pi/2}^{\pi/2}\left(\int_{2x^2}^{\pi|x|}(\cos x+\sin x)\,dy\right)dx = \int_{-\pi/2}^{\pi/2}(\cos x+\sin x)(\pi|x|-2x^2)\,dx$

$= \int_{-\pi/2}^{\pi/2} \cos x\,\pi|x| - 2x^2\cos x + \sin \pi|x| - 2x^2 \sin x\,)\,dx =$

$= \int_{-\pi/2}^{\pi/2} \cos x\,\pi|x|\,dx - \int_{-\pi/2}^{\pi/2} 2x^2 \cos x\,dx + \int_{-\pi/2} \sin \pi|x|\,dx -$

$- \int_{-\pi/2}^{\pi/2} 2x^2 \sin x\,dx = \int_{-\pi/2}^{0} -\cos x\,\pi x\,dx + \int_{0}^{\pi/2} \cos x\,\pi x\,dx -$

$- \int_{-\pi/2}^{\pi/2} 2x^2 \cos x\,dx + \int_{-\pi/2}^{0} -\sin x\,\pi x\,dx + \int_{0}^{\pi/2} \sin x\,\pi x\,dx -$

$- \int_{\pi/2}^{\pi/2} 2x^2 \sin x\,dx$ (1)

Υπολογίζω ένα-ένα τα ολοκληρώματα:

▶ $\int_{\pi/2}^{0} -\cos x\,\pi x = \left[-\sin x\,\pi x\right]_{-\pi/2}^{0} + \int_{-\pi/2}^{0} \pi \sin x\,dx =$

$\frac{\pi^2}{2} + \left[-\pi\cos x\right]_{-\pi/2}^{0} = \frac{\pi^2}{2} - \pi$

▶ $\int_{0}^{\pi/2} \cos x\,\pi x\,dx = \left[\sin x\,\pi x\right]_{0}^{\pi/2} - \int_{0}^{\pi/2} \pi \sin x\,dx =$

$\frac{\pi^2}{2} + \left[-\pi\cos x\right]_{0}^{\pi/2} = \frac{\pi^2}{2} + \pi$

▶ $\int_{-\pi/2}^{\pi/2} 2x^2 \cos x\,dx = \left[\sin x\,2x^2\right]_{-\pi/2}^{\pi/2} - \int_{-\pi/2}^{\pi/2} 4x \sin x\,dx =$

$= \frac{\pi^2}{2} + \frac{\pi^2}{2} - 4\left(\left[x\cos x\right]_{-\pi/2}^{\pi/2} + \int_{-\pi/2}^{\pi/2} \cos x\,dx\right) =$

$\pi^2 - 4\left[\sin x\right]_{-\pi/2}^{\pi/2} = \pi^2 - 4(1+1) = \pi^2 - 8.$

▶ $\int_{-\pi/2}^{0} -\sin x\,\pi x\,dx = \left[\cos x\,\pi x\right]_{-\pi/2}^{0} - \int_{-\pi/2}^{0} \cos x\cdot\pi\,dx = -\pi\left[\sin x\right]_{-\pi/2}^{0} = -\pi$

▸ $\int_0^{\pi/2} \sin x \, \pi x \, dx = [-\cos x \, \pi x]_0^{\pi/2} - \int_0^{\pi/2} \pi \cos x \, dx =$

$= -\pi \, [\sin x]_0^{\pi/2} = -\pi$

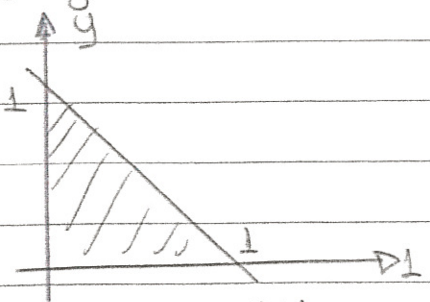
▸ $\int_{-\pi/2}^{\pi/2} 2x^2 \sin x \, dx = [-2x^2 \cos x]_{-\pi/2}^{\pi/2} - \int_{-\pi/2}^{\pi/2} -4x \cos x \, dx =$

$= [\sin x \, 4x]_{-\pi/2}^{\pi/2} - \int_{-\pi/2}^{\pi/2} 4x \cos x \, dx = 2\pi - 2\pi + 4 (\cos x)_{-\pi/2}^{\pi/2} = 0$

Άρα: $\cancel{\frac{\pi^2}{2}} - \cancel{\pi} + \cancel{\frac{\pi^2}{2}} + \cancel{\pi} - \pi^2 + 8 - \cancel{\pi} - \pi - 0 = 8 - 2\pi$

Άσκηση 4

$\int_0^1 \int_0^{1-x} e^{(1-y)^2} \, dy \, dx$ $\qquad 0 \le y \le -x+1$, $\quad 0 \le x \le 1$



$0 \le y \le 1$

$y = 1-x$

$\int_0^1 \int_0^{1-y} e^{(1-y)^2} \, dx \, dy$ $\qquad y = 1-x$, $\quad x = 1-y$

$\int_0^1 e^{(1-y)^2} \int_0^{1-y} dx \, dy = \int_0^1 e^{(1-y)^2} (1-y) \, dy =$

$\int_0^1 e^{x^2} x \, dx = \frac{x^2}{2} e^{x^2} - \int_0^1 \frac{x^2}{2} e^{x^2} 2x \, dx =$

$\left[\frac{x^2}{2} e^{x^2}\right]_0^1 - 0 = \frac{1}{2} e - 0 = \frac{e}{2}$ ✓

$= \left[-\frac{e^{(1-y)^2}}{2}\right]_0^1 = -\frac{e^0}{2} + \frac{e}{2} = -\frac{1}{2} + \frac{e}{2} = \frac{e-1}{2}$ ✓

Κατεύθυνση: Μαθηματική

Μαρία Κοτρώτσιου math5286 3/10/16

Άσκηση 7

Ασκ 2η Έστω D το χωρίο $D = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le x^2\}$ και

Τ ο μετασχηματισμός $T(x,y) = (x+y,\ x-y)$

α) Νδο η Τ είναι "1-1"

Έστω ότι Τ είναι "1-1". Θα πρέπει να ισχύει:

$T((x_1,y_1)) = T((x_2,y_2)) \Leftrightarrow (x_1+y_1,\ x_1-y_1) = (x_2+y_2,\ x_2-y_2) \Leftrightarrow$

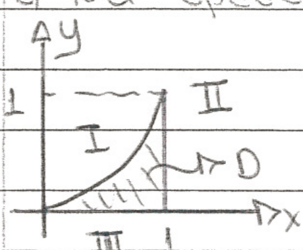
$$\left.\begin{array}{l} x_1+y_1 = x_2+y_2 \\ x_1-y_1 = x_2-y_2 \end{array}\right\} \Rightarrow \left.\begin{array}{l} x_1 = x_2+y_2-y_1 \\ x_1 = y_1+x_2-y_2 \end{array}\right\} \Rightarrow$$

$\cancel{x_2}+y_2-y_1 = y_1+\cancel{x_2}-y_2 \Leftrightarrow \cancel{2}y_2 = \cancel{2}y_1 \Leftrightarrow \boxed{y_2 = y_1}$ ①

Άρα $x_1 = \cancel{y_1}+x_2-\cancel{y_1} \xRightarrow{\text{από ①}} \Leftrightarrow \boxed{x_1 = x_2}$ ②

Άρα από ①, ② → $T((x_1,y_1)) = T((x_2,y_2)) \Leftrightarrow$ Η Τ είναι "1-1". ■

β) Να βρεθεί και να σχεδιαστεί το χωρίο T(D).



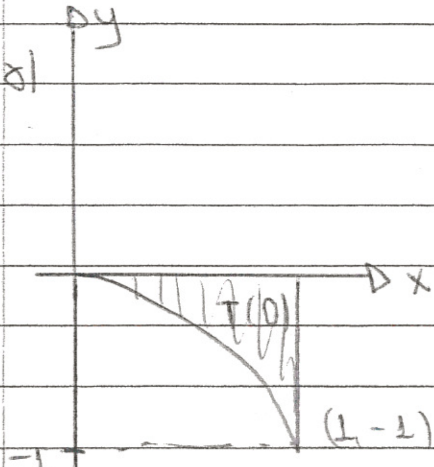
$\mathrm{I}: (t,t^2) \xrightarrow{T} (t,0) = (0,0) + t(1,0)$

$\mathrm{II}: (1,t) \xrightarrow{T} (1,\ t-1) = (1,-1) + t(0,1)$

$\mathrm{III}: (t,0) \xrightarrow{T} (t,\ -t^2)$

$T(D) = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le -x^2\}$

γ)



$x = u$

$y - x^2 = v \Rightarrow y = v + x^2$

Τελικά $T^{-1}(D) = (x,y) =$

$= (x,\ y+x^2)$

Για να υπολογίσω το $\int_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} 1 \, dx$ ολοκληρώνω ως προς x και θεωρώ τα z και y σταθερές.

Άρα, $\left[x\right]_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} = 2\sqrt{z^2-y^2}$

Άρα η (1) ⇝ $\int_0^2 \left(\int_{-z}^{z} 2\sqrt{z^2-y^2}\, dy \right) dz = \int_0^2 \left(2 \int_{-z}^{z} \sqrt{z^2-y^2}\, dy \right) dz$

Για να υπολογίσω το $\int_{-z}^{z} \sqrt{z^2-y^2}\, dy$ θεωρώ το z ως σταθερό και ολοκληρώνω ως προς y.

$dy = z\cos t \, dt$

Θέτω $y = z\sin t$ επειδή ισχύει $\cos^2 t + \sin^2 t = 1 \Leftrightarrow$

$z^2 = z^2\cos^2 t + z^2 \sin^2 t \Leftrightarrow$

$z^2 \cos^2 t = z^2 - z^2 \sin^2 t$

Άρα έχω: $\int_{-\pi/2}^{\pi/2} z \sqrt{z^2 - z^2\sin^2 t}\, dt =$

$= z \int_{-\pi/2}^{\pi/2} |z| \sqrt{(1-\sin^2 t)} \cos t \, dt = z^2 \int_{-\pi/2}^{\pi/2} \sqrt{\cos^2 t} \cos t \, dt =$

$= z^2 \int_{-\pi/2}^{\pi/2} \frac{1+\cos 2t}{2} = z^2 \int_{-\pi/2}^{\pi/2} \frac{1}{2} dt + \frac{z^2}{2} \int_{-\pi/2}^{\pi/2} \cos 2t \, dt =$

$= \frac{z^2}{2} \left[t \right]_{-\pi/2}^{\pi/2} + \frac{z^2}{2} \left[\overset{-1}{\cos \pi} - \overset{-1}{\cos(-\pi)} \right] =$

$= \frac{z^2}{2}\pi + \frac{z^2}{1} \overset{0}{\cancel{(-1-(-1))}} = \frac{z^2 \pi}{2}$

Άρα η (2) ⇝ $2 \int_0^2 z^2 \frac{\pi}{2} dz = \frac{\cancel{2}\pi}{\cancel{2}} \int_0^2 z^2 dz = \frac{\cancel{2}\pi}{\cancel{2}} \left[\frac{z^3}{3} \right]_0^2 =$

$= \frac{\pi \cancel{2}}{\cancel{2}} \frac{2^3}{3} - \frac{\pi}{\cancel{2}} \cancel{0} = \frac{2^3 \pi}{3} = \frac{8\pi}{3}$

Τώρα θέτω $x = \sin t$ Για $x=1$: $t = \pi/2$

$dx = \cos t \, dt$ Για $x=-1$: $t = -\pi/2$

Άρα η ① $\rightsquigarrow$ $4\int_{-\pi/2}^{\pi/2} \sqrt{1-\sin^2 t}\,\cos t\, dt - \frac{4}{3}\int_{-\pi/2}^{\pi/2} \left(\sqrt{1-\sin^2 t}\right)^3 \cos t \, dt -$

$- 4\int_{-\pi/2}^{\pi/2} \sin^2 t \sqrt{1-\sin^2 t}\, \cos t \, dt =$

$= 4\int_{-\pi/2}^{\pi/2} \cos^2 t \, dt - \frac{4}{3}\int_{-\pi/2}^{\pi/2} \cos^4 t \, dt - 4\int_{-\pi/2}^{\pi/2} \sin^2 t \cos^2 t \, dt =$

$= 4\int_{-\pi/2}^{\pi/2} \frac{1+\cos 2t}{2} \; (\alpha) - \frac{4}{3}\int_{-\pi/2}^{\pi/2} \frac{1}{8}\cos 4t + \frac{1}{2}\cos 2t + \frac{3}{8}\, dt \; (\beta) - 4\int_{-\pi/2}^{\pi/2} (\sin t \cos t)^2 \, dt \; (\gamma)$

Το α $\rightsquigarrow$ • $4\int_{-\pi/2}^{\pi/2} \frac{1}{2}\, dt + 4\int_{-\pi/2}^{\pi/2} \frac{\cos 2t}{2}\, dt = \frac{4}{2}\left[t\right]_{-\pi/2}^{\pi/2} + \frac{\cancel{4}}{\cancel{2}}\left[\frac{\sin 2t}{\cancel{2}}\right]_{-\pi/2}^{\pi/2} =$

$= 2\left[\frac{\pi}{2} + \frac{\pi}{2}\right] + \sin\frac{2\pi}{2} - \sin\left(-\frac{2\pi}{2}\right) = 2\pi + \cancel{\sin\pi}^{0} - \cancel{\sin(-\pi)}^{0} =$

$= 2\pi$

Το β $\rightsquigarrow$ • $-\frac{\cancel{4}}{3}\frac{1}{\cancel{8}_2}\int_{-\pi/2}^{\pi/2} \cos 4t \, dt - \frac{4}{3}\frac{1}{2}\int_{-\pi/2}^{\pi/2} \cos 2t \, dt - \frac{\cancel{4}}{\cancel{3}}\frac{\cancel{3}}{8}\int_{-\pi/2}^{\pi/2} dt =$

$= -\frac{1}{6}\left[\frac{\sin 4t}{4}\right]_{-\pi/2}^{\pi/2} - \frac{4}{6}\left[\frac{\sin 2t}{2}\right]_{-\pi/2}^{\pi/2} - \frac{1}{2}\left[t\right]_{-\pi/2}^{\pi/2} =$

$= -\frac{1}{24}\left(\cancel{\sin 2\pi - \sin(-2\pi)}^{0}\right) - \frac{4}{12}\left(\cancel{\sin\pi - \sin(-\pi)}^{0}\right) - \frac{1}{2}\left(\frac{\pi}{2} + \frac{\pi}{2}\right) =$

$= -\frac{1}{24}\cdot 0 - \frac{4}{12}\cdot 0 - \frac{1}{2}\frac{\cancel{2}\pi}{\cancel{2}} = -\frac{\pi}{2}$

Το γ $\rightsquigarrow$ • $-4\int_{-\pi/2}^{\pi/2} (\sin t \cos t)^2 \, dt = -2\int (2\sin t \cos t)(\sin t \cos t)\, dt =$

$= -\int (\sin 2t)(\sin 2t)\, dt = -\int_{-\pi/2}^{\pi/2} (\sin 2t)^2 \, dt =$

$= -\left[-\frac{\cos 2t}{2}(\sin 2t)\right]_{-\pi/2}^{\pi/2} + \int_{-\pi/2}^{\pi/2} -\frac{\cos 2t}{\cancel{2}}\cos 2t \, \cancel{2}\, dt =$

$$= -\left[-\frac{\pi}{2} \cos\frac{\pi}{2} \cancel{(\sin \pi)} + \frac{\cos(-\pi)}{2} \cancel{\sin(-\pi)} \right] - \int_{-\pi/2}^{\pi/2} (\cos 2t)^2 \, dt =$$

$$= \int_{-\pi/2}^{\pi/2} (\cos 2t)^2 \, dt = -\cancel{\left[\frac{\sin 2t}{2} \cos 2t \right]_{-\pi/2}^{\pi/2}} + \int_{-\pi/2}^{\pi/2} \frac{\sin(2t)}{2} (-\sin 2 \ldots$$

$$= - \int_{-\pi/2}^{\pi/2} (\sin 2t)^2 \, dt$$

Άρα $-4\int_{-\pi/2}^{\pi/2} (\sin t \cos t)^2 \, dt = -\frac{\pi}{2}$

Οπότε η ⑤ ⇒ $2\pi - \frac{\pi}{2} - \frac{\pi}{2} = \frac{2\pi - \cancel{2}\pi}{2} = \pi$

Άρα ο ζητούμενος όγκος ισούται με π

Άσκηση 4 $D = \{ (x, y, z) : 0 \leq x \leq 1, \ 0 \leq y \leq 1, \ 0 \leq z \leq xy \}$

α) Ο όγκος του D είναι $\int_0^1 \int_0^1 \int_0^{xy} 1 \, dz) dy) dx =$

$$= \int_0^1 \left(\int_0^1 [z]_0^{xy} dy \right) dx = \int_0^1 \left(\int_0^1 xy \, dy \right) dx =$$

$$= \int_0^1 \left(x \left[\frac{y^2}{2} \right]_0^1 \right) dx = \int_0^1 x \frac{1}{2} \, dx = \frac{1}{2} \int_0^1 \left(\frac{x^2}{2} \right) dx =$$

$$= \frac{1}{2} \left[\frac{x^2}{2} \right]_0^1 = \frac{1}{2} \frac{1}{2} = \frac{1}{4}$$

β) $\int_0^1 \left(\int_0^1 \left(\int_0^{xy} x \, dz \right) dy \right) dx = \int_0^1 \left(\int_0^1 x [z]_0^{xy} \right) dy) dx$

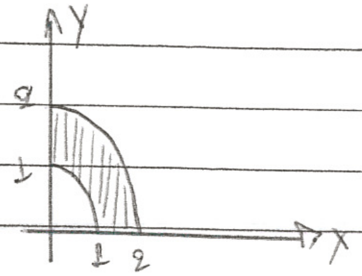
$$= \int_0^1 \left(\int_0^1 x^2 y \, dy \right) dx = \int_0^1 \left(x^2 \left[\frac{y^2}{2} \right]_0^1 \right) dx = \int_0^1 \frac{x^2}{2} \, dx$$

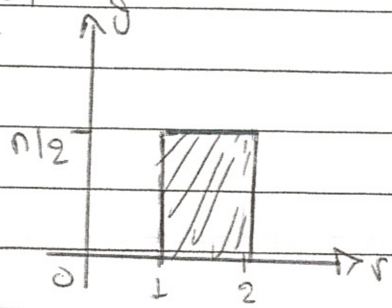
$$= \frac{1}{2} \int_0^1 x^2 \, dx = \frac{1}{2} \left[\frac{x^3}{3} \right]_0^1 = \frac{1}{2} \frac{1}{3} = \frac{1}{6}$$

Άσκηση 9

$D = \{ 1 \le x^2 + y^2 \le 2^2 , x \geq 0 , y \geq 0 \}$

$T(r,\theta) = (r\cos\theta, r\sin\theta)$





$D^* = \{ (r,\theta) \mid 1 \le r \le 2 , 0 \le \theta \le \pi/2 \}$

$T(r_1,\theta_1) = T(r_2,\theta_2) \Rightarrow (r_1\cos\theta_1, r_1\sin\theta_1) = (r_2\cos\theta_2, r_2\sin\theta_2)$

$\Rightarrow r_1\cos\theta_1 = r_2\cos\theta_2$ (1)

$r_1\sin\theta_1 = r_2\sin\theta_2$ (2)

$\theta \in [0,\pi/2]$: $\frac{(2)}{(1)} \Rightarrow \frac{\sin\theta_1}{\cos\theta_1} = \frac{\sin\theta_2}{\cos\theta_2} \Rightarrow \tan\theta_1 = \tan\theta_2$

$\Rightarrow \theta_1 = \theta_2$ Άρα και $r_1 = r_2$

Αν $\theta_1 = \theta_2 = \pi/2$: $r_1\cos\frac{\pi}{2} = r_2\cos\frac{\pi}{2} \Rightarrow 0=0$ που ισχύει

$r_1\sin\pi/2 = r_2\sin\pi/2 \Rightarrow r_1 = r_2$

Αν $\theta_1 = \frac{\pi}{2}$, $\theta_2 \in [0,\pi/2)$: Αν $r_1 \neq r_2$ τότε:

$r_1\cos\frac{\pi}{2} = 0 \neq r_2\cos\theta_2$ καθώς $r_2\cos\theta_2 > 0$. Άρα για

$(r_1,\theta_1) \neq (r_2,\theta_2) \Rightarrow r_1\cos\theta_1 \neq r_2\cos\theta_2 \Rightarrow T(r_1,\theta_1) \neq T(r_2,\theta_2)$

Όμοια αν $\theta_1 \in [0,\pi/2)$ και $\theta_2 = \pi/2$

Επομένως ο μετασχηματισμός T είναι "1-1" στο $[0,\pi/2]$.

Άσκηση 6

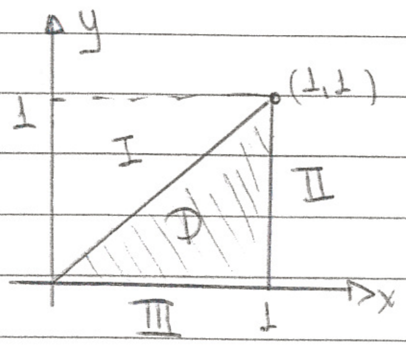
$D = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le x\}$ $T(x,y) = (x+y,\ x-y)$

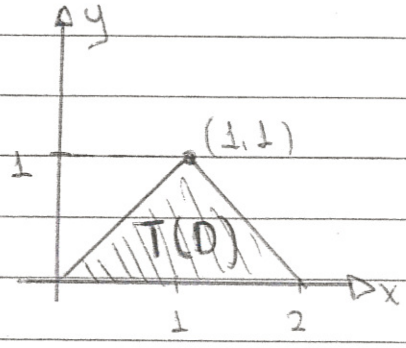
α) $T(x_1,y_1) = T(x_2,y_2) \Rightarrow (x_1+y_1,\ x_1-y_1) = (x_2+y_2,\ -y_2+x_2) \Rightarrow$

$$\left.\begin{array}{l} x_1+y_1 = x_2+y_2 \\ x_1-y_1 = x_2-y_2 \end{array}\right\} \Rightarrow \left.\begin{array}{l} y_1+\cancel{x_2}-y_2+y_1 = \cancel{x_2}+y_2 \\ x_1 = y_1+x_2-y_2 \end{array}\right\} \Rightarrow \cancel{2}y_1 = \cancel{2}y_2$$

$\Rightarrow y_1 = y_2$ Άρα και $x_1 = x_2$ Άρα ο T είναι "1-1".

β)





$\mathrm{I}: (t,t) \xrightarrow{T} (2t,0) = (0,0) + t(2,0)$

$\mathrm{II}: (1,t) \xrightarrow{T} (1+t,\ 1-t) = (1,1) + t(1,-1)$

$\mathrm{III}: (t,0) \xrightarrow{T} (t,t) = (0,0) + t(1,1)$

$T(D) = \{(x,y) \mid 0 \le x \le 2,\ x \le y \le 2-x\}$

γ) $u = x+y \Rightarrow x = u-y \Rightarrow x = u-x+v \Rightarrow x = \frac{u+v}{2}$

$x-y = v \Rightarrow y = x-v$

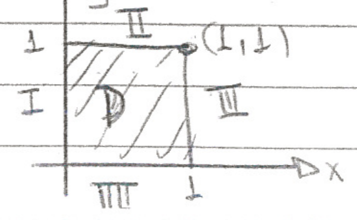
Άρα $y = \frac{u+v}{2} - v \Rightarrow y = \frac{u-v}{2}$

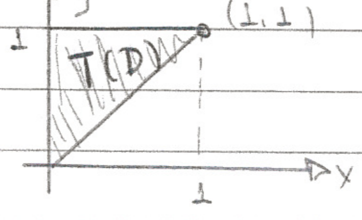
Τελικά $T^{-1}(x,y) = \left(\frac{x+y}{2},\ \frac{x-y}{2}\right)$

Άσκηση 8

$D = [0,1] \times [0,1]$ με $T(x,y) = (xy,\ y) = (u,v)$

α)





$I: (0,t) \xrightarrow{T} (0,t)$

$II: (t,1) \xrightarrow{T} (t,1)$

$III: (1,t) \xrightarrow{T} (t,t) = (0,0) + t(1,1)$

$IIII: (t,0) \xrightarrow{T} (0,0)$

$T(D) = \{ (x,y) \mid 0 \leq x \leq 1, \ x \leq y \leq 1 \}$

β) Αν $y \in (0,1]$: $T(x_1, y_1) = T(x_2, y_2) \Rightarrow$

$(x_1 y_1, y_1) = (x_2 y_2, y_2) \Rightarrow \left.\begin{matrix} x_1 \cancel{y_1} = x_2 \cancel{y_2} \\ y_1 = y_2 \end{matrix}\right\} \Rightarrow \begin{matrix} x_1 = x_2 \\ y_1 = y_2 \end{matrix}$

Αν $y=0$ τότε αναγκαστικά $x=0$: διότι $T(0,0) \neq (0,0)$

Άρα ο T είναι "1-1"

Άσκηση 2 Να υπολογίσω το:

$x+y=1$

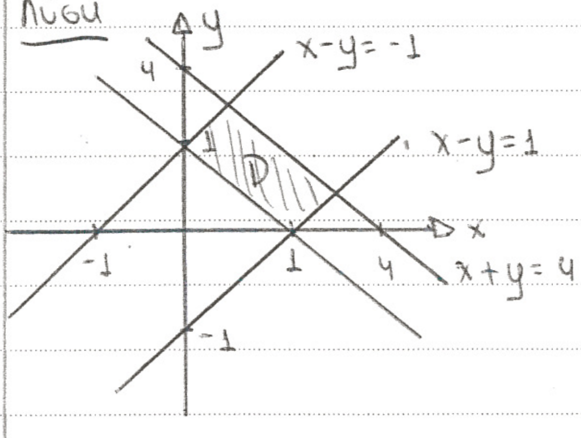
$x+y=+4$

$x-y=-1$

$x-y=1$

$$\iint_D (x+y)^2 e^{x-y}\, dx dy \quad (2)$$ με τις κατάλληλες αλλαγές συντεταγμένων.

Λύση



Θέτω $\left.\begin{array}{l} u = x+y \\ v = x-y \end{array}\right\} \Rightarrow$

(1) $\left.\begin{array}{l} x = v+y \\ x = u-y \end{array}\right\} \Rightarrow v+y = u-y \Rightarrow$

$2y = u-v \Rightarrow y = \frac{u-v}{2} \Rightarrow$

$$\boxed{y = \frac{u}{2} - \frac{v}{2}}$$

Άρα η (1) $\Rightarrow x = v + \frac{u}{2} - \frac{v}{2} \Rightarrow$

$$\boxed{x = \frac{u}{2} + \frac{v}{2}}$$

Άρα η (2) $\Rightarrow \iint_{D^*} u^2 e^v \left| \frac{d(x,y)}{d(u,v)} \right| du\, dv$

Η ιακωβιανή του μετασχηματισμού είναι $\begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{vmatrix} =$

$$= \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{2} - \frac{1}{2} = -\frac{2}{4} = -\frac{1}{2}$$

Οπότε έχω: $\displaystyle\int_1^4 \left(\int_{-1}^{1} \frac{1}{2} u^2 e^v dv \right) du = \int_1^4 \frac{1}{2} u^2 [e^v]_{-1}^{1}\, du =$

$$\int_1^4 \frac{1}{2}u^2(e^1-e^{-1})\,du = \frac{1}{2}(e^1-e^{-1})\left[\frac{u^3}{3}\right]_1^4 =$$

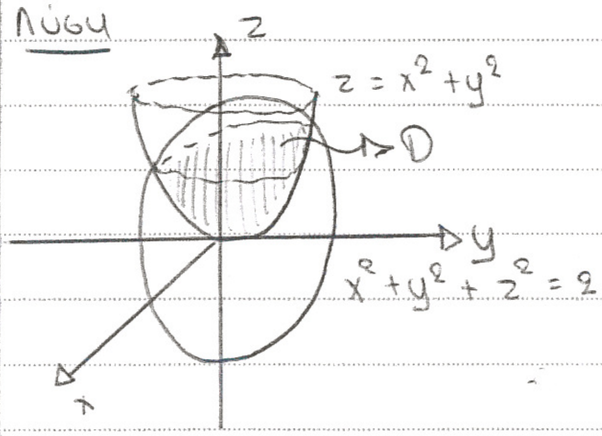
$$= \frac{1}{2}(e^1-e^{-1})\,\frac{64-1}{3} = \frac{63}{6}(e^1-e^{-1}) = \frac{21}{2}(e-e^{-1}).\ \blacksquare$$

Άσκηση 5

$z = x^2+y^2$ Να βρω τον όγκο ανάμεσα στις

$x^2+y^2+z^2=2$ επιφάνειες

Λύση



Αρχικά λύνω το σύστημα για να βρω κοινά σημεία

$z^2+z-2=0 \Leftrightarrow$

$(z-1)(z+2)=0 \Leftrightarrow$

$z=1$ ή $z=-2$

δεκτή — απορρίπτεται

Άρα για $z=1$ έχω $1=x^2+y^2$ (μοναδιαίος κύκλος)

Οπότε $\iiint_\Omega dx\,dy\,dz = \iint_D \left(\int_{x^2+y^2}^{\sqrt{2-x^2-y^2}} dt\right)dx\,dy =$

$$= \iint_D \sqrt{2-x^2-y^2} - x^2 - y^2\ dx\,dy \quad ①$$

Τώρα κάνω αλλαγή σε πολικές συντεταγμένες και θέτω $x = r\cos\theta$ με $0 \le r \le 1$

$y = r\sin\theta$ με $0 \le \theta \le 2\pi$

Άρα η ① $\Rightarrow \iint_{D^*}\left(\sqrt{2-r^2} - r^2\right) r\,dr\,d\theta =$

$$u_1^2 - u_2^2 = \frac{u_2^2 v_2^2}{u_1^2} - v_2^2 \Rightarrow u_1^2 - u_2^2 = v_2^2\left(\frac{u_2^2}{u_1^2} - 1\right) \Rightarrow$$

$$\Rightarrow u_1^2 - u_2^2 = v_2^2\left(\frac{u_2^2 - u_1^2}{u_1^2}\right) \Rightarrow u_1^2 - u_2^2 = -\frac{v_2^2}{u_1^2}(u_1^2 - u_2^2)$$

Προφανώς $\frac{v_2^2}{u_1^2} \neq 0$ καθώς $u_1 \in [1,2]$ και $v \in [1,3]$

Άρα $u_1^2 - u_2^2 = 0$ ή $u_1^2 - u_2^2 \neq 0$

- Αν $u_1^2 - u_2^2 = 0$: $u_1^2 = u_2^2 \Rightarrow u_1 = u_2$ για $u_1 \in [1,2]$

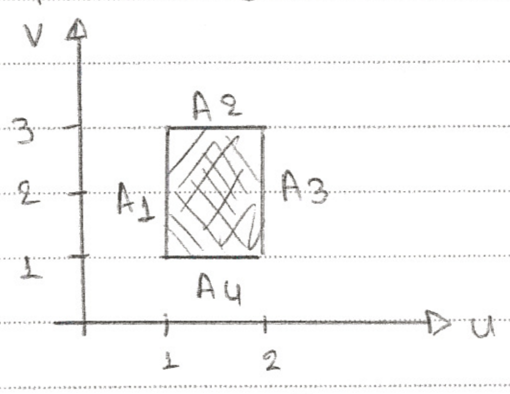
  $v_1 = \frac{\cancel{u_2} v_2}{\cancel{u_1}} \xrightarrow[u_1 = u_2]{} v_1 = v_2$

- Αν $u_1^2 - u_2^2 \neq 0$: $u_1^2 - u_2^2 = -\frac{v_2^2}{u_1^2}(u_1^2 - u_2^2) \Rightarrow$

  $1 = -\frac{v_2^2}{u_1^2} \Rightarrow u_1^2 = -v_2^2$ άτοπο

Άρα η T είναι "1-1"

β)

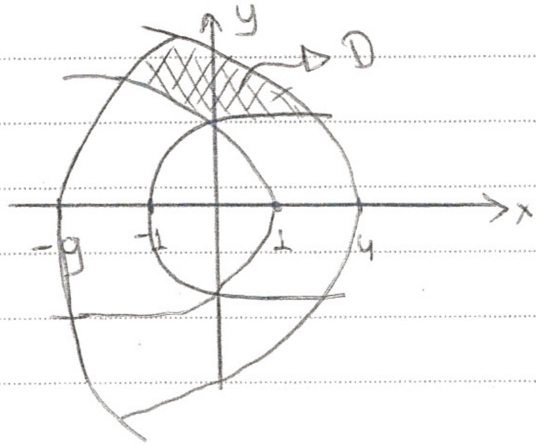


$A_1 : (1, t) \xrightarrow{T} (1 - t^2, 2t)$

$A_2 : (t, 3) \xrightarrow{T} (t^2 - 9, 6t)$

$A_3 : (2, t) \xrightarrow{T} (4 - t^2, 4t)$

$A_4 : (t, 1) \xrightarrow{T} (t^2 - 1, 2t)$



Κορυφές

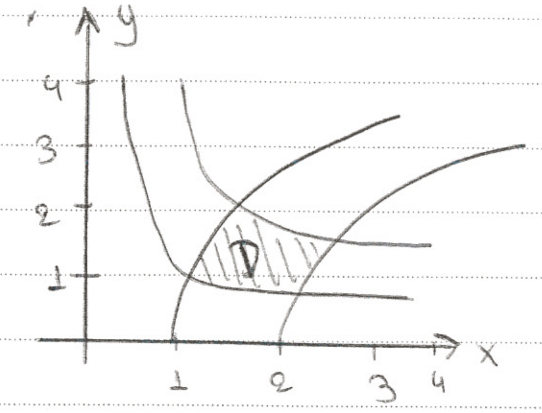
$(1,1) \xrightarrow{T} (0,2)$

$(1,3) \xrightarrow{T} (-8,6)$

$(2,1) \xrightarrow{T} (3,4)$

$(2,3) \xrightarrow{T} (-5,12)$

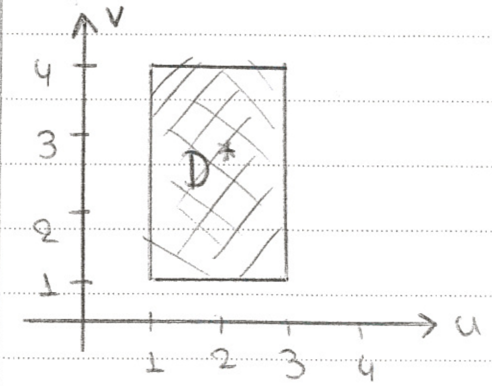
Άσκηση 8



Έστω $T(x,y) = (xy, x^2-y^2) = (u,v)$

Άρα το χωρίο D^* που προκύπτει από το μετασχηματισμό είναι το

$D^* = \{(u,v):\ 1 \le u \le 3,\ 1 \le v \le 4\}$



Θέτω $u = xy$, $v = x^2 - y^2$

$$\frac{d(x,y)}{d(u,v)} = \frac{1}{\frac{d(u,v)}{d(x,y)}}$$ Αρκεί $\frac{d(u,v)}{d(x,y)} \neq 0$

Όμως $$\frac{d(u,v)}{d(x,y)} = \begin{vmatrix} \frac{du}{dx} & \frac{du}{dy} \\ \frac{dv}{dx} & \frac{dv}{dy} \end{vmatrix} = \begin{vmatrix} x & x \\ 2x & -2y \end{vmatrix} =$$

$$= 2y^2 - 2x^2 = -2(x^2+y^2) \neq 0$$

Άρα $$\frac{d(x,y)}{d(u,v)} = \frac{-1}{2(x^2+y^2)}$$

Οπότε: $$\iint_D (x^2+y^2)\,dx\,dy = \int_1^3 \int_1^4 \cancel{(x^2+y^2)} \frac{1}{2\cancel{(x^2+y^2)}}\,dv\,du =$$

$$= \frac{1}{2}\int_1^3 \int_1^4 1\,dv\,du = \frac{1}{2}\int_1^3 [v]_1^4\,du = \frac{1}{2}\int_1^3 (4-1)\,du =$$

$$= \frac{1}{2}\int_1^3 3\,du = \frac{1}{2}\,3\,[u]_1^3 = \frac{3}{2}(3-1) = \frac{3\cdot\cancel{2}}{\cancel{2}} = 3 \blacksquare$$

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Άσκηση 1

Έστω D ο μοναδιαίος δίσκος στο επίπεδο. Να υπολογίσετε το ολοκλήρωμα $\iint_D (1+x^2+y^2)^{3/2}\,dx\,dy$

Λύση

Κάνοντας αλλαγή μεταβλητών σε πολικές συντεταγμένες έχουμε: $x = r\cos\theta$ και $y = r\sin\theta$

Άρα, $\iint_D (1+x^2+y^2)^{3/2}\,dx\,dy = \iint_{D^*} \left(1+(r\cos\theta)^2+(r\sin\theta)^2\right)^{3/2} \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta$

$= \iint_{D^*} (1+r^2\cos^2\theta + r^2\sin^2\theta)^{3/2} \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta =$

$= \iint_{D^*} \left(1+r^2\overbrace{(\cos^2\theta+\sin^2\theta)}^{1}\right)^{3/2} \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta =$

$= \iint_{D^*} (1+r^2)^{3/2} \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta.$ (1)

Η Ιακωβιανή του μετασχηματισμού είναι $\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| =$

$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = \cos\theta\, r\cos\theta + r\sin\theta\sin\theta =$

$= r\cos^2\theta + r\sin^2\theta = r(\cos^2\theta+\sin^2\theta) = r\cdot 1 = r$

Για να βρούμε τα όρια του ολοκληρώματος έχουμε:

$(x,y) = (r\cos\theta, r\sin\theta)$

Αφού D είναι ο μοναδιαίος δίσκος τότε $(x,y)\in D$

$\Rightarrow x^2+y^2 \le 1 \Rightarrow r^2\cos^2\theta + r^2\sin^2\theta \le 1 \Rightarrow r^2 \le 1$

Αλλά κάθε σημείο (x,y) του μοναδιαίου δίσκου γράφεται στη μορφή $(r\cos\theta, r\sin\theta)$ για τα οποία $0\le r\le 1$ και $0\le\theta\le 2\pi$

Άρα η ① $\to \int_0^{2\pi}\int_0^1 (1+r^2)^{3/2}\, r \, dr\, d\theta =$

$$= \int_0^{2\pi} \left(\int_0^1 r \sqrt{(1+r^2)^2 (1+r^2)}\, dr \right) d\theta =$$

$$= \int_0^{2\pi} \left(\int_0^1 r (1+r^2) \sqrt{1+r^2}\, dr \right) d\theta =$$

$$= \int_0^{2\pi} \left(\int_1^2 t\sqrt{t}\, \frac{dt}{2} \right) d\theta = \int_0^{2\pi} \left(\frac{1}{2}\int_1^2 t\sqrt{t}\, dt \right) d\theta =$$

$$= \int_0^{2\pi} \left(\frac{1}{2}\int_1^2 t \cdot t^{1/2}\, dt \right) d\theta =$$

Θέτω $1+r^2 = t$

$dt = 2r\, dr \Leftrightarrow$

$r\, dr = \frac{dt}{2}$

Όταν $r=0$: $t=1$

Όταν $r=1$: $t=2$

$$= \int_0^{2\pi} \left(\frac{1}{2}\int_1^2 t^{3/2}\, dt \right) d\theta =$$

$$= \int_0^{2\pi} \left(\frac{1}{2} \left[\frac{t^{5/2}}{5/2} \right]_1^2 \right) d\theta =$$

$$= \int_0^{2\pi} \left(\frac{1}{2} \frac{2^{5/2}}{5/2} - \frac{1}{5/2} \right) d\theta = \int_0^{2\pi} \frac{1}{2} \left(\frac{2^{5/2} - 1}{5/2} \right) d\theta =$$

$$= \int_0^{2\pi} \frac{1}{\cancel{2}} \frac{\cancel{2}}{5} \left(2^{5/2} - 1 \right) d\theta = \frac{1}{5}\left(\sqrt{2^5} - 1 \right) \int_0^{2\pi} d\theta =$$

$$= \frac{\sqrt{32}-1}{5} \left[\theta \right]_0^{2\pi} = \left(\frac{\sqrt{2\cdot 16} - 1}{5} \right) 2\pi = \left(\frac{4\sqrt{2} - 1}{5} \right) 2\pi =$$

$$= \frac{2\pi}{5} \left(4\sqrt{2} - 1 \right) \blacksquare$$

Άσκηση 3

Αν D είναι η μοναδιαία μπάλα στον $\mathbb{R}^3$ υπολογίστε το ολοκλήρωμα $\iiint_D \frac{dxdydz}{\sqrt{2+x^2+y^2+z^2}}$ (1) με τη χρήση κατάλληλης αλλαγής συντεταγμένων.

Λύση

Κάνοντας αλλαγή μεταβλητών σε σφαιρικές συντεταγμένες έχουμε:

$z = \rho\cos\phi \qquad 0 \le \phi \le \pi$

$x = \rho\sin\phi\cos\theta \qquad 0 \le \theta \le 2\pi$

$y = \rho\sin\phi\sin\theta \qquad 0 \le \rho \le 1$

Άρα η (1) $\rightarrow$ $\iiint_D \frac{dxdydz}{\sqrt{2+x^2+y^2+z^2}} dxdydz =$

$$= \iiint_{D^*} \frac{d\rho d\phi d\theta}{\sqrt{2+\rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta+\rho^2\cos^2\phi}} \left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right| =$$

$$= \iiint_{D^*} \frac{\rho^2\sin\phi}{\sqrt{2+\rho^2(\sin^2\phi\cos^2\theta+\sin^2\phi\sin^2\theta+\cos^2\phi)}} d\rho d\phi d\theta =$$

$$= \iiint_{D^*} \frac{\rho^2\sin\phi\, d\rho d\phi d\theta}{\sqrt{2+\rho^2(\sin^2\phi\underbrace{(\cos^2\theta+\sin^2\theta)}_{=1}+\cos^2\phi)}}$$

Η ιακωβιανή του μετασχηματισμού είναι:

$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = -\rho^2\sin\phi$

$$= \iiint_{D^*} \frac{\rho^2\sin\phi\, d\rho d\phi d\theta}{\sqrt{2+\rho^2\underbrace{(\sin^2\phi+\cos^2\phi)}_{=1}}} =$$

Αλλά $\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right| = \rho^2\sin\phi$.

$$= \iiint_{D^*} \frac{\rho^2\sin\phi}{\sqrt{2+\rho^2}} d\rho d\phi d\theta = \int_0^1\left(\int_0^\pi\left(\int_0^{2\pi}\frac{\rho^2\sin\phi}{\sqrt{2+\rho^2}}d\theta\right)d\phi\right)d\rho$$

$$= \int_0^1 \left(\int_0^\pi \frac{\rho^2}{\sqrt{2+\rho^2}} \sin\phi \left([\theta]_0^{2\pi} \right) d\phi \right) d\rho =$$

$$= \int_0^1 \left(\int_0^\pi 2\pi \, \frac{\rho^2}{\sqrt{2+\rho^2}} \sin\phi \, d\phi \right) d\rho =$$

$$= \int_0^1 2\pi \, \frac{\rho^2}{\sqrt{2+\rho^2}} \left([-\cos\phi]_0^\pi \right) d\rho =$$

$$= \int_0^1 2\pi \, \frac{\rho^2}{\sqrt{2+\rho^2}} \left(-\cos\pi - (\cos 0) \right) d\rho =$$

$$= \int_0^1 2\pi \, \frac{\rho^2}{\sqrt{2+\rho^2}} \left(-(-1) - (-1) \right) d\rho =$$

$$= \int_0^1 2\pi \, \frac{\rho^2}{\sqrt{2+\rho^2}} (1+1) \, d\rho = 4\pi \int_0^1 \frac{\rho^2}{\sqrt{2+\rho^2}} \, d\rho$$

$$= 4\pi \int_0^1 \rho^2 \, \frac{1}{\sqrt{\rho^2+2}} \, d\rho = 4\pi \int_0^1 \rho^2 \left(\ln\left(\rho + \sqrt{\rho^2+2}\right) \right)' d\rho =$$

$$= 4\pi \left[\ln\left(\rho+\sqrt{\rho^2+2}\right) \rho^2 \right]_0^1 - 4\pi \int_0^1 \ln\left(\rho + \sqrt{\rho^2+2}\right) 2\rho \, d\rho =$$

$$= 4\pi \left[\ln\left(1+\sqrt{3}\right) \right] - 8\pi \int_0^1 \rho \left(\ln\left(\rho+\sqrt{\rho^2+2}\right) \right) d\rho =$$

$$= 4\pi \int_0^1 \rho \cdot \frac{1}{\sqrt{\rho^2+2}} \, d\rho = 4\pi \int_0^1 \rho \cdot \left(\sqrt{2+\rho^2} \right)' d\rho =$$

$$4\pi \left(\left[\rho\sqrt{2+\rho^2} \right]_0^1 - \int_0^1 \sqrt{\rho^2+2} \, d\rho \right) = \ldots$$

Άσκηση 2 Να υπολογίσω το:

$x+y=1$

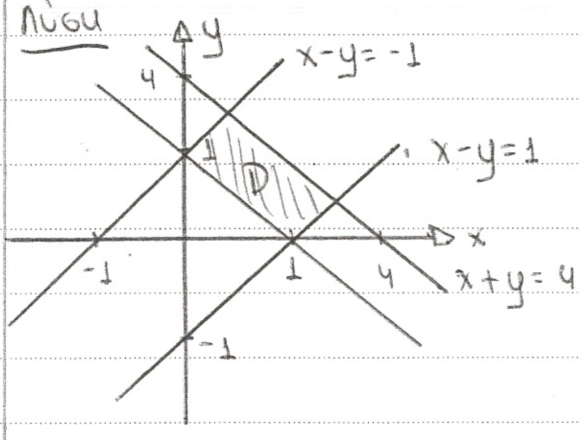
$x+y=+4$

$x-y=-1$

$x-y=1$

$$\iint_D (x+y)^2 e^{x-y}\,dx\,dy \quad (2)$$ με τις κατάλληλες αλλαγές συντεταγμένων.

Λύση



Θέτω $\left.\begin{aligned} u &= x+y \\ v &= x-y \end{aligned}\right\} \Rightarrow$

(1) $\left.\begin{aligned} x &= v+y \\ x &= u-y \end{aligned}\right\} \Rightarrow v+y = u-y \Rightarrow$

$2y = u-v \Rightarrow y = \frac{u-v}{2} \Rightarrow$

$$\boxed{y = \frac{u}{2} - \frac{v}{2}}$$

Άρα η (1) $\Rightarrow x = v + \frac{u}{2} - \frac{v}{2} \Rightarrow$

$$\boxed{x = \frac{u}{2} + \frac{v}{2}}$$

Άρα η (2) $\Rightarrow \iint_{D^*} u^2 e^v \left| \frac{d(x,y)}{d(u,v)} \right| du\,dv$

Η Ιακωβιανή του μετασχηματισμού είναι $\begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{vmatrix} =$

$= \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{2} - \frac{1}{2} = -\frac{2}{4} = -\frac{1}{2}$

Οπότε έχω: $\int_1^4 \left(\int_{-1}^1 \frac{1}{2} u^2 e^v \, dv \right) du = \int_1^4 \frac{1}{2} u^2 [e^v]_{-1}^1 \, du =$

11/9/16

Άσκηση 1

Έστω D ο μοναδιαίος δίσκος στο επίπεδο. Να υπολογίσετε το ολοκλήρωμα $\iint_D (1+x^2+y^2)^{3/2} dx\,dy$

Λύση

Κάνοντας αλλαγή μεταβλητών σε πολικές συντεταγμένες έχουμε: $x = r\cos\theta$ και $y = r\sin\theta$

Άρα, $\iint_D (1+x^2+y^2)^{3/2} dx\,dy = \iint_{D^*} (1+(r\cos\theta)^2+(r\sin\theta)^2)^{3/2} \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta$

$= \iint_{D^*} (1+r^2\cos^2\theta + r^2\sin^2\theta)^{3/2} \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta =$

$= \iint_{D^*} \left(1+r^2 \overbrace{(\cos^2\theta+\sin^2\theta)}^{1}\right)^{3/2} \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta =$

$= \iint_{D^*} (1+r^2)^{3/2} \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta$. ①

Η Ιακωβιανή του μετασχηματισμού είναι $\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| =$

$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = \cos\theta\, r\cos\theta + r\sin\theta\sin\theta =$

$= r\cos^2\theta + r\sin^2\theta = r(\cos^2\theta+\sin^2\theta) = r\cdot 1 = r$

Για να βρούμε τα άκρα του ολοκληρώματος έχουμε:

$(x,y) = (r\cos\theta, r\sin\theta)$

Αφού D είναι ο μοναδιαίος δίσκος τότε $(x,y)\in D$

$\Rightarrow x^2+y^2 \le 1 \Rightarrow r^2\cos^2\theta + r^2\sin^2\theta \le 1 \Rightarrow r^2 \le 1$

Αλλά κάθε σημείο (x,y) του μοναδιαίου δίσκου γράφεται στη μορφή $(r\cos\theta, r\sin\theta)$ για τα οποία $0\le r\le 1$ και $0\le\theta\le 2\pi$

Άρα η ① $\to \displaystyle\int_0^{2\pi}\int_0^1 (1+r^2)^{3/2}\, r\, dr\, d\theta =$

$$= \int_0^{2\pi}\left(\int_0^1 r\sqrt{(1+r^2)^2(1+r^2)}\, dr\right) d\theta =$$

$$= \int_0^{2\pi}\left(\int_0^1 r(1+r^2)\sqrt{1+r^2}\, dr\right) d\theta =$$

$$= \int_0^{2\pi}\left(\int_1^2 t\sqrt{t}\,\frac{dt}{2}\right) d\theta = \int_0^{2\pi}\left(\frac{1}{2}\int_1^2 t\sqrt{t}\, dt\right) d\theta =$$

$$= \int_0^{2\pi}\left(\frac{1}{2}\int_1^2 t\cdot t^{1/2}\, dt\right) d\theta =$$

θέτω $1+r^2=t$

$dt = 2r\,dr \Leftrightarrow$

$r\,dr = \frac{dt}{2}$

Όταν $r=0$: $t=1$

Όταν $r=1$: $t=2$

$$= \int_0^{2\pi}\left(\frac{1}{2}\int_1^2 t^{3/2}\, dt\right) d\theta =$$

$$= \int_0^{2\pi}\left(\frac{1}{2}\left[\frac{t^{5/2}}{5/2}\right]_1^2\right) d\theta =$$

$$= \int_0^{2\pi}\left(\frac{1}{2}\left(\frac{2^{5/2}}{5/2} - \frac{1}{5/2}\right)\right) d\theta = \int_0^{2\pi}\frac{1}{2}\left(\frac{2^{5/2}-1}{5/2}\right) d\theta =$$

$$= \int_0^{2\pi}\frac{1}{\cancel{2}}\,\frac{\cancel{2}}{5}\left(2^{5/2}-1\right) d\theta = \frac{1}{5}\left(\sqrt{2^5}-1\right)\int_0^{2\pi} d\theta =$$

$$= \frac{\sqrt{32}-1}{5}\left[\theta\right]_0^{2\pi} = \left(\frac{\sqrt{2\cdot 16}-1}{5}\right)2\pi = \left(\frac{4\sqrt{2}-1}{5}\right)2\pi =$$

$$= \frac{2\pi}{5}\left(4\sqrt{2}-1\right) \quad \blacksquare$$

Άσκηση 2

Να υπολογίσω το:

$x+y=1$

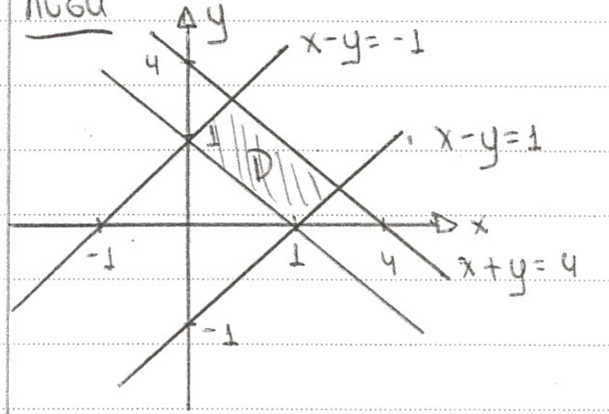
$x+y=+4$

$x-y=-1$

$x-y=1$

$$\iint_D (x+y)^2 e^{x-y}\, dx dy \quad (2)$$ με τις κατάλληλες αλλαγές συντεταγμένων.

Λύση



Θέτω $\left.\begin{aligned} u &= x+y \\ v &= x-y \end{aligned}\right\} \Rightarrow$

(1) $\left.\begin{aligned} x &= v+y \\ x &= u-y \end{aligned}\right\} \Rightarrow v+y = u-y \Rightarrow$

$2y = u-v \Rightarrow y = \frac{u-v}{2} \Rightarrow$

$$\boxed{y = \frac{u}{2} - \frac{v}{2}}$$

Άρα η (1) $\Rightarrow x = v + \frac{u}{2} - \frac{v}{2} \Rightarrow$

$$\boxed{x = \frac{u}{2} + \frac{v}{2}}$$

Άρα η (2) $\Rightarrow \iint_{D^*} u^2 e^v \left| \frac{d(x,y)}{d(u,v)} \right| du\, dv$

Η Ιακωβιανή του μετασχηματισμού είναι $\begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{vmatrix} =$

$= \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{2} - \frac{1}{2} = -\frac{2}{4} = -\frac{1}{2}$

Οπότε έχω: $\int_1^4 \left(\int_{-1}^1 \frac{1}{2} u^2 e^v dv \right) du = \int_1^4 \frac{1}{2} u^2 [e^v]_{-1}^1 du =$

$$\int_1^4 \frac{1}{2}u^2(e^1-e^{-1})\,du = \frac{1}{2}(e^1-e^{-1})\left[\frac{u^3}{3}\right]_1^4 =$$

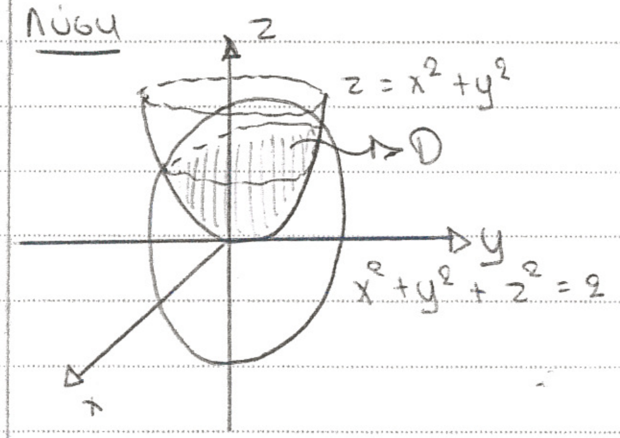
$$= \frac{1}{2}(e^1-e^{-1})\frac{64-1}{3} = \frac{63}{6}(e^1-e^{-1}) = \frac{21}{2}(e-e^{-1}) \blacksquare$$

Άσκηση 5

$z = x^2+y^2$ — Να βρω τον όγκο ανάμεσα στις επιφάνειες

$x^2+y^2+z^2 = 2$

Λύση



Αρχικά λύνω το σύστημα για να βρω κοινά σημεία

$z^2+z-2=0 \Leftrightarrow$

$(z-1)(z+2)=0 \Leftrightarrow$

$z=1$ ή $z=-2$

δεκτή, απορρίπτεται

Άρα για $z=1$ έχω $1 = x^2+y^2$ (μοναδιαίος κύκλος)

Οπότε $$\iiint_\Omega dx\,dy\,dz = \iint_D \left(\int_{x^2+y^2}^{\sqrt{2-x^2-y^2}} dt \right) dx\,dy =$$

$$= \iint_D \sqrt{2-x^2-y^2} - x^2 - y^2 \; dx\,dy \quad (1)$$

Τώρα κάνω αλλαγή σε πολικές συντεταγμένες και θέτω $x = r\cos\theta$ με $0 \le r \le 1$

$y = r\sin\theta$ με $0 \le \theta \le 2\pi$

Άρα η (1) $\rightarrow \iint_{D^*} \left(\sqrt{2-r^2} - r^2\right) r\,dr\,d\theta =$

$$= \int_0^1 \left(\int_0^{2\pi} r\sqrt{2-r^2} - r^3 \, d\theta \right) dr =$$

$$= 2\pi \int_0^1 r\sqrt{2-r^2}\, dr - 2\pi \int_0^1 r^3 \, dr =$$

$$= 2\pi \int_2^1 \sqrt{t} \left(-\frac{dt}{2}\right) - 2\pi \left[r^4/4 \right]_0^1 =$$

θέτω $2 - r^2 = t$

$dt = -2r\,dr$

$r\,dr = -\frac{dt}{2}$

Για $r=0 : t=2$

Για $r=1 : t=1$

$$= -\frac{2\pi}{2}(-1)\int_1^2 \sqrt{t}\, dt - \frac{2\pi}{4} =$$

$$= \pi \int_1^2 \sqrt{t}\, dt - \frac{\pi}{2} =$$

$$= \pi \int_1^2 t^{1/2}\, dt - \frac{\pi}{2} = \pi \left[\frac{t^{3/2}}{3/2} \right]_1^2 - \frac{\pi}{2} =$$

$$= \pi \frac{2^{3/2} - 1}{3/2} - \frac{\pi}{2} = \frac{2\pi}{3}(\sqrt{8} - 1) - \frac{\pi}{2} =$$

$$= \frac{2\pi}{3}(2\sqrt{2} - 1) - \frac{\pi}{2} = \frac{8\pi\sqrt{2}}{6} - \frac{6}{6} - \frac{3\pi}{6} = \frac{8\pi\sqrt{2} - 6 - 3\pi}{6}$$ ■

Άσκηση 4

α) $T(u,v) = (u^2 - v^2, 2uv) = (x,y)$

α) Για να είναι "1-1" πρέπει: $T(u_1, v_1) = T(u_2, v_2) \Leftrightarrow$

$(u_1^2 - v_1^2, 2u_1v_1) = (u_2^2 - v_2^2, 2u_2v_2) \Leftrightarrow$

$$\left. \begin{array}{l} u_1^2 - v_1^2 = u_2^2 - v_2^2 \\ \cancel{2}u_1v_1 = \cancel{2}u_2v_2 \end{array} \right\} \Rightarrow \left. \begin{array}{l} (u_1 - v_1)(u_1 + v_1) - (u_2 - v_2)(u_2 + v_2) = 0 \\ u_1v_1 - u_2v_2 = 0 \Leftrightarrow v_1 = \frac{u_2v_2}{u_1} \; (1) \end{array} \right\} \Rightarrow$$

με $u_1 \in [1,2]$

$$u_1^2 - v_1^2 = u_2^2 - v_2^2 \overset{(1)}{\Rightarrow} u_1^2 - \frac{u_2^2 v_2^2}{u_1^2} = u_2^2 - v_2^2 \Rightarrow$$

$$u_1^2 - u_2^2 = \frac{u_2^2 v_2^2}{u_1^2} - v_2^2 \Rightarrow u_1^2 - u_2^2 = v_2^2\left(\frac{u_2^2}{u_1^2} - 1\right) \Rightarrow$$

$$\Rightarrow u_1^2 - u_2^2 = v_2^2\left(\frac{u_2^2 - u_1^2}{u_1^2}\right) \Rightarrow u_1^2 - u_2^2 = -\frac{v_2^2}{u_1^2}(u_1^2 - u_2^2)$$

Προφανώς $\frac{v_2^2}{u_1^2} \neq 0$ καθώς $u_1 \in [1,2]$ και $v \in [1,3]$

Άρα $u_1^2 - u_2^2 = 0$ ή $u_1^2 - u_2^2 \neq 0$

▶ Αν $u_1^2 - u_2^2 = 0$: $u_1^2 = u_2^2 \Rightarrow u_1 = u_2$ για $u_1 \in [1,2]$

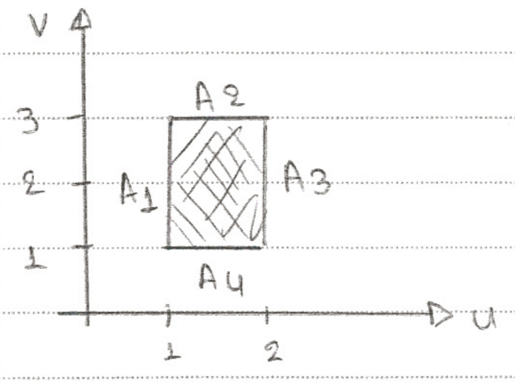
$v_1 = \frac{\cancel{u_2} v_2}{\cancel{u_1}} \xrightarrow[u_1 = u_2]{} v_1 = v_2$

▶ Αν $u_1^2 - u_2^2 \neq 0$: $u_1^2 - u_2^2 = -\frac{v_2^2}{u_1^2}(u_1^2 - u_2^2) \Rightarrow$

$1 = -\frac{v_2^2}{u_1^2} \Rightarrow u_1^2 = -v_2^2$ άτοπο ↯

Άρα η T είναι "1-1"

Β)

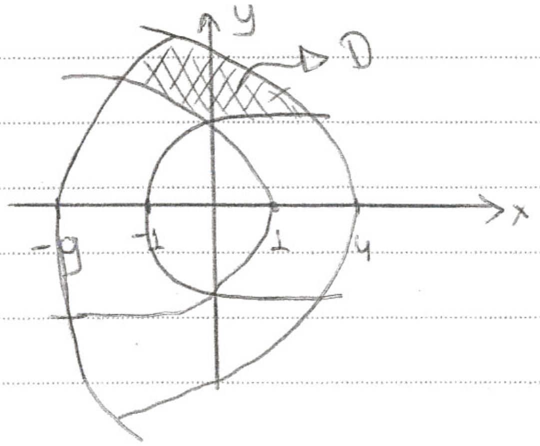


$A_1: (1,t) \xrightarrow{T} (1-t^2, 2t)$

$A_2: (t,3) \xrightarrow{T} (t^2-9, 6t)$

$A_3: (2,t) \xrightarrow{T} (4-t^2, 4t)$

$A_4: (t,1) \xrightarrow{T} (t^2-1, 2t)$



Κορυφές

$(1,1) \xrightarrow{T} (0,2)$

$(1,3) \xrightarrow{T} (-8,6)$

$(2,1) \xrightarrow{T} (3,4)$

$(2,3) \xrightarrow{T} (-5,12)$

$x = u^2 - v^2 \quad , \quad y = 2uv \quad \begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} =$

$= 4(u^2+v^2)$

Άρα $\iint_D 1 \, dx dy = \int_1^3 \int_1^2 4(u^2+v^2) \, du dv =$

$= \int_1^3 4 \left[\frac{u^3}{3} + uv^2 \right]_1^2 dv = 4 \int_1^3 \left(\frac{8}{3} + 2v^2 - \frac{1}{3} - v^2 \right) dv =$

$= 4 \int_1^3 \left(\frac{7}{3} + v^2 \right) dv = 4 \left[\frac{7}{3} v + \frac{v^3}{3} \right]_1^3 = 4 \left[7 + 9 - \frac{7}{3} - \frac{1}{3} \right] =$

$= 4(16 - 8/3) = 4 \left(\frac{48-8}{3} \right) = 4 \frac{40}{3} = \frac{160}{3}$ ■

Άσκηση 6

$x^2 + y^2 \leq \frac{1}{5} z^2$ και $0 \leq z \leq 5 + \sqrt{5 - x^2 - y^2}$

$z = 5 + \sqrt{5 - x^2 - y^2} \Rightarrow z = 5 + \sqrt{5 - \frac{1}{5} z^2} \Rightarrow (z-5)^2 = 5 - \frac{1}{5} z^2 \Rightarrow$

$z^2 - 10z + 25 = 5 - \frac{1}{5} z^2 \Rightarrow 5z^2 - 50z + 125 = 25 - z^2 \Rightarrow$

$6z^2 - 50z + 100 = 0 \qquad z_{1,2} = \begin{cases} \frac{60}{12} = 5 \\ \frac{40}{12} = \frac{10}{3} < 5 \text{ απορρίπτεται} \end{cases}$

Άρα $x^2 + y^2 \leq \frac{1}{5} z^2 \Rightarrow x^2 + y^2 \leq 5$

Οπότε: $\displaystyle\int_{-\sqrt{5}}^{\sqrt{5}} \int_{-\sqrt{5-x^2}}^{\sqrt{5-x^2}} \int_{\sqrt{5(x^2+y^2)}}^{5+\sqrt{5-x^2-y^2}} 1 \, dz \, dy \, dx =$

$$= \int_{-\sqrt{5}}^{\sqrt{5}} \int_{-\sqrt{5-x^2}}^{\sqrt{5-x^2}} 5+\sqrt{5-x^2-y^2} - \sqrt{5(x^2+y^2)}\, dy\, dx =$$

$$= \int_0^{\sqrt{5}} \int_0^{2\pi} 5+\sqrt{5-r^2} - \sqrt{5r^2}\,) r\, d\theta\, dr =$$

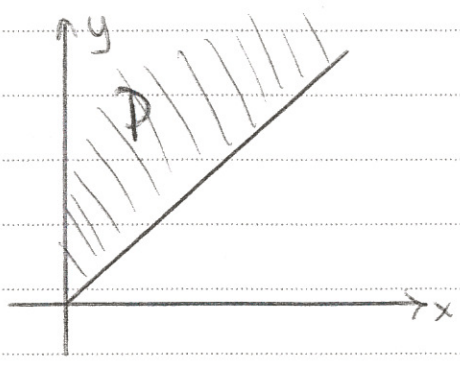
$$= 2\pi \int_0^{\sqrt{5}} (5r + r\sqrt{5-r^2} - r^2\sqrt{5})\, dr =$$

$$= 2\pi \left[5\frac{r^2}{2} - \frac{(5-r^2)^{3/2}}{3} - \sqrt{5}\frac{r^3}{3} \right]_0^{\sqrt{5}} =$$

$$= 2\pi \left(\frac{5 \cdot 5}{2} - \frac{\sqrt{5}(\sqrt{5})^3}{3} + \frac{5^{3/2}}{3} \right) = 2\pi \left(\frac{25}{2} - \frac{25}{3} + \frac{5\sqrt{5}}{3} \right)$$

$$= 2\pi \left(\frac{25}{6} + \frac{5\sqrt{5}}{3} \right) = \pi \left(\frac{25}{3} + \frac{10\sqrt{5}}{3} \right) = \frac{\pi}{3}(25+10\sqrt{5}) \blacksquare$$

Άσκηση 7



$$\int_0^{\infty} \int_x^{\infty} x e^{-y^3}\, dy\, dx =$$

$$= \int_0^{\infty} \int_0^{y} x e^{-y^3}\, dx\, dy =$$

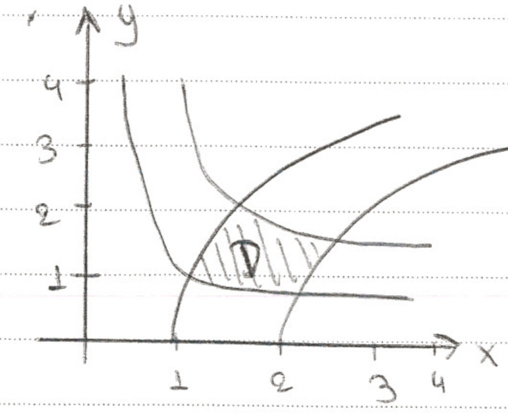
$$= \int_0^{\infty} \left[\frac{x^2}{2} e^{-y^3} \right]_0^{y} dy =$$

$$= \int_0^{\infty} \frac{y^2}{2} e^{-y^3}\, dy = \frac{1}{2} \lim_{c \to \infty} \int_0^{c} y^2 e^{-y^3}\, dy =$$

$$= \frac{1}{2} \lim_{c \to +\infty} \left[\frac{e^{-y^3}}{-3} \right]_0^{c} = \frac{1}{2} \lim_{c \to \infty} \left(\frac{e^{-c^3}}{-3} + \frac{e^0}{3} \right) =$$

$$= \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6} \blacksquare$$

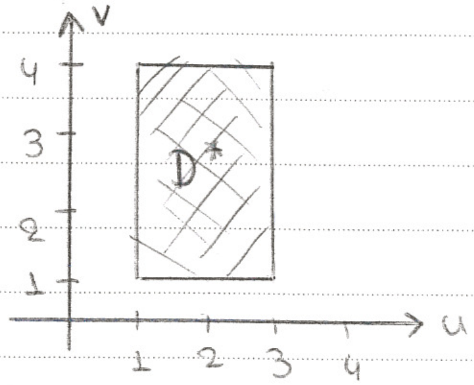
Άσκηση 8



Έστω $T(x,y)=(xy,\,x^2-y^2)=(u,v)$

Άρα το χωρίο D^* που προκύπτει από το μετασχηματισμό είναι το

$D^*=\{(u,v):\ 1\le u\le 3,\ 1\le v\le 4\}$



Θέτω $u=xy$, $v=x^2-y^2$

$$\frac{d(x,y)}{d(u,v)}=\frac{1}{\frac{d(u,v)}{d(x,y)}}$$ Αρκεί $\frac{d(u,v)}{d(x,y)}\neq 0$

Όμως $$\frac{d(u,v)}{d(x,y)}=\begin{vmatrix}\frac{du}{dx} & \frac{du}{dy}\\ \frac{dv}{dx} & \frac{dv}{dy}\end{vmatrix}=\begin{vmatrix} x & x\\ 2x & -2y\end{vmatrix}=$$

$$=2y^2-2x^2=-2(x^2+y^2)\neq 0$$

Άρα $$\frac{d(x,y)}{d(u,v)}=-\frac{1}{2(x^2+y^2)}$$

Οπότε: $$\iint_D (x^2+y^2)\,dx\,dy=\int_1^3\int_1^4 \cancel{(x^2+y^2)}\,\frac{1}{2\cancel{(x^2+y^2)}}\,dv\,du=$$

$$=\frac{1}{2}\int_1^3\int_1^4 1\,dv\,du=\frac{1}{2}\int_1^3 [v]_1^4\,du=\frac{1}{2}\int_1^3(4-1)\,du=$$

$$=\frac{1}{2}\int_1^3 3\,du=\frac{1}{2}\,3\,[u]_1^3=\frac{3}{2}(3-1)=\frac{3\cdot\cancel{2}}{\cancel{2}}=3$$ ■