Positive solutions of a Neumann Problem with competing critical nonlinearities

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Abstract

We present existence results for a Neumann problem involving critical Sobolev nonlinearities both on the right hand side of the equation and at the boundary condition. Positive solutions are obtained through constrained minimization on the Nehari manifold. Our approach is based on the concentration compactness principle of P. L. Lions and M. Struwe.

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1 Introduction

In this paper we are concerned with the Neumann problem

$$\begin{cases} -\Delta u + \lambda u = N(N-2)Q(x)u^{\frac{N+2}{N-2}}, & \text{in} \quad \Omega, \\ u > 0 \quad \text{in} \quad \bar{\Omega}, \qquad \frac{\partial u}{\partial \nu} = P(x)u^{\frac{N}{N-2}}, & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbf{R}^N$ is a bounded domain with smooth boundary and $\lambda \geq 0$ is a parameter. The constant $N(N-2)$ appearing in (1.1) has been introduced for technical convenience and can be absorbed in $Q(x)$. We denote by $2^* = \frac{2N}{N-2}$ and $q = \frac{2(N-1)}{N-2}$ the critical Sobolev exponents for the embedding of $H^1(\Omega)$ into $L^{2^*}(\Omega)$, and $H^1(\Omega)$ into $L^q(\partial\Omega)$, respectively, for $N \geq 3$. Both embeddings are continuous but not compact. Throughout the paper we

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assume that Q and P are continuous functions on $\bar{\Omega}$ and $\partial\Omega$ respectively. In addition, Q is a positive function on $\bar{\Omega}$. Further assumptions on Q and P will be formulated later.

Semilinear elliptic problems with critical Sobolev nonlinearities have received considerable attention after the pioneering work of Brezis and Nirenberg [9] on the constant coefficient Dirichlet counterpart of (1.1), but also in connection with the Yamabe problem, see e.g., [17], [20] and references therein.

If $P(x) \equiv 0$, then solutions of (1.1) are obtained as minimizers of the variational problem

$$S_\lambda = \inf_{u\in H^1(\Omega)\setminus\{0\}} \frac{\int_\Omega (|\nabla u|^2 + \lambda u^2)\,dx}{\left(N(N-2)\int_\Omega Q(x)|u|^{2^*}\,dx\right)^{\frac{2}{2^*}}}. \tag{1.2}$$

If $N(N-2)Q(x) \equiv 1$ and $P(x) \equiv 0$, problem (1.1) has an extensive literature, see e.g., [1]–[6], [10]–[13], [14], [26]. The first existence results in this case were obtained by Adimurthi - Mancini [1], Adimurthi - Yadava [5] and X.J. Wang [26]. If S denotes the best Sobolev constant, and

$$S_\lambda < S/2^{\frac{2}{N}}\,, \tag{1.3}$$

for some $\lambda > 0$, then problem (1.1) admits a least energy solution. The validity of this inequality can be verified by using a special family of functions, namely, the Talenti instantons centered at a point on the boundary $\partial\Omega$ with positive mean curvature. It turns out that solutions exist for any $\lambda > 0$, under the assumption that $\partial\Omega$ has at least one point with positive mean curvature.

If $Q(x) \not\equiv$ *constant* and $P(x) \equiv 0$, problem (1.1) has been studied by Chabrowski and Willem [11]. Assuming that $Q(x) > 0$, and setting $Q_m := \max_{x\in\partial\Omega} Q(x)$ and $Q_M := \max_{x\in\bar{\Omega}} Q(x)$, the authors showed that if

$$S_\lambda < (S/2^{\frac{2}{N}})\ \min\{2^{\frac{2}{N}}/(N(N-2)Q_M)^{2/2^*},\ 1/(N(N-2)Q_m)^{2/2^*}\}, \tag{1.4}$$

then a least energy solution exists. If $N(N-2)Q(x) \equiv 1$ then (1.4) reduces to (1.3). Condition (1.4) ensures that no concentration due to the presence of the critical exponent can occur in a minimizing sequence of (1.2), and therefore one can extract a subsequence that converges to a solution of (1.1). In case $Q_M \le 2^{\frac{2}{N-2}}Q_m$, inequality (1.4) is satisfied for any $\lambda \ge 0$ provided Q_m is attained at a point $x_0 \in \partial\Omega$ with positive mean curvature. On the other hand, if $Q_M > 2^{\frac{2}{N-2}}Q_m$, then least energy solutions exist for $\lambda \in (0, \lambda_0)$, for some $\lambda_0 > 0$, and in general, no least energy solutions exist for large values of λ.

Our purpose in this work is to obtain existence results in the case where $Q(x) \not\equiv$ *constant* and $P(x) \not\equiv 0$. Since P is not anymore identically zero, the critical nonlinearity on the boundary comes into play. We note that, in the case where Q and P are nonzero constants, some existence results for $\lambda = 0$ are provided by Pierotti and Terracini [22], among other things.

Elliptic problems that resemble to (1.1), involving two critical exponents, arise in a natural way in geometry, see Escobar [18], Han and Li [19, 20], Ambrosetti, Li, Malchiodi [7], and Djadli, Malchiodi, Ould Ahmedou [16] and references therein. Consider for instance the problem of prescribing the scalar curvature and the boundary mean curvature of the standard half three sphere, by conformally deforming its standard metric. After a conformal transformation that sends the half sphere to the upper half space, the problem is reduced to finding positive solutions (with finite energy) of

$$\begin{cases} -4\frac{N-1}{N-2}\Delta u = K(x)u^{\frac{N+2}{N-2}}, & \text{in} \quad \mathbf{R}^N_+, \\ \quad -\frac{N}{N-2}\frac{\partial u}{\partial x_N} = H(x)u^{\frac{N}{N-2}}, & \text{on} \quad \partial\mathbf{R}^N_+, \end{cases} \tag{1.5}$$

with K, H smooth functions and $K(x) > 0$. Problems (1.1) and (1.5) are the same if we identify Ω with $\mathbf{R}^N_+$, we note however that in our case the geometry of $\partial\Omega$ plays an important role. In [16] the authors among other things proved the existence of positive solutions for (1.5) in the case $N = 3$, under suitable conditions on K and H. Their method is based on Bahri's theory of the critical points at infinity.

Our approach is based on the concentration compactness principle of Lions [21] and Struwe [23]. We will obtain solutions of (1.1) as critical points of the functional

$$\mathcal{J}_\lambda(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + \lambda u^2)\,dx - \frac{N(N-2)}{2^*}\int_\Omega Q(x)|u|^{2^*}\,dx - \frac{1}{q}\int_{\partial\Omega} P(x)|u|^q\,dS_x, \tag{1.6}$$

which is a C^1 functional on $H^1(\Omega)$. The Fréchet derivative of $\mathcal{J}_\lambda$ is given by

$$< \mathcal{J}'_\lambda(u), \phi >= \int_\Omega (\nabla u\cdot\nabla\phi + \lambda u\phi)\,dx - N(N-2)\int_\Omega Q(x)|u|^{2^*-2}u\phi\,dx - \int_{\partial\Omega} P(x)|u|^{q-2}u\phi\,dS_x, \tag{1.7}$$

for every $\phi \in H^1(\Omega)$. To find critical points we consider the following constrained minimization problem

$$c_\lambda = \inf_{u\in\mathcal{M}_\lambda} \mathcal{J}_\lambda(u), \qquad \mathcal{M}_\lambda = \{u \in H^1(\Omega);\ u \not\equiv 0, \quad < \mathcal{J}'_\lambda(u), u >= 0\}. \tag{1.8}$$

We note that in the case $P(x) \equiv 0$ the two infima in (1.2) and (1.8) are related by

$$c_\lambda = \frac{1}{N}S_\lambda^{\frac{N}{2}}.$$

In order to determine the energy level of $\mathcal{J}_\lambda$ below which the Palais–Smale conditions hold, we use the Neumann problem in a half space:

$$\begin{cases} -\Delta u = N(N-2)u^{\frac{N+2}{N-2}}, & \text{in} \quad \mathbf{R}^N_+, \\ u > 0 \quad \text{in} \quad \mathbf{R}^N_+, \qquad -\frac{\partial u(x',0)}{\partial x_N} = cu^{\frac{N}{N-2}} & \text{on} \quad \partial\mathbf{R}^N_+ = \mathbf{R}^{N-1}, \end{cases} \tag{1.9}$$

where c is a constant. In what follows we shall use the notation $x = (x', x_N)$, $x' \in \mathbf{R}^{N-1}$. It is known (see [15], [25]) that all nonnegative non-zero solutions of (1.9) are given by

$$u(x) = \left(\frac{\varepsilon}{\varepsilon^2 + |x' - x'_0|^2 + (x_N + \varepsilon c(N-2)^{-1})^2}\right)^{\frac{N-2}{2}}, \tag{1.10}$$

with $\varepsilon > 0$, $x'_0 \in \mathbf{R}^{N-1}$. For simplicity we assume that $x'_0 = 0$.

A simple scaling argument shows that the problem

$$\begin{cases} -\Delta u = N(N-2)au^{\frac{N+2}{N-2}}, & \text{in} \quad \mathbf{R}^N_+, \\ u > 0 \quad \text{in} \quad \mathbf{R}^N_+, \qquad -\frac{\partial u(x',0)}{\partial x_N} = bu^{\frac{N}{N-2}} & \text{on} \quad \partial\mathbf{R}^N_+ = \mathbf{R}^{N-1}, \end{cases} \tag{1.11}$$

where $a > 0$ and b are constants, has solutions of the form

$$U_\varepsilon(x) = a^{-\frac{N-2}{4}} \left(\frac{\varepsilon}{\varepsilon^2 + |x'|^2 + (x_N + \varepsilon\mu(N-2)^{-1})^2} \right)^{\frac{N-2}{2}}, \qquad \mu = \frac{b}{\sqrt{a}}, \quad \varepsilon > 0. \tag{1.12}$$

Note that

$$U_\varepsilon(x) = \varepsilon^{-\frac{N-2}{2}} U_1\left(\frac{x}{\varepsilon}\right).$$

The functional, as well as the solution manifold associated with problem (1.11), are given respectively by

$$J_{a,b}(u) = \frac{1}{2}\int_{\mathbf{R}^N_+} |\nabla u|^2\, dx - \frac{aN(N-2)}{2^*}\int_{\mathbf{R}^N_+} |u|^{2^*}\, dx - \frac{b}{q}\int_{\mathbf{R}^{N-1}} |u(x',0)|^q\, dx', \tag{1.13}$$

and

$$M_{a,b} = \{u \in \mathcal{D}^{1,2}(\mathbf{R}^N_+);\ u \not\equiv 0, \quad < J'_{a,b}(u), u >= 0\}. \tag{1.14}$$

Here $\mathcal{D}^{1,2}(\mathbf{R}^N_+)$ is the Sobolev space

$$\mathcal{D}^{1,2}(\mathbf{R}^N_+) = \{u;\ \nabla u \in L^2(\mathbf{R}^N_+), \quad u \in L^{2^*}(\mathbf{R}^N_+)\}, \tag{1.15}$$

equipped with the norm

$$\|u\|^2_{\mathcal{D}^{1,2}(\mathbf{R}^N_+)} = \int_{\mathbf{R}^N_+} |\nabla u|^2\, dx. \tag{1.16}$$

The paper is organized as follows. In Section 2 we find the energy level of $\mathcal{J}_\lambda$, $\lambda > 0$, below which a least energy solution exists (see Theorem 2.4). This theorem gives the existence of solutions provided the infimum of the functional $\mathcal{J}_\lambda$ on the Nehari manifold satisfies condition (2.22). Condition (2.22) is the analogue of (1.4) and is reduced to that when $P(x) \equiv 0$. Section 3 is devoted to the verification of (2.22). In Theorem 3.1 we formulate conditions guaranteeing the existence of least energy solutions for every $\lambda > 0$ whereas in Theorem 3.2 conditions are provided under which least energy solutions exist for small positive values of λ. Finally Section 4 is devoted to establishing existence of least energy solutions in the case $\lambda = 0$, see Theorem 4.4.

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2 Constrained minimization

For future use we need to compute the infimum of $J_{a,b}(u)$ when $u \in M_{a,b}$. We then have

Lemma 2.1 *There holds*

$$\inf_{u \in M_{a,b}} J_{a,b}(u) = \frac{\Pi}{2} a^{-\frac{N-2}{2}} K\left(\frac{b}{\sqrt{a}}\right), \tag{2.1}$$

where the constant Π *is given by*

$$\Pi := \int_{\mathbf{R}^{N-1}} \frac{dx'}{(1+|x'|^2)^{N-1}} = \frac{\pi^{\frac{N}{2}}}{2^{N-2}\Gamma(\frac{N}{2})}, \tag{2.2}$$

and the function $K(\cdot)$ *is defined by*

$$K(\mu) := (N-2)\int_{\frac{\mu}{N-2}}^{\infty} \frac{dt}{(1+t^2)^{\frac{N+1}{2}}} + \frac{\mu}{N-1}\left(1+\frac{\mu^2}{(N-2)^2}\right)^{-\frac{N-1}{2}}; \tag{2.3}$$

we note that $K(\mu)$ *is strictly decreasing in* $\mu \in (-\infty, \infty)$.

Proof: Since all solutions to problem (1.11) are given by $U_\varepsilon(x)$ (see (1.12)), the infimum (2.1) is equal to $J_{a,b}(U_\varepsilon)$. Since $< J'_{a,b}(U_\varepsilon), U_\varepsilon >= 0$, we have that for any $\varepsilon > 0$,

$$\int_{\mathbf{R}^N_+} |\nabla U_\varepsilon|^2\,dx = aN(N-2)\int_{\mathbf{R}^N_+} |U_\varepsilon|^{2^*}\,dx + b\int_{\mathbf{R}^{N-1}} |U_\varepsilon(x',0)|^q\,dx'. \tag{2.4}$$

In view of this, we can write $J_{a,b}(U_\varepsilon)$ (see (1.13)) as

$$J_{a,b}(U_\varepsilon) = a(N-2)\int_{\mathbf{R}^N_+} |U_\varepsilon|^{2^*}\,dx + \frac{b}{2(N-1)}\int_{\mathbf{R}^{N-1}} |U_\varepsilon(x',0)|^q\,dx'. \tag{2.5}$$

We first compute the last integral in (2.5).

$$\begin{aligned}\int_{\mathbf{R}^{N-1}} |U_\varepsilon(x',0)|^q\,dx' &= a^{-\frac{N-1}{2}}\varepsilon^{N-1}\int_{\mathbf{R}^{N-1}} \frac{dx'}{(\varepsilon^2+(x_N+\varepsilon\mu(N-2)^{-1}+|x'|^2)^{N-1}} \\ &= a^{-\frac{N-1}{2}}\left(1+\mu^2(N-2)^{-2}\right)^{-\frac{N-1}{2}}\int_{\mathbf{R}^{N-1}} \frac{dy'}{(1+|y'|^2)^{N-1}} \\ &= a^{-\frac{N-1}{2}}\left(1+\mu^2(N-2)^{-2}\right)^{-\frac{N-1}{2}}\Pi. \end{aligned} \tag{2.6}$$

We next compute the first integral in (2.5)

$$\int_{\mathbf{R}^N_+} |U_\varepsilon|^{2^*}\,dx = a^{-\frac{N}{2}}\int_0^\infty\int_{\mathbf{R}^{N-1}} \frac{\varepsilon^N\,dx'\,dx_N}{(\varepsilon^2+(x_N+\varepsilon\mu(N-2)^{-1})^2+|x'|^2)^N}.$$

To proceed we use the following identity which is easily proved using polar coordinates followed by an integration by parts:

$$\int_{\mathbf{R}^{N-1}} \frac{dx'}{(c+|x'|^2)^N} = \frac{1}{2}c^{-\frac{N+1}{2}}\int_{\mathbf{R}^{N-1}} \frac{dz'}{(1+|z'|^2)^{N-1}} = \frac{\Pi}{2}c^{-\frac{N+1}{2}}, \qquad c>0.$$

Using this identity with $c = \varepsilon^2 + (x_N + \varepsilon\mu(N-2)^{-1})^2 > 0$ we calculate

$$\int_{\mathbf{R}^N_+} |U_\varepsilon|^{2^*}\,dx = \frac{\Pi}{2}a^{-\frac{N}{2}}\int_{\frac{\mu}{N-2}}^{\infty} \frac{dt}{(1+t^2)^{\frac{N+1}{2}}}. \tag{2.7}$$

Relation (2.1) then follows from (2.5), (2.6) and (2.7). Finally, differentiating $K(\mu)$ we find that

$$K'(\mu) = -\frac{N-2}{N-1}\left(1+\frac{\mu^2}{(N-2)^2}\right)^{-\frac{N-1}{2}} < 0,$$

which proves its monotonicity. An easy calculation shows that

$$K(-\infty) = 2K(0), \qquad K(0) = \frac{2^{N-2}(N-2)\Gamma^2\left(\frac{N}{2}\right)}{\Gamma(N)}, \qquad K(\infty) = 0. \tag{2.8}$$

///

We next prove a variation of Lemma 2.1. Let $D \subset \mathbf{R}^N$ be a bounded domain with a smooth boundary such that $\partial\mathbf{R}^N_+ \cap D \neq \emptyset$ we denote by $V^1_0(\mathbf{R}^N_+, D)$ the Sobolev space defined by

$$V^1_0(\mathbf{R}^N_+, D) = \{u \in H^1(\mathbf{R}^N_+ \cap D);\ u = 0 \quad \text{on} \quad \mathbf{R}^N_+ \cap \partial D\}.$$

For $u \in V^1_0(\mathbf{R}^N_+, D)$ we set

$$J_{\mathbf{R}^N_+, D;a,b}(u) = \frac{1}{2}\int_{\mathbf{R}^N_+} |\nabla u|^2\, dx - \frac{aN(N-2)}{2^*}\int_{\mathbf{R}^N_+} |u|^{2^*}\, dx - \frac{b}{q}\int_{\partial\mathbf{R}^N_+ \cap D} |u|^q\, dx',$$

and

$$M_{\mathbf{R}^N_+, D;a,b} = \{u \in V^1_0(\mathbf{R}^N_+, D);\ u \not\equiv 0, \quad < J'_{\mathbf{R}^N_+, D;a,b}(u), u >= 0\}.$$

We then have

Lemma 2.2 *Let $D \subset \mathbf{R}^N$ be a bounded domain with a smooth boundary such that $\partial\mathbf{R}^N_+ \cap D \neq \emptyset$. Then*

$$\inf_{u \in M_{\mathbf{R}^N_+, D;a,b}} J_{\mathbf{R}^N_+, D;a,b}(u) = \frac{\Pi}{2} a^{-\frac{N-2}{2}} K\left(\frac{b}{\sqrt{a}}\right)$$

Proof: By standard arguments we have

$$\inf_{u \in M_{\mathbf{R}^N_+, D;a,b}} J_{\mathbf{R}^N_+, D;a,b}(u) = \inf_{u \in V^1_0(\mathbf{R}^N_+, D)} \max_{t \geq 0} J_{\mathbf{R}^N_+, D;a,b}(tu).$$

Given any such domain D, it is clear that there exist balls B_r and B_R, of radii r, R respectively, centered at $x_0 \in \partial\mathbf{R}^N_+$ such that $\mathbf{R}^N_+ \cap B_r \subset \mathbf{R}^N_+ \cap D \subset \mathbf{R}^N_+ \cap B_R$. It follows easily that $V^1_0(\mathbf{R}^N_+, B_r) \subset V^1_0(\mathbf{R}^N_+, D) \subset V^1_0(\mathbf{R}^N_+, B_R)$ and

$$\inf_{u \in M_{\mathbf{R}^N_+, B_r;a,b}} J_{\mathbf{R}^N_+, B_r;a,b}(u) \geq \inf_{u \in M_{\mathbf{R}^N_+, D;a,b}} J_{\mathbf{R}^N_+, D;a,b}(u) \geq \inf_{u \in M_{\mathbf{R}^N_+, B_R;a,b}} J_{\mathbf{R}^N_+, B_R;a,b}(u).$$

By a simple scaling argument

$$\inf_{u \in M_{\mathbf{R}^N_+, B_r;a,b}} J_{\mathbf{R}^N_+, B_r;a,b}(u) = \inf_{u \in M_{\mathbf{R}^N_+, B_R;a,b}} J_{\mathbf{R}^N_+, B_R;a,b}(u).$$

Indeed, if $u(x)$ is a test function for the first infimum, then $u_\lambda(x) = \lambda^{\frac{N-2}{2}} u(\lambda x)$ is a test function for the second infimum, with $\lambda = r/R$, and both functionals take on the same value. The result then follows from Lemma 2.1. ///

We next establish an inequality which will allow us to control the concentration on $\partial\Omega$ of Palais–Smale sequences for $\mathcal{J}_\lambda$ (see (1.6)).

First, we introduce some notation analogous to the case with constant coefficients. If $\Sigma \subset \mathbf{R}^N$ and $D \subset \mathbf{R}^N$ are domains with smooth boundaries such that $\partial\Sigma \cap D \neq \emptyset$ we denote by $V_0^1(\Sigma, D)$ the Sobolev space defined by

$$V_0^1(\Sigma, D) = \{u \in H^1(\Sigma \cap D);\ u = 0 \quad \text{on} \quad \Sigma \cap \partial D\}.$$

For $u \in V_0^1(\Sigma, D)$ we set

$$\mathcal{J}_{\Sigma,D;\lambda}(u) = \frac{1}{2}\int_\Sigma (|\nabla u|^2 + \lambda u^2)\,dx - \frac{N(N-2)}{2^*}\int_\Sigma Q(x)|u|^{2^*}\,dx - \frac{1}{q}\int_{\partial\Sigma\cap D} P(x)|u|^q\,dS_x. \tag{2.9}$$

In the special case $\lambda = 0$, we write $\mathcal{J}_{\Sigma,D;0}(u) = \mathcal{J}_{\Sigma,D}(u)$. We also set

$$\mathcal{M}_{\Sigma,D;\lambda} = \{u \in V_0^1(\Sigma, D);\ u \not\equiv 0, \quad < \mathcal{J}'_{\Sigma,D;\lambda}(u), u >= 0\}, \qquad \mathcal{M}_{\Sigma,D} = \mathcal{M}_{\Sigma,D;0}. \tag{2.10}$$

We finally set

$$\mathcal{C}(x) := \frac{\Pi}{2}(Q(x))^{-\frac{N-2}{2}} K\left(\frac{P(x)}{\sqrt{Q(x)}}\right). \tag{2.11}$$

We then have

Proposition 2.3 *Let $x \in \partial\Omega$ and denote by $B_r(x)$ the ball of radius r centered at x. Then*

$$\lim_{r\downarrow 0} \inf_{u\in\mathcal{M}_{\Omega,B_r(x);\lambda}} \mathcal{J}_{\Omega,B_r(x);\lambda}(u) = \mathcal{C}(x), \tag{2.12}$$

where $\mathcal{J}_{\Omega,B_r(x);\lambda}$, $\mathcal{M}_{\Omega,B_r(x);\lambda}$ and $\mathcal{C}(x)$ are defined in (2.9), (2.10) and (2.11), respectively.

Proof: We will divide the proof into various steps. Let us fix an $x_0 \in \partial\Omega$.

Step 1: As usual we have that

$$\inf_{u\in\mathcal{M}_{\Omega,B_r(x_0);\lambda}} \mathcal{J}_{\Omega,B_r(x_0);\lambda}(u) = \inf_{u\in V_0^1(\Omega,B_r(x_0))} \max_{t\geq 0} \mathcal{J}_{\Omega,B_r(x_0);\lambda}(tu). \tag{2.13}$$

Clearly,

$$f(t) := \mathcal{J}_{\Omega,B_r(x_0);\lambda}(tu) = \frac{\alpha}{2}t^2 - \frac{\beta N(N-2)}{2^*}t^{\frac{2N}{N-2}} - \frac{\gamma}{q}t^{\frac{2(N-1)}{N-2}}.$$

with

$$\alpha = \int_{\Omega\cap B_r(x_0)} (|\nabla u|^2 + \lambda u^2)dx, \quad \beta = \int_{\Omega\cap B_r(x_0)} Q(x)|u|^{2^*}dx, \quad \gamma = \int_{\partial\Omega\cap B_r(x_0)} P(x)|u|^q dS_x. \tag{2.14}$$

An elementary analysis shows that $f(0) = 0$, $f(+\infty) = -\infty$ and that $f(t)$ has a unique positive maximum at a point $t_M = t_M(\alpha, \beta, \gamma)$. We denote the maximum value of $f(t)$ by $\phi(\alpha, \beta, \gamma) := f(t_M(\alpha, \beta, \gamma))$. Hence,

$$\max_{t\geq 0} \mathcal{J}_{\Omega, B_r(x);\lambda}(tu) = \phi(\alpha, \beta, \gamma), \tag{2.15}$$

with α, β, γ as defined in (2.14). It is easy to check that the function ϕ is increasing in α and decreasing in β and γ.

Step 2: Since u is zero on $\Omega \cap \partial B_r(x_0)$ we have by the Poincaré inequality that

$$\int_{\Omega\cap B_r(x_0)} u^2 dx \leq cr^2 \int_{\Omega\cap B_r(x_0)} |\nabla u|^2 dx,$$

for some constant $c > 0$ independent of r. Also, from the continuity of P and Q we have $Q(x_0) - \varepsilon(r) \leq Q(x) \leq Q(x_0) + \varepsilon(r)$ for $x \in \Omega \cap B_r(x)$ and $|P(x) - P(x_0)| \leq \varepsilon(r)$ for $x \in \partial\Omega \cap B_r(x)$, with $\varepsilon(r) \to 0$ as $r \to 0$. In view of these and the monotonicity properties of ϕ, we have

$$\phi(\alpha^-, \beta^-, \gamma^-) \leq \phi(\alpha, \beta, \gamma) \leq \phi(\alpha^+, \beta^+, \gamma^+), \tag{2.16}$$

with

$$\alpha^\pm = (1 \pm cr^2) \int_{\Omega\cap B_r(x)} |\nabla u|^2 dx, \qquad \beta^\pm = (Q(x_0) \mp \varepsilon(r)) \int_{\Omega\cap B_r(x)} |u|^{2^*} dx,$$

$$\gamma^\pm = (P(x_0) \mp \varepsilon(r)) \int_{\partial\Omega\cap B_r(x)} |u|^q dS_x.$$

Step 3: To relate $\Omega \cap B_r(x_0)$ for small r with the half space, we use a change of variables that straightens the boundary. We may assume for convenience that $x_0 = 0$ and that the part $B(0, r) \cap \partial\Omega$ of the boundary is given by $h(x') = \frac{1}{2}\sum_{i=1}^{N-1} a_i x_i^2 + o(|x'|^2)$ for $|x'| < r$, where a_i, $i = 1, \ldots, N-1$, denote the principal curvatures of $\partial\Omega$ at 0. Let T be a transformation $y' = x'$, $y_N = x_N - h(x')$, which is smooth and invertible. We denote by $v(y)$ and by $\tilde{B}_r$ the images of B_r and u under T, respectively. After some standard calculations we find that

$$\phi(\tilde{\alpha}^-, \tilde{\beta}^-, \tilde{\gamma}^-) \leq \phi(\alpha, \beta, \gamma) \leq \phi(\tilde{\alpha}^+, \tilde{\beta}^+, \tilde{\gamma}^+), \tag{2.17}$$

with

$$\tilde{\alpha}^\pm = (1 \pm \tilde{\varepsilon}(r)) \int_{\mathbf{R}^N_+\cap \tilde{B}_r} |\nabla v|^2 dy, \qquad \tilde{\beta}^\pm = (Q(x_0) \mp \tilde{\varepsilon}(r)) \int_{\mathbf{R}^N_+\cap \tilde{B}_r} |v|^{2^*} dy,$$

$$\tilde{\gamma}^\pm = (P(x_0) \mp \tilde{\varepsilon}(r)) \int_{\partial\mathbf{R}^N_+\cap \tilde{B}_r} |v|^q dy'.$$

Here $\tilde{\varepsilon}(r)$ is some positive function such that $\tilde{\varepsilon}(r) \to 0$ as $r \to 0$. Consequently,

$$\inf_{v\in V_0^1(\mathbf{R}^N_+, \tilde{B}_r)} \phi(\tilde{\alpha}^-, \tilde{\beta}^-, \tilde{\gamma}^-) \leq \inf_{u\in V_0^1(\Omega, B_r(x_0))} \phi(\alpha, \beta, \gamma) \leq \inf_{v\in V_0^1(\mathbf{R}^N_+, \tilde{B}_r)} \phi(\tilde{\alpha}^+, \tilde{\beta}^+, \tilde{\gamma}^+).$$

Step 4: To complete the proof we will show that

$$\lim_{r\downarrow 0}\inf_{v\in V_0^1(\mathbf{R}^N_+,\tilde{B}_r)}\phi(\tilde{\alpha}^+,\tilde{\beta}^+,\tilde{\gamma}^+)=\mathcal{C}(x_0), \tag{2.18}$$

and similarly for $\phi(\tilde{\alpha}^-,\tilde{\beta}^-,\tilde{\gamma}^-)$.

We let $(1+\tilde{\varepsilon}(r))^{1/2}v(y)=w(y)$. Then

$$\tilde{\alpha}^+=\int_{\mathbf{R}^N_+\cap\tilde{B}_r}|\nabla w|^2dy,\qquad \tilde{\beta}^+=(Q(x_0)-\tilde{\varepsilon}(r))(1+\tilde{\varepsilon}(r))^{2^*/2}\int_{\mathbf{R}^N_+\cap\tilde{B}_r}|w|^{2^*}dy,$$

$$\tilde{\gamma}^+=(P(x_0)-\tilde{\varepsilon}(r))(1+\tilde{\varepsilon}(r))^{q/2}\int_{\partial\mathbf{R}^N_+\cap\tilde{B}_r}|w|^q dy'. \tag{2.19}$$

Then, using Lemma 2.2 with $D=\tilde{B}_r$, $a=(Q(x_0)-\tilde{\varepsilon}(r))(1+\tilde{\varepsilon}(r))^{2^*/2}$ and $b=(P(x_0)+\tilde{\varepsilon}(r))(1+\tilde{\varepsilon}(r))^{q/2}$ we get

$$\begin{aligned}\inf_{v\in V_0^1(\mathbf{R}^N_+,\tilde{B}_r)}\phi(\tilde{\alpha}^+,\tilde{\beta}^+,\tilde{\gamma}^+) &= \inf_{w\in V_0^1(\mathbf{R}^N_+,\tilde{B}_r)}\phi(\tilde{\alpha}^+,\tilde{\beta}^+,\tilde{\gamma}^+)\\ &= \inf_{w\in V_0^1(\mathbf{R}^N_+,\tilde{B}_r)}\max_{t\geq 0}J_{\mathbf{R}^N_+,\tilde{B}_r;a,b}(tw)\\ &= \inf_{w\in M_{\mathbf{R}^N_+,\tilde{B}_r;a,b}}J_{\mathbf{R}^N_+,\tilde{B}_r;a,b}(w)\\ &= \frac{\Pi}{2}a^{-\frac{N-2}{2}}K\left(\frac{b}{\sqrt{a}}\right).\end{aligned}$$

Taking now the limit as $r\to 0$ and noting that $a\to Q(x_0)$ and $b\to P(x_0)$, (2.18) follows and this completes the proof. ///

Let S denote the best constant for the for the critical Sobolev imbedding in $\mathbf{R}^N$, $N\geq 3$, that is

$$S=\inf\left\{\frac{\int_{\mathbf{R}^N}|\nabla u|^2}{\int_{\mathbf{R}^N}|u|^{2^*}};\quad u\not\equiv 0,\quad u\in L^{2^*}(\mathbf{R}^N),\quad \nabla u\in L^2(\mathbf{R}^N)\right\}.$$

It is known from [24] that

$$S=\pi N(N-2)\left(\frac{\Gamma(\frac{N}{2})}{\Gamma(N)}\right)^{\frac{2}{N}}. \tag{2.20}$$

We now prove the main existence Theorem. We set

$$Q_M:=\max_{x\in\bar{\Omega}}Q(x),$$

and

$$S_\infty:=\min\Big(\inf_{x\in\partial\Omega}\mathcal{C}(x),\ \frac{S^{\frac{N}{2}}}{N(N(N-2)Q_M)^{\frac{N-2}{2}}}\Big). \tag{2.21}$$

We then have

Theorem 2.4 *If for some positive constant λ,*

$$c_\lambda = \inf_{u \in \mathcal{M}_\lambda} \mathcal{J}_\lambda(u) < S_\infty, \tag{2.22}$$

then c_λ is achieved and in particular problem (1.1) has a solution.

Proof: <u>*Step 1:*</u> (Positivity of $c_\lambda > 0$ and boundedness of the minimizing sequence). First we check that $c_\lambda > 0$. The Sobolev space $H^1(\Omega)$ is equipped with the norm

$$\|u\|^2 = \int_\Omega (|\nabla u|^2 + \lambda u^2)\, dx, \qquad \lambda > 0.$$

Let $u \in \mathcal{M}_\lambda$. Then by the Sobolev inequalities

$$\|u\|^2 = N(N-2) \int_\Omega Q(x)|u(x)|^{2^*}\, dx + \int_{\partial\Omega} P(x)|u|^q\, dS_x \le C(\|u\|^{2^*} + \|u\|^q).$$

This implies that there exists $\delta > 0$ such that

$$\|u\| \ge \delta \ \text{ for every } u \in \mathcal{M}_\lambda.$$

Therefore for $u \in \mathcal{M}_\lambda$ we have

$$\begin{aligned} \mathcal{J}_\lambda(u) &= \mathcal{J}_\lambda(u) - \frac{1}{q} < \mathcal{J}'_\lambda(u),\ u > \\ &= \frac{1}{2(N-1)}\|u\|^2 + \frac{(N-2)^2}{2(N-1)} \int_\Omega Q(x)|u|^{2^*}\, dx \ge \frac{\delta^2}{2(N-1)} \end{aligned}$$

and hence $c_\lambda > 0$. Let $\{u_m\}$ be a minimizing sequence for c_λ. By standard arguments (see e.g., Theorem 2.2 in [9]) we have that

$$\mathcal{J}_\lambda(u_m) \to c_\lambda, \qquad \text{and} \qquad \mathcal{J}'_\lambda(u_m) \to 0 \quad \text{in} \quad H^{-1}(\Omega). \tag{2.23}$$

We claim that the sequence $\{u_m\}$ is bounded in $H^1(\Omega)$. To see this, we first note that

$$\begin{aligned} \mathcal{J}_\lambda(u_m) &- \frac{1}{q} < \mathcal{J}'_\lambda(u_m),\ u_m > \\ &= \frac{1}{2(N-1)}\|u_m\|^2 + \frac{(N-2)^2}{2(N-1)} \int_\Omega Q|u_m|^{2^*} dx. \end{aligned} \tag{2.24}$$

It then follows from (2.23), (2.24) that (as $m \to \infty$)

$$\frac{1}{2(N-1)}\|u_m\|^2 + \frac{(N-2)^2}{2(N-1)} \int_\Omega Q|u_m|^{2^*} dx \le c_\lambda + \frac{1}{q}\|\mathcal{J}'_\lambda(u_m)\|_{H^{-1}(\Omega)}\|u_m\| + o(1). \tag{2.25}$$

Since $Q > 0$, we necessarily have that $\|u_m\| \le C$, as claimed. As a consequence we obtain

$$\frac{1}{2(N-1)}\|u_m\|^2 + \frac{(N-2)^2}{2(N-1)} \int_\Omega Q|u_m|^{2^*} dx = c_\lambda + o(1). \tag{2.26}$$

Step 2: (Concentration–compactness properties of minimizing sequence). Since $\{u_m\}$ is bounded in $H^1(\Omega)$, passing to a subsequence (still denoted by $\{u_m\}$), we have, for some $u \in H^1(\Omega)$,

$$\begin{aligned} u_m &\rightharpoonup u \text{ weakly in } H^1(\Omega), \\ u_m &\to u \text{ a.e. in } \Omega, \\ u_m &\to u \text{ in } L^p(\Omega), \quad \text{for } 2 \le p < 2^*, \\ u_m &\to u \text{ in } L^r(\partial\Omega), \quad \text{for } 2 \le r < q. \end{aligned}$$

By the concentration–compactness principle ([21]) there also exists an at most countable set of points $\{x_j\}$, $j \in J$, such that the following convergence results (in the sense of measures) hold

$$\begin{aligned} |u_m|^{2^*} dx &\rightharpoonup d\nu = |u|^{2^*} dx + \sum_{j\in J} \delta_{x_j}\nu_j, \qquad x_j \in \bar{\Omega}, \\ |u_m|^q dS_x &\rightharpoonup d\bar{\nu} = |u|^q dS_x + \sum_{j\in J} \delta_{x_j}\bar{\nu}_j, \qquad x_j \in \partial\Omega, \\ |\nabla u_m|^2 dx &\rightharpoonup d\mu \ge |\nabla u|^2 dx + \sum_{j\in J} \delta_{x_j}\mu_j, \qquad x_j \in \bar{\Omega}. \end{aligned}$$

Here δ_{x_j} denotes the Dirac mass centered at x_j and ν_j, $\bar{\nu}_j$ and μ_j are nonnegative constants. In addition, the following inequality holds

$$S\,\nu_j^{2/2^*} \le \mu_j, \qquad \text{if} \qquad x_j \in \Omega, \tag{2.27}$$

with S as in (2.20).

We next derive some relations between ν_j, $\bar{\nu}_j$ and μ_j. Let $\phi_\delta(x)$ be a smooth cutoff function such that $0 \le \phi_\delta(x) \le 1$, $\phi_\delta(x) = 1$ for $|x-x_j| < \delta$ and $\phi_\delta(x) = 0$ for $|x-x_j| > 2\delta$.

Since $\mathcal{J}'_\lambda(u_m) \to 0$ in $H^{-1}(\Omega)$ and $\|u_m\| < C$, it follows that $< \mathcal{J}'_\lambda(u_m), u_m\phi_\delta^2 > \to 0$, that is, as $m \to \infty$,

$$\begin{aligned} \int_\Omega (|\nabla u_m|^2\phi_\delta^2 + 2u_m\phi_\delta\nabla u_m\cdot\nabla\phi_\delta + \lambda u_m^2\phi_\delta^2)\,dx &= \\ = N(N-2)\int_\Omega Q(x)|u_m|^{2^*}\phi_\delta^2\,dx \;+\; \int_{\partial\Omega} P(x)|u_m|^q\phi_\delta^2\,dS_x + o(1). \end{aligned} \tag{2.28}$$

Using the previous convergence results and letting $m \to \infty$ first, and then $\delta \to 0$ we get

$$\mu_j = N(N-2)Q(x_j)\nu_j, \qquad \text{if} \quad x_j \in \Omega \tag{2.29}$$

$$\mu_j = N(N-2)Q(x_j)\nu_j + P(x_j)\bar{\nu}_j, \quad \text{if} \quad x_j \in \partial\Omega. \tag{2.30}$$

Step 3: Here we will show that no concentration of $\{u_m\}$ occurs. That is, we will show that $\nu_j = \bar{\nu}_j = \mu_j = 0$, for all $j \in J$.

Passing to the limit ($m \to \infty$) in (2.26) and using the convergence properties of $\{u_m\}$ we get

$$\begin{aligned} c_\lambda \;\ge\; & \frac{1}{2(N-1)}\int_\Omega (|\nabla u|^2 + \lambda u^2)dx + \frac{(N-2)^2}{2(N-1)}\int_\Omega Q(x)|u|^{2^*}dx \\ & + \frac{1}{2(N-1)}\sum_{x_j\in\bar{\Omega}}\mu_j + \frac{(N-2)^2}{2(N-1)}\sum_{x_j\in\bar{\Omega}} Q(x_j)\nu_j. \end{aligned} \tag{2.31}$$

We first show that there is no concentration at interior points. Assuming that $\nu_k > 0$ for some $x_k \in \Omega$ we will reach a contradiction. From (2.31) we get that

$$c_\lambda \geq \frac{1}{2(N-1)}\mu_k + \frac{(N-2)^2}{2(N-1)}Q(x_k)\nu_k. \tag{2.32}$$

From (2.29) and (2.32) we get that

$$c_\lambda \geq \frac{1}{N}\mu_k. \tag{2.33}$$

On the other, hand from (2.22) and (2.29) we have

$$c_\lambda < \frac{S^{\frac{N}{2}}}{N(Q(x_k)N(N-2))^{\frac{N-2}{2}}} = \frac{1}{N}S^{\frac{N}{2}}\left(\frac{\nu_k}{\mu_k}\right)^{\frac{N-2}{2}}. \tag{2.34}$$

From (2.33) and (2.34) we obtain $S\nu_k^{2/2^*} > \mu_k$, which contradicts (2.27). Hence, $\mu_j = \nu_j = 0$ for all $x_j \in \Omega$ and the concentration at interior points is excluded.

We next assume that a concentration occurs at a boundary point $x_k \in \partial\Omega$. Let $B_r(x_k)$ be a ball centered at x_k of radius r, with r sufficiently small, and $\Omega_r = \Omega \cap B_r(x_k)$. Also let ϕ_δ, $2\delta < r$, be the test function used before, centered at x_k.

It is clear that for each m and r there exist constants $t_{m,r} > 0$, depending also on δ, such that

$$\begin{aligned}\int_{\Omega_r}(|\nabla(u_m\phi_\delta)|^2 + \lambda(u_m\phi_\delta)^2)dx \quad &= \\ = t_{m,r}^{2^*-2}N(N-2)\int_{\Omega_r}Q|u_m|^{2^*}\phi_\delta^{2^*}dx \quad &+ \quad t_{m,r}^{q-2}\int_{\partial\Omega\cap B_r(x_k)}P|u_m|^q\phi_\delta^q dS_x.\end{aligned} \tag{2.35}$$

In the notation of Proposition 2.3, (2.35) is equivalent to $t_{m,r}u_m\phi_\delta \in \mathcal{M}_{\Omega,B_r(x_k);\lambda}$. We claim that there exist constants A and B such that $0 < A < t_{m,r} < B$. Indeed, if we let $m \to \infty$ and then $r \to 0$, the left hand side of (2.35) tends to a constant $M_k \geq \mu_k > 0$, whereas the integrals in the right hand side remain bounded. Hence, $t_{m,r}$ cannot approach to zero. Since $2^* - 2 > q - 2 > 0$ and Q is positive, $t_{m,r}$ cannot escape to infinity either and the claim is proved.

We may therefore assume that $t_{m,r} \to \bar{t}$, $0 < \bar{t} < \infty$, as $m \to \infty$ and $r \to 0$. Taking the limits in (2.35) we get

$$\mu_k = \bar{t}^{2^*-2}N(N-2)Q(x_k)\nu_k + \bar{t}^{q-2}P(x_k)\bar{\nu}_k. \tag{2.36}$$

Combining this with (2.30), we conclude that $\bar{t} = 1$.

Since $t_{m,r}u_m\phi_\delta \in \mathcal{M}_{\Omega,B_r(x);\lambda}$, it follows from Proposition 2.3, that as $r \to 0$,

$$\mathcal{J}_{\Omega,B_r(x_k);\lambda}(t_{m,r}u_m\phi_\delta) \geq \inf_{u\in\mathcal{M}_{\Omega,B_r(x_k);\lambda}} \mathcal{J}_{\Omega,B_r(x_k);\lambda}(u) = \mathcal{C}(x_k) + o(1). \tag{2.37}$$

Letting first $m \to \infty$ and then $r \to 0$ in (2.37) (see (2.9) for the definition of $\mathcal{J}_{\Omega,B_r(x_k);\lambda}$, and using the convergence properties of $\{u_m\}$ and $\{t_{m,r}\}$ we get:

$$\frac{1}{2}\mu_k - \frac{N(N-2)}{2^*}Q(x_k)\nu_k - \frac{1}{q}P(x_k)\bar{\nu}_k \geq \mathcal{C}(x_k). \tag{2.38}$$

Replacing $P(x_k)\bar{\nu}_k$ from (2.36) (with $\bar{t}=1$, there) in (2.38), we get that

$$\frac{1}{2(N-1)}\mu_k + \frac{(N-2)^2}{2(N-1)}Q(x_k)\nu_k \geq \mathcal{C}(x_k). \tag{2.39}$$

It then follows from (2.31) that $c_\lambda \geq \mathcal{C}(x_k)$, which contradicts (2.22). Hence the concentration at boundary points is also excluded and $\nu_j = \bar{\nu}_j = \mu_j = 0$, for all $j \in J$.
<u>*Step 4:*</u> (Completion of the proof.) We have that

$$\begin{aligned} c_\lambda + o(1) &= \mathcal{J}_\lambda(u_m) - \frac{1}{2} < \mathcal{J}'_\lambda(u_m),\ u_m > \\ &= (N-2)\int_\Omega Q|u_m|^{2^*}dx + \frac{1}{2(N-1)}\int_{\partial\Omega} P|u_m|^q dS_x. \end{aligned} \tag{2.40}$$

Since $\nu_j = \bar{\nu}_j = 0$, both terms in the right hand side of (2.40) converge strongly, and

$$c_\lambda = (N-2)\int_\Omega Q|u|^{2^*}dx + \frac{1}{2(N-1)}\int_{\partial\Omega} P|u|^q dS_x.$$

Recalling that c_λ is strictly positive we conclude that $u \not\equiv 0$. On the other hand, we have $< \mathcal{J}'_\lambda(u_m),\ u > \to 0$, whence $< \mathcal{J}'_\lambda(u),\ u >= 0$. Consequently u is on the Nehari manifold, that is $u \in \mathcal{M}_\lambda$. It then follows easily from (2.31) (with $\mu_j = \nu_j = 0$ there) that u is a minimizer for c_λ and $\{u_m\}$ converges strongly in $H^1(\Omega)$ to u.

Clearly, $|u|$ is also a minimizer and therefore we may assume that $u \geq 0$, whereas by regularity theory it is a classical solution. By the maximum principle $u > 0$ in $\bar{\Omega}$. This completes the proof of the Theorem. ///

Remark If $P(x) \equiv 0$ on $\partial\Omega$, then

$$\begin{aligned} \mathcal{C}(x) &= \frac{1}{2}\Pi K(0)Q(x)^{-\frac{N-2}{2}} = \frac{1}{2}(N-2)\pi^{\frac{N}{2}}\frac{\Gamma(\frac{N}{2})}{\Gamma(N)}Q(x)^{-\frac{N-2}{2}} \\ &= \frac{1}{2}\frac{S^{\frac{N}{2}}}{N^{\frac{N}{2}}(N-2)^{\frac{N-2}{2}}}Q(x)^{-\frac{N-2}{2}} = \frac{S^{\frac{N}{2}}}{2N(N(N-2)Q(x))^{\frac{N-2}{2}}}. \end{aligned}$$

Hence

$$\inf_{x\in\partial\Omega}\mathcal{C}(x) = \frac{S^{\frac{N}{2}}}{2N(N(N-2)Q_m)^{\frac{N-2}{2}}},$$

where $Q_m = \min_{x\in\partial\Omega} Q(x)$. In this case the condition (2.22) takes the form

$$c_\lambda = \inf_{u\in\mathcal{M}_\lambda}\mathcal{J}_\lambda(u) < \min\Big(\frac{S^{\frac{N}{2}}}{2N(N(N-2)Q_m)^{\frac{N-2}{2}}}, \frac{S^{\frac{N}{2}}}{N(N(N-2)Q_M)^{\frac{N-2}{2}}}\Big),$$

and we recover the result from the paper [11].

In Theorem 2.5, below, we examine the dependence of c_λ on λ.

Theorem 2.5 *For $\lambda > 0$, c_λ is a nondecreasing function such that*

$$0 < c_\lambda \leq S_\infty,$$

and in addition,

$$\lim_{\lambda\to\infty} c_\lambda = S_\infty.$$

Proof: Let $0 < \lambda_1 < \lambda_2$ and $u \in \mathcal{M}_{\lambda_2}$. Then, there exists $s \in (0,1)$ such that $su \in \mathcal{M}_{\lambda_1}$. We then have

$$\begin{aligned} c_{\lambda_1} &\leq \mathcal{J}_{\lambda_1}(su) = \frac{s^2}{2(N-1)} \int_\Omega \left(|\nabla u|^2 + \lambda_1 u^2 \right) dx + \frac{(N-2)^2 s^{2^*}}{2(N-1)} \int_\Omega Q(x)|u|^{2^*} dx \\ &\leq \mathcal{J}_{\lambda_1}(u) \leq \mathcal{J}_{\lambda_2}(u). \end{aligned}$$

Since this holds for every $u \in \mathcal{M}_{\lambda_2}$, we get $c_{\lambda_1} \leq c_{\lambda_2}$.

To establish the second part of our assertion we argue by contradiction. Let $\{\lambda_m\}$ be an increasing sequence of positive numbers with $\lim_{m\to\infty} \lambda_m = \infty$. Assume that $\lim_{m\to\infty} c_{\lambda_m} < S_\infty$. Then for every λ_m there exists a least energy solution $u_m = u_{\lambda_m}$ of (1.1) such that $c_{\lambda_m} = \mathcal{J}_{\lambda_m}(u_m) < S_\infty$ and $\mathcal{J}'_{\lambda_m}(u_m) = 0$. Then, the sequence $\{u_m\}$ is bounded in $H^1(\Omega)$ and we may assume that $u_m \rightharpoonup u$ in $H^1(\Omega)$. Also, $\int_\Omega u_m^2\, dx = O(\frac{1}{\lambda_m})$, and therefore $u_m \to 0$ in $L^2(\Omega)$ and $u_m \rightharpoonup 0$ in $H^1(\Omega)$.

The rest of the proof is quite similar to the proof of Theorem 2.4 with the following modifications. Relation (2.28) remains the same, but relations (2.29) and (2.30) are true as inequalities:

$$\begin{aligned} \mu_j &\leq N(N-2)Q(x_j)\nu_j, & \text{if} \quad x_j \in \Omega, \\ \mu_j &\leq N(N-2)Q(x_j)\nu_j + P(x_j)\bar{\nu}_j, & \text{if} \quad x_j \in \partial\Omega. \end{aligned}$$

This difference stems from the fact that we have no longer "good" control on the term $\lambda_m \int u_m^2$. The rest of the argument that excludes concentration at interior points remains the same.

To exclude concentration at a boundary point, we first note that (2.35) remains valid whereas (2.36) becomes the inequality,

$$\mu_k \leq \bar{t}^{2^*-2} N(N-2)Q(x_k)\nu_k + \bar{t}^{q-2} P(x_k)\bar{\nu}_k.$$

To establish that $\bar{t} = 1$, we subtract (2.28) from (2.35) (in order to kill the bad term) and then pass to the limits to arrive at

$$0 \leq N(N-2)Q(x_k)\nu_k + P(x_k)\bar{\nu}_k = \bar{t}^{2^*-2} N(N-2)Q(x_k)\nu_k + \bar{t}^{q-2} P(x_k)\bar{\nu}_k,$$

from which it follows that $\bar{t} = 1$.

For $u \in \mathcal{M}_\lambda$, we will use the following expression for the functional $\mathcal{J}_\lambda(u)$:

$$\mathcal{J}_\lambda(u) = (N-2) \int_\Omega Q|u|^{2^*} dx + \frac{1}{2(N-1)} \int_{\partial\Omega} P|u|^q dS_x. \tag{2.41}$$

Starting from $c_{\lambda_m} = \mathcal{J}_{\lambda_m}(u_m)$ and taking the limit $m \to \infty$, we conclude that

$$\mathcal{C}(x_k) \geq S_\infty > \lim_{m\to\infty} c_{\lambda_m} = (N-2) \sum_{x_j \in \partial\Omega} Q(x_j)\nu_j + \frac{1}{2(N-1)} \sum_{x_j \in \partial\Omega} P(x_j)\bar{\nu}_j. \tag{2.42}$$

Using the monotonicity of $\inf_{u \in \mathcal{M}_{\Omega, B_r(x_k);\lambda}} \mathcal{J}_{\Omega, B_r(x_k);\lambda}(u)$ with respect to λ, we obtain the analogue of (2.37) for large m such that $\lambda_m \geq \lambda$,

$$\mathcal{J}_{\Omega, B_r(x_k);\lambda_m}(t_{m,r} u_m \phi_\delta) \geq \inf_{u \in \mathcal{M}_{\Omega, B_r(x_k);\lambda}} \mathcal{J}_{\Omega, B_r(x_k);\lambda}(u) = \mathcal{C}(x_k) + o(1).$$

Taking now the limits, $m \to \infty$ and $r \to 0$ we get

$$(N-2)Q(x_k)\nu_k + \frac{1}{2(N-1)}P(x_k)\bar{\nu}_k \geq \mathcal{C}(x_k),$$

which contradicts (2.42). Thus, concentration at the boundary is also excluded.

It follows that $\{u_m\}$ converges strongly in $H^1(\Omega)$ to zero. This in its turn implies that $c_{\lambda_m} \to 0$, as $\lambda_m \to \infty$, contradicting the fact that c_λ is positive and nondecreasing. ///

3 Existence for $\lambda > 0$

This Section is devoted to the verification of the inequality (2.22) of Theorem 2.4. First we consider the case where

$$\min_{x\in\partial\Omega} \mathcal{C}(x) \leq \frac{S^{\frac{N}{2}}}{N(Q_M N(N-2))^{\frac{N-2}{2}}}, \tag{3.1}$$

and therefore $S_\infty = \min_{x\in\partial\Omega} \mathcal{C}(x)$. Using the values of Π, K and S (see (2.2), (2.8), (2.20)), we see that

$$\frac{S^{\frac{N}{2}}}{N(Q_M N(N-2))^{\frac{N-2}{2}}} = \frac{\Pi}{2} Q_M^{-\frac{N-2}{2}} K(-\infty). \tag{3.2}$$

Hence, recalling the definition of $\mathcal{C}(x)$ (see (2.11)), inequality (3.1) is equivalent to

$$\min_{x\in\partial\Omega} \left[Q(x)^{-\frac{N-2}{2}} K\left(\frac{P(x)}{\sqrt{Q(x)}}\right)\right] \leq (\max_{x\in\bar{\Omega}} Q(x))^{-\frac{N-2}{2}} K(-\infty), \tag{3.3}$$

with $K(\mu)$ defined in Lemma 2.1. Since $K(\mu)$ is a decreasing function, it is easy to see that if $Q(x)$ takes its maximum value on the boundary $\partial\Omega$, then (3.3) is always satisfied. In particular if Q is constant, (3.3) is satisfied.

We next have

Theorem 3.1 *Suppose that (3.3) (or, equivalently, (3.1)) holds and let $x_0 \in \partial\Omega$, the point where $\mathcal{C}(x)$ takes on its minimum value, that is, $\mathcal{C}(x_0) = \min_{x\in\partial\Omega} \mathcal{C}(x)$. We assume that both the functions Q and P are differentiable at the point x_0 and we denote by $\frac{\partial Q}{\partial \nu}(x_0)$ the outward normal derivative and by $H(x_0)$ the mean curvature of $\partial\Omega$ at x_0. Then, problem (1.1) has a solution for every $\lambda > 0$, provided that:*
(i) in case $N = 3$, $H(x_0) > 0$,
(ii) in case $N \geq 4$, $H(x_0) \geq 0$, $\frac{\partial Q}{\partial \nu}(x_0) \leq 0$ and $H(x_0) - \frac{\partial Q}{\partial \nu}(x_0) > 0$.

Proof: For simplicity we may assume that $x_0 = 0$. Let $U_\varepsilon(x)$ be the solution of (1.11) given by (1.12) with $a = Q(0)$ and $b = P(0)$. It is enough to show that for a fixed $\lambda > 0$, $\max_{t\geq 0} \mathcal{J}_\lambda(tU_\varepsilon) < \min_{x\in\partial\Omega} \mathcal{C}(x) = \mathcal{C}(0)$.

Since Q, P are differentiable at $x_0 = 0$ we have that as $x \to 0$, $x \in \Omega$,

$$|Q(x) - Q(0) - \nabla Q(0) \cdot x| = o(|x|), \tag{3.4}$$

and a similar relation for P.

Using the explicit form of U_ε as well as (3.4) we compute

$$
\begin{aligned}
\max_{t\geq 0}\mathcal{J}_\lambda(tU_\varepsilon) &= \mathcal{J}_\lambda(t_\varepsilon U_\varepsilon) = \frac{t_\varepsilon^2}{2}\int_\Omega (|\nabla U_\varepsilon|^2+\lambda U_\varepsilon^2)dx \\
&-\frac{t_\varepsilon^{2^*}N(N-2)}{2^*}\int_\Omega Q(x)U_\varepsilon^{2^*}dx-\frac{t_\varepsilon^q}{q}\int_{\partial\Omega}P(x)U_\varepsilon^q dS_x \\
&= \frac{t_\varepsilon^2}{2}\int_\Omega |\nabla U_\varepsilon|^2dx+\frac{\lambda t_\varepsilon^2}{2}O(\varepsilon^2)-\frac{t_\varepsilon^{2^*}N(N-2)}{2^*}Q(0)\int_\Omega U_\varepsilon^{2^*}dx \\
&-\frac{t_\varepsilon^{2^*}N(N-2)}{2^*}\int_\Omega \nabla Q(0)\cdot xU_\varepsilon^{2^*}dx \\
&-\frac{t_\varepsilon^q}{q}P(0)\int_{\partial\Omega}U_\varepsilon^q dS_x-\frac{t_\varepsilon^q}{q}\int_{\partial\Omega}\nabla P(0)\cdot xU_\varepsilon^q dS_x \qquad (3.5)\\
&-\frac{t_\varepsilon^{2^*}N(N-2)}{2^*}\int_\Omega (Q(x)-Q(0)-\nabla Q(0)\cdot x)U_\varepsilon^{2^*}dx \\
&-\frac{t_\varepsilon^q}{q}\int_{\partial\Omega}(P(x)-P(0)-\nabla P(0)\cdot x)U_\varepsilon^q dS_x \\
&= \frac{t_\varepsilon^2}{2}\int_\Omega |\nabla U_\varepsilon|^2dx-\frac{t_\varepsilon^{2^*}N(N-2)}{2^*}Q(0)\int_\Omega U_\varepsilon^{2^*}dx-\frac{t_\varepsilon^q}{q}P(0)\int_{\partial\Omega}U_\varepsilon^q dS_x \\
&-\frac{t_\varepsilon^{2^*}N(N-2)}{2^*}\int_\Omega \nabla Q(0)\cdot xU_\varepsilon^{2^*}dx-\frac{t_\varepsilon^q}{q}\int_{\partial\Omega}\nabla P(0)\cdot xU_\varepsilon^q dS_x \\
&+(t_\varepsilon^{2^*}+t_\varepsilon^q)\,o(\varepsilon)+\lambda t_\varepsilon^2 O(\varepsilon^2).
\end{aligned}
$$

In the above calculations we used the estimate $\int_\Omega U_\varepsilon^2 dx = O(\varepsilon^2)$ as well as

$$\int_\Omega (Q(x)-Q(0)-\nabla Q(0)\cdot x)U_\varepsilon^{2^*}dx = o(\varepsilon), \qquad \text{as} \quad \varepsilon\to 0, \tag{3.6}$$

and a similar estimate for the boundary term. The L^2 norm of U_ε is quite easily estimated using the scaling property of $U_\varepsilon(x)=\varepsilon^{-\frac{N-2}{2}}U_1(x/\varepsilon)$. We next show how the estimate (3.6) is obtained. The corresponding estimate for the boundary integral is quite similar.

Given $\eta>0$ we choose $\delta(\eta)>0$ so that $|Q(x)-Q(0)-\nabla Q(0)\cdot x|\leq \eta|x|$ for $|x|\leq\delta(\eta)$. Then

$$
\begin{aligned}
\int_\Omega |Q(x)-Q(0)-\nabla Q(0)\cdot x|U_\epsilon^{2^*}dx &\leq \eta\int_{\Omega\cap(|x|<\delta(\eta))}|x|U_\epsilon^{2^*}dx + C\int_{\Omega\cap(|x|\geq\delta(\eta))}U_\varepsilon^{2^*}dx \\
&\leq C\Big(\epsilon\eta+\epsilon^N\int_{\delta(\eta)}^\infty \frac{r^{N-1}}{r^{2N}}\,dr\Big) = C\Big(\epsilon\eta+\frac{\epsilon^N}{\delta(\eta)^N}\Big).
\end{aligned}
$$

Hence

$$\limsup_{\epsilon\to 0}\frac{1}{\epsilon}\int_\Omega |Q(x)-Q(0)-\nabla Q(0)\cdot x|U_\epsilon^{2^*}dx\leq C\eta.$$

Since η is arbitrary, this limit is equal to 0 and (3.6) follows.

Since $t_\varepsilon U_\varepsilon\in\mathcal{M}_\lambda$, we similarly have that

$$\int_\Omega (|\nabla U_\varepsilon|^2+\lambda U_\varepsilon^2)dx = t_\varepsilon^{2^*-2}N(N-2)\int_\Omega Q(x)U_\varepsilon^{2^*}dx + t_\varepsilon^{q-2}\int_{\partial\Omega}P(x)U_\varepsilon^q dS_x$$

$$= t_\varepsilon^{2^*-2} N(N-2) Q(0) \int_\Omega U_\varepsilon^{2^*} dx + t_\varepsilon^{q-2} P(0) \int_{\partial\Omega} U_\varepsilon^q dS_x \tag{3.7}$$
$$+ t_\varepsilon^{2^*-2} N(N-2) \int_\Omega \nabla Q(0)\cdot x U_\varepsilon^{2^*} dx + t_\varepsilon^{q-2} \int_{\partial\Omega} \nabla P(0)\cdot x U_\varepsilon^q dS_x$$
$$+ (1 + t_\varepsilon^{2^*-2})\, o(\varepsilon).$$

Now it follows easily from this that as $\varepsilon \to 0$, t_ε stays bounded away from zero and infinity, and consequently we may assume that $t_\varepsilon \to \bar t > 0$. Letting $\varepsilon \to 0$ in (3.7) we get

$$\int_{\mathbf{R}^N_+} |\nabla U_1|^2 = \bar t^{2^*-2} N(N-2) Q(0) \int_{\mathbf{R}^N_+} U_1^{2^*} dx + \bar t^{q-2} P(0) \int_{\mathbf{R}^{N-1}} U_1^q dx'.$$

Combining this with (2.4) we conclude that $\bar t = 1$ and $t_\varepsilon = 1 + o(1)$ as $\varepsilon \to 0$.

We now continue with the estimate of $\mathcal{J}_\lambda(t_\varepsilon U_\varepsilon)$. We rewrite (3.5) as

$$\begin{aligned}
\mathcal{J}_\lambda(t_\varepsilon U_\varepsilon) \;&=\; \frac{t_\varepsilon^2}{2}\int_{\mathbf{R}^N_+} |\nabla U_\varepsilon|^2\,dx - \frac{t_\varepsilon^{2^*}}{2^*} N(N-2)Q(0)\int_{\mathbf{R}^N_+} U_\varepsilon^{2^*}\,dx \\
&-\; \frac{t_\varepsilon^q}{q} P(0) \int_{\mathbf{R}^{N-1}} U_\varepsilon(x',0)^q\,dx' \\
&-\; \frac{t_\varepsilon^{2^*} N(N-2)}{2^*} \int_{\mathbf{R}^N_+} \nabla Q(0)\cdot x U_\varepsilon^{2^*}\,dx - \frac{t_\varepsilon^q}{q}\int_{\mathbf{R}^{N-1}} \nabla P(0)\cdot x U_\varepsilon^q\,dx' \\
&-\; \frac{t_\varepsilon^2}{2} K_1(\varepsilon) + \frac{t_\varepsilon^{2^*}}{2^*} Q(0) N(N-2) K_2(\varepsilon) + \frac{t_\varepsilon^q}{q} P(0) K_3(\varepsilon), \\
&+\; \frac{t_\varepsilon^{2^*}}{2^*} N(N-2)\Lambda_2(\varepsilon) + \frac{t_\varepsilon^q}{q}\Lambda_3(\varepsilon) + o(\varepsilon),
\end{aligned} \tag{3.8}$$

where

$$K_1(\varepsilon) := \int_{\mathbf{R}^N_+} |\nabla U_\varepsilon|^2\,dx - \int_\Omega |\nabla U_\varepsilon|^2\,dx,$$
$$K_2(\varepsilon) := \int_{\mathbf{R}^N_+} U_\varepsilon^{2^*}\,dx - \int_\Omega U_\varepsilon^{2^*}\,dx,$$
$$K_3(\varepsilon) := \int_{\mathbf{R}^{N-1}} U_\varepsilon(x',0)^q\,dx' - \int_{\partial\Omega} U_\varepsilon^q\,dS_x,$$

and

$$\Lambda_2(\varepsilon) := \int_{\mathbf{R}^N_+} \nabla Q(0)\cdot x U_\varepsilon^{2^*}\,dx - \int_\Omega \nabla Q(0)\cdot x U_\varepsilon^{2^*}\,dx,$$
$$\Lambda_3(\varepsilon) := \int_{\mathbf{R}^{N-1}} \nabla P(0)\cdot x U_\varepsilon(x',0)^q\,dx' - \int_{\partial\Omega} \nabla P(0)\cdot x U_\varepsilon^q\,dS_x.$$

We next estimate the integrals $K_i(\varepsilon), \Lambda_i(\varepsilon)$. We represent $\partial\Omega$ near $x_0 = 0$ as $x_N = h(x') = \frac{1}{2}\sum_{i=1}^{N-1} a_i x_i^2 + o(|x'|^2) = g(x') + o(|x'|^2)$, where the a_i are the principal curvatures of $\partial\Omega$ at 0. By our assumptions we have that $H(0) = \frac{1}{N-1}\sum_{i=1}^{N-1} a_i \ge 0$.

We first estimate $K_1(\varepsilon)$. We will use the following easily verifiable relations

$$\left| \int_{|x'|<\delta} dx' \int_{g(x')}^{h(x')} |\nabla U_\varepsilon|^2\,dx_N dx' \right| = \begin{cases} o(\varepsilon), & N \ge 4 \\ o(\varepsilon|\ln\varepsilon|), & N = 3. \end{cases}$$

Also,
$$\int_{\mathbf{R}^N\cap\{|x'|>\delta\}} |\nabla U_\varepsilon|^2\,dx = O(\varepsilon^{N-2}), \quad N\ge 3.$$

We then have for $N\ge 4$,

$$\begin{aligned} K_1(\varepsilon) &= \int_{\mathbf{R}^N_+} |\nabla U_\varepsilon|^2\,dx - \int_\Omega |\nabla U_\varepsilon|^2\,dx \\ &= \int_{|x'|<\delta} dx' \int_0^{g(x')} |\nabla U_\varepsilon|^2\,dx_N + o(\varepsilon) \\ &= \int_{\mathbf{R}^{N-1}} dx' \int_0^{g(x')} |\nabla U_\varepsilon|^2\,dx_N + o(\varepsilon) \\ &= \int_{\mathbf{R}^{N-1}} dx' \int_0^{\varepsilon g(x')} |\nabla U_1|^2\,dx_N + o(\varepsilon). \end{aligned}$$

¿From this we deduce by L'Hospital rule that

$$K_1(\varepsilon) = K_1\varepsilon + o(\varepsilon), \qquad N\ge 4, \tag{3.9}$$

where

$$\begin{aligned} K_1 &= a^{-\frac{N-2}{2}}(N-2)^2 \int_{\mathbf{R}^{N-1}} \frac{(|x'|^2+\bar\mu^2)g(x')}{(1+|x'|^2+\bar\mu^2)^N}\,dx' \\ &= \frac{1}{2}a^{-\frac{N-2}{2}}(N-2)^2 H(0) \int_{\mathbf{R}^{N-1}} \frac{(|x'|^2+\bar\mu^2)|x'|^2}{(1+|x'|^2+\bar\mu^2)^N}\,dx', \end{aligned} \tag{3.10}$$

and
$$\bar\mu := \frac{P(0)}{(N-2)\sqrt{Q(0)}}.$$

We next obtain a more explicit form for the integral in (3.10). We set

$$D := \frac{\int_{\mathbf{R}^{N-1}} \frac{(|x'|^2+\bar\mu^2)|x'|^2}{\left(1+|x'|^2+\bar\mu^2\right)^N}\,dx'}{\int_{\mathbf{R}^{N-1}} \frac{|x'|^2}{\left(1+|x'|^2+\bar\mu^2\right)^N}\,dx'}.$$

The integral in the denominator can be easily computed

$$\int_{\mathbf{R}^{N-1}} \frac{|x'|^2}{\left(1+|x'|^2+\bar\mu^2\right)^N}\,dx' = \frac{\Pi}{2}(1+\bar\mu^2)^{-\frac{N-1}{2}}, \tag{3.11}$$

with Π as defined in Lemma 2.1. We then have

$$D = \frac{\int_0^\infty \frac{(r^2+\bar\mu^2)r^N}{\left(1+r^2+\bar\mu^2\right)^N}\,dr}{\int_0^\infty \frac{r^N}{\left(1+r^2+\bar\mu^2\right)^N}\,dr} = (1+\bar\mu^2)\frac{\int_0^\infty \frac{r^{N+2}}{\left(1+r^2\right)^N}\,dr}{\int_0^\infty \frac{r^N}{(1+r^2)^N}\,dr} + \bar\mu^2.$$

An integration by parts (see [26], p. 297) shows that

$$\int_0^\infty \frac{s^{N+2}}{(1+s^2)^N}\,ds = \frac{N+1}{N-3}\int_0^\infty \frac{s^N}{(1+s^2)^N}\,ds, \;\; N\ge 4.$$

Hence
$$D = (1+\bar\mu^2)\frac{N+1}{N-3} + \bar\mu^2.$$
and therefore
$$K_1 = \frac{\Pi}{4} a^{-\frac{N-2}{2}} (N-2)^2 H(0) \left(\frac{N+1}{N-3} + \frac{2(N-1)}{N-3}\bar\mu^2 \right) (1+\bar\mu^2)^{-\frac{N-1}{2}}, \qquad N \geq 4. \quad (3.12)$$

We now consider the case $N = 3$. We split the term $|\nabla U_\varepsilon|^2$ as follows
$$|\nabla U_\varepsilon|^2 = \frac{a^{-\frac{1}{2}}\varepsilon}{(\varepsilon^2 + |x'|^2 + (x_N + \varepsilon\bar\mu)^2)^2} - \frac{a^{-\frac{1}{2}}\varepsilon^3}{(\varepsilon^2 + |x'|^2 + (x_N + \varepsilon\bar\mu)^2)^3}.$$
We then have
$$\begin{aligned} K_1(\varepsilon) &= \int_{\mathbf{R}^2} dx' \int_0^{g(x')} |\nabla U_\varepsilon|^2 \, dx_3 + o(\varepsilon|\ln\varepsilon|) \\ &= \int_{\mathbf{R}^2} dx' \int_0^{g(x')} \frac{a^{-\frac{1}{2}}\varepsilon}{(\varepsilon^2 + |x'|^2 + (x_3 + \varepsilon\bar\mu)^2)^2} \, dx_3 \\ &- \int_{\mathbf{R}^2} dx' \int_0^{g(x')} \frac{a^{-\frac{1}{2}}\varepsilon^3}{\left(\varepsilon^2 + |x'|^2 + (x_3 + \varepsilon\bar\mu)^2\right)^3} \, dx_3 + o(\varepsilon|\ln\varepsilon|) \\ &=: A_1(\varepsilon) - A_2(\varepsilon) + o(\varepsilon|\ln\varepsilon|). \end{aligned}$$
Since $|g(x')| \leq c|x'|^2$, we have
$$|A_2(\varepsilon)| \leq C\varepsilon^3 \int_{\mathbf{R}^2} \frac{|x'|^2}{(\varepsilon^2 + |x'|^2)^3} \, dx' = C\varepsilon \int_{\mathbf{R}^2} \frac{|x'|^2}{(1+|x'|^2)^3} \, dx' = O(\varepsilon).$$

Concerning A_1 we have that
$$\begin{aligned} -\frac{1}{\varepsilon} A_1(\varepsilon) + A_1'(\varepsilon) &= -4a^{-\frac{1}{2}} \int_{\mathbf{R}^2} dx' \int_0^{g(x')} \frac{\varepsilon(\varepsilon + (x_3 + \varepsilon\bar\mu)\bar\mu)}{\left(\varepsilon^2 + |x'|^2 + (x_3 + \varepsilon\bar\mu)^2\right)^3} \, dx_3 \\ &= -\frac{4a^{-\frac{1}{2}}}{\varepsilon} \int_{\mathbf{R}^2} \int_0^{\varepsilon g(x')} \frac{1 + (x_3 + \bar\mu)\bar\mu}{\left(1 + |x'|^2 + (x_3 + \bar\mu)^2\right)^3} \, dx_3. \end{aligned}$$
¿From this we deduce that
$$\lim_{\varepsilon\to 0} (-\frac{1}{\varepsilon} A_1(\varepsilon) + A_1'(\varepsilon)) = -4a^{-\frac{1}{2}} \int_{\mathbf{R}^2} \frac{(1+\bar\mu^2) g(x')}{(1+|x'|^2+\bar\mu^2)^3} \, dx_3.$$
It follows now easily that $\lim_{\varepsilon\to 0} \frac{A_1(\varepsilon)}{\varepsilon} = +\infty$, and by L'Hospital's rule we have
$$\lim_{\varepsilon\to 0} \frac{\frac{1}{\varepsilon} A_1(\varepsilon)}{\ln\varepsilon} = \lim_{\varepsilon\to 0} (-\frac{1}{\varepsilon} A_1(\varepsilon) + A_1'(\varepsilon)) = -4a^{-\frac{1}{2}} \int_{\mathbf{R}^2} \frac{(1+\bar\mu^2) g(x')}{(1+|x'|^2+\bar\mu^2)^3} \, dx_3.$$

Whence,
$$K_1(\varepsilon) = \hat{K}_1 \varepsilon |\ln\varepsilon| + o(\varepsilon|\ln\varepsilon|), \qquad N = 3, \quad (3.13)$$

where

$$\begin{aligned}\hat{K}_1 &= 4a^{-\frac{1}{2}}\int_{\mathbf{R}^2}\frac{(1+\bar{\mu}^2)g(x')}{(1+|x'|^2+\bar{\mu}^2)^3}\,dx' \\ &= 2a^{-\frac{1}{2}}H(0)\int_{\mathbf{R}^2}\frac{(1+\bar{\mu}^2)|x'|^2}{(1+|x'|^2+\bar{\mu}^2)^3}\,dx'.\end{aligned} \tag{3.14}$$

We will similarly estimate $K_2(\varepsilon)$. Using the relations,

$$\left|\int_{|x'|<\delta}dx'\int_{h(x')}^{g(x')}U_\varepsilon^{2^*}\,dx_N\right| = o(\varepsilon), \quad \text{and} \quad \int_{\mathbf{R}^N\cap\{|x'|>\delta\}}U_\varepsilon^{2^*}\,dx = O(\varepsilon^N),$$

we write for $N \geq 3$,

$$\begin{aligned}K_2(\varepsilon) &= \int_{|x'|<\delta}dx'\int_0^{g(x')}U_\varepsilon^{2^*}\,dx_N + o(\varepsilon), \\ &= \int_{\mathbf{R}^{N-1}}dx'\int_0^{g(x')}U_\varepsilon^{2^*}\,dx_N + o(\varepsilon), \\ &= a^{-\frac{N}{2}}\int_{\mathbf{R}^{N-1}}dx'\int_0^{\varepsilon g(x')}\frac{dx_N}{(1+|x'|^2|+\bar{\mu}^2)^N} + o(\varepsilon).\end{aligned}$$

Hence

$$K_2(\varepsilon) = K_2\varepsilon + o(\varepsilon), \qquad N \geq 3, \tag{3.15}$$

where

$$\begin{aligned}K_2 &= a^{-\frac{N}{2}}\int_{\mathbf{R}^{N-1}}\frac{g(x')}{(1+|x'|^2+\bar{\mu}^2)^N}\,dx' \\ &= \frac{1}{2}a^{-\frac{N}{2}}H(0)\int_{\mathbf{R}^{N-1}}\frac{|x'|^2}{(1+|x'|^2+\bar{\mu}^2)^N}\,dx' \\ &= \frac{\Pi}{4}a^{-\frac{N}{2}}H(0)(1+\bar{\mu}^2)^{-\frac{N-1}{2}}.\end{aligned} \tag{3.16}$$

Analogous calculations show that

$$\begin{aligned}\int_{\mathbf{R}^N_+}\nabla Q(0)\cdot xU_\varepsilon^{2^*}\,dx &= \varepsilon a^{-\frac{N}{2}}Q_{x_N}(0)\left[\frac{1}{2(N-1)}\int_{\mathbf{R}^{N-1}}\frac{dx'}{(1+|x'|^2+\bar{\mu}^2)^{N-1}}\right. \\ &\qquad \left. -\,\bar{\mu}\int_{\mathbf{R}^N_+}\frac{dx}{(1+|x'|^2+(x_N+\bar{\mu})^2)^N}\right], \qquad (3.17) \\ &= \frac{\Pi}{2}\varepsilon a^{-\frac{N}{2}}Q_{x_N}(0)\left[\frac{1}{N-1}(1+\bar{\mu}^2)^{-\frac{N-1}{2}} - \bar{\mu}\int_{\bar{\mu}}^\infty\frac{dt}{(1+t^2)^{\frac{N+1}{2}}}\right],\end{aligned}$$

with Π as defined in Lemma 2.1. It is easy to check that

$$\int_{\mathbf{R}^N_+}\nabla P(0)\cdot xU_\varepsilon^q\,dx = 0. \tag{3.18}$$

We also obtain that

$$\Lambda_2(\varepsilon) = o(\varepsilon), \qquad N \geq 3. \tag{3.19}$$

Finally, we estimate $K_3(\varepsilon)$:

$$\begin{aligned} K_3(\varepsilon) &= \int_{\mathbf{R}^{N-1}} U_\varepsilon(x',0)^q\,dx' - \int_{\partial\Omega\cap(|x'|<\delta)} U_\varepsilon(x)^q\,dS_x - \int_{\partial\Omega\cap(|x'|>\delta)} U_\varepsilon(x)^q\,dS_x \\ &= \int_{\mathbf{R}^{N-1}} U_\varepsilon(x',0)^q\,dx' - \int_{\partial\Omega\cap(|x'|<\delta)} U_\varepsilon(x)^q\,dS_x + O(\varepsilon^{N-1}). \end{aligned}$$

We now estimate the surface integral

$$\begin{aligned} \int_{\partial\Omega\cap(|x'|<\delta)} U_\varepsilon(x)^q\,dS_x &= a^{-\frac{N-1}{2}} \int_{|x'|<\delta} \frac{\varepsilon^{N-1}(1+|\nabla h(x')|^2)^{\frac{1}{2}}}{(\varepsilon^2+|x'|^2+(h(x')+\varepsilon\bar\mu)^2)^{N-1}}\,dx' \\ &= a^{-\frac{N-1}{2}} \int_{\mathbf{R}^{N-1}} \frac{\varepsilon^{N-1}(1+|\nabla h(x')|^2)^{\frac{1}{2}}}{(\varepsilon^2+|x'|^2+(h(x')+\varepsilon\bar\mu)^2)^{N-1}}\,dx' + O(\varepsilon^{N-1}) \\ &= a^{-\frac{N-1}{2}} \int_{\mathbf{R}^{N-1}} \frac{(1+|\nabla h(\varepsilon x')|^2)^{\frac{1}{2}}}{(1+|x'|^2+(\varepsilon^{-1}h(\varepsilon x')+\bar\mu)^2)^{N-1}}\,dx' + O(\varepsilon^{N-1}). \end{aligned}$$

Combining together the last two relations we get

$$\begin{aligned} K_3(\varepsilon) &= a^{-\frac{N-1}{2}} \left[\int_{\mathbf{R}^{N-1}} \frac{1}{(1+|x'|^2+\bar\mu^2)^{N-1}}\,dx' \right. \\ &- \left. \int_{\mathbf{R}^{N-1}} \frac{(1+|\nabla h(\varepsilon x')|^2)^{\frac{1}{2}}}{(1+|x'|^2+(\varepsilon^{-1}h(\varepsilon x')+\bar\mu)^2)^{N-1}}\,dx' \right] + O(\varepsilon^{N-1}). \end{aligned}$$

Since

$$(1+|\nabla h(\varepsilon x')|^2)^{\frac{1}{2}} = 1 + \frac{\theta|\nabla h(\varepsilon x')|^2}{(1+\theta^2|\nabla h(\varepsilon x')|^2)^{\frac{1}{2}}},$$

for some $0<\theta<1$, we have

$$\begin{aligned} & K_3(\varepsilon) = a^{-\frac{N-1}{2}} \left[\int_{\mathbf{R}^{N-1}} \frac{1}{(1+|x'|^2+\bar\mu^2)^{N-1}}\,dx' \right. \\ - & \left. \int_{\mathbf{R}^{N-1}} \frac{dx'}{(1+|x'|^2+(\varepsilon^{-1}h(\varepsilon x')+\bar\mu)^2)^{N-1}} \right] \\ - & a^{-\frac{N-1}{2}} \int_{\mathbf{R}^{N-1}} \frac{\theta|\nabla h(\varepsilon x')|^2}{(1+|x'|^2+(\varepsilon^{-1}h(\varepsilon x')+\bar\mu)^2)^{N-1}(1+\theta^2|\nabla h(\varepsilon x')|^2)^{\frac{1}{2}}}\,dx' \\ =: & B(\varepsilon) - C(\varepsilon). \end{aligned}$$

We now observe that $|\nabla|h(\varepsilon x')| \le c\varepsilon|x'|$ and $\varepsilon^{-1}|h(\varepsilon x')| \le c\varepsilon|x'|^2$. Hence

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}C(\varepsilon) = 0.$$

We now examine the integral $B(\varepsilon)$. We let $L(x') = 1+|x'|^2+\bar\mu^2$ and $M(\varepsilon,x') = 1+|x'|^2+(\varepsilon^{-1}h(\varepsilon x')+\bar\mu)^2$. Then

$$\frac{\partial M(\varepsilon,x')}{\partial\varepsilon} = 2(\varepsilon^{-1}h(\varepsilon,x')+\bar\mu)(-\varepsilon^{-2}h(\varepsilon x')+\nabla h(\varepsilon x')\cdot x')$$

and by L'Hospital's rule, we have

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}B(\varepsilon)=\lim_{\varepsilon\to 0}a^{-\frac{N-1}{2}}\int_{\mathbf{R}^{N-1}}\frac{M(\varepsilon,x')^{N-1}-L(x')^{N-1}}{\varepsilon L(x')^{N-1}M(\varepsilon,x')^{N-1}}\,dx'\\
&=\lim_{\varepsilon\to 0}a^{-\frac{N-1}{2}}\int_{\mathbf{R}^{N-1}}\frac{(N-1)\frac{\partial M(\varepsilon,x')}{\partial\varepsilon}M(\varepsilon,x')^{N-2}\,dx'}{L(x')^{N-1}M(\varepsilon,x')^{N-1}+\varepsilon(N-1)L(x')^{N-1}\frac{\partial M(\varepsilon,x')}{\partial\varepsilon}M(\varepsilon,x')^{N-2}}\\
&=a^{-\frac{N-1}{2}}\int_{\mathbf{R}^{N-1}}\frac{-2(N-1)\bar\mu g(x')L(x')^{N-2}}{L(x')^{2N-2}}\,dx'\\
&=-2a^{-\frac{N-1}{2}}\bar\mu(N-1)\int_{\mathbf{R}^{N-1}}\frac{g(x')}{(1+|x'|^2+\bar\mu^2)^N}\,dx'\\
&=-a^{-\frac{N-1}{2}}\bar\mu H(0)(N-1)\int_{\mathbf{R}^{N-1}}\frac{|x'|^2}{(1+|x'|^2+\bar\mu^2)^N}\,dx'.
\end{aligned}
$$

Consequently we have

$$
K_3(\varepsilon)=-K_3\varepsilon+o(\varepsilon),\qquad N\geq 3, \tag{3.20}
$$

with

$$
\begin{aligned}
K_3 &= (N-1)a^{-\frac{N-1}{2}}\bar\mu H(0)\int_{\mathbf{R}^{N-1}}\frac{|x'|^2}{(1+|x'|^2+\bar\mu^2)^N}\,dx'\\
&= \frac{\Pi}{2}(N-1)a^{-\frac{N-1}{2}}H(0)\bar\mu(1+\bar\mu^2)^{-\frac{N-1}{2}}.
\end{aligned} \tag{3.21}
$$

Similar calculations show that

$$
\Lambda_3(\varepsilon)=o(\varepsilon),\qquad N\geq 3. \tag{3.22}
$$

We can now continue with the estimate of $\mathcal{J}_\lambda(t_\varepsilon U_\varepsilon)$, see (3.8). The first three integrals in the right hand side of (3.8) are bounded above by $\mathcal{C}(0)$; cf Lemma 2.1. Therefore, recalling also that $t_\varepsilon=1+o(1)$ as $\varepsilon\to 0$, we have for $N\geq 3$,

$$
\begin{aligned}
\mathcal{J}_\lambda(t_\varepsilon U_\varepsilon) \;\leq\; & \mathcal{C}(0)-\frac{(N-2)^2}{2}\int_{\mathbf{R}^N_+}\nabla Q(0)\cdot xU_\varepsilon^{2^*}\,dx-\frac{1}{2}(1+o(1))K_1(\varepsilon)\\
&+\frac{(N-2)^2}{2}Q(0)K_2(\varepsilon)+\frac{N-2}{2(N-1)}P(0)K_3(\varepsilon)+o(\varepsilon).
\end{aligned} \tag{3.23}
$$

We first consider the case $N\geq 4$. Using the asymptotics of $K_i(\varepsilon)$, $i=1,2,3$, (see (3.9), (3.12), (3.15), (3.16), (3.20) and (3.21)) as well as of the gradient term in (3.23) (see (3.17)) we obtain after some straightforward calculations

$$
\begin{aligned}
\mathcal{J}_\lambda(t_\varepsilon U_\varepsilon) \;\leq\; & \mathcal{C}(0)-\frac{\Pi}{4}\frac{(N-2)^2}{N-3}Q^{-\frac{N-2}{2}}(0)\times\\
&\left[\frac{(N-3)Q_{x_N}(0)}{Q(0)}\left(\frac{(1+\bar\mu^2)^{-\frac{N-1}{2}}}{N-1}-\bar\mu\int_{\bar\mu}^{\infty}\frac{dt}{(1+t^2)^{\frac{N+1}{2}}}\right)\right.\\
&\left.+2H(0)(1+(N-2)\bar\mu^2)(1+\bar\mu^2)^{-\frac{N-1}{2}}\right]\varepsilon+o(\varepsilon).
\end{aligned} \tag{3.24}
$$

We note that

$$\Lambda(\bar{\mu}) := \frac{(1+\bar{\mu}^2)^{-\frac{N-1}{2}}}{N-1} - \bar{\mu}\int_{\bar{\mu}}^{\infty} \frac{dt}{(1+\bar{\mu}^2)^{\frac{N+1}{2}}} > 0,$$

since $\Lambda(+\infty) = 0$ and $\Lambda'(\bar{\mu}) < 0$ for all $\bar{\mu} \in (-\infty, \infty)$. Noting that $\frac{\partial Q(0)}{\partial \nu} = -Q_{x_N}(0)$, it follows that under the assumptions of the Theorem, $\mathcal{J}_\lambda(t_\varepsilon U_\varepsilon) < \mathcal{C}(0)$ for ε small, and therefore a solution exists.

We next consider the case $N = 3$. The result now follows easily by noticing in (3.23) that the term containing $K_1(\varepsilon)$ (see (3.13) and (3.14)) is negative and of order $O(\varepsilon|\ln\varepsilon|)$, whereas with the exception of $\mathcal{C}(0)$, the other terms are of order $O(\varepsilon)$.

The proof of the Theorem is now complete. ///

Remark When $N \geq 4$, (3.24) gives a sharper criterion for obtaining existence of solutions. For example, when $P(x) \equiv 0$ (Neumann problem), we have that $b = P(0) = 0$ and $\bar{\mu} = 0$, and (3.24) takes the form

$$\mathcal{J}_\lambda(t_\varepsilon U_\varepsilon) \leq \mathcal{C}(0) - \frac{\Pi}{4}\frac{(N-2)^2}{N-3} Q^{-\frac{N-2}{2}}(0)\left[\frac{(N-3)Q_{x_N}(0)}{Q(0)} + 2H(0)\right]\varepsilon + o(\varepsilon). \quad (3.25)$$

In order to have $\mathcal{J}_\lambda(t_\varepsilon U_\varepsilon) < \mathcal{C}(0)$ one needs the quantity in the brackets to be positive, which is equivalent to

$$2H(x_0)Q(x_0) - (N-3)\frac{\partial Q(x_0)}{\partial \nu} > 0.$$

We finally consider the case

$$S_\infty = \frac{S^{\frac{N}{2}}}{N(Q_M N(N-2))^{\frac{N-2}{2}}} < \min_{x\in\partial\Omega} \mathcal{C}(x), \quad (3.26)$$

and we present sufficient conditions for the existence of a solution to problem (1.1).

Theorem 3.2 *Assume that (3.26) holds. Suppose that either*
(i) $\int_{\partial\Omega} P(x)dx \geq 0$, *or else that*
(ii) $\int_{\partial\Omega} P(x)dx < 0$ *and in addition*

$$\frac{\left|\int_{\partial\Omega} P(x)dS_x\right|^N Q_M^{\frac{N-2}{2}}}{\left(\int_\Omega Q(x)dx\right)^{N-1}} < 2(N-1)N^{\frac{N}{2}}(N-2)^{\frac{N-2}{2}} S^{\frac{N}{2}}. \quad (3.27)$$

Then, there exists a $\lambda_0 > 0$ *such that for* $0 \leq \lambda < \lambda_0$,

$$c_\lambda = \inf_{u\in\mathcal{M}_\lambda} \mathcal{J}_\lambda(u) < S_\infty. \quad (3.28)$$

In particular, problem (1.1) has a solution for $\lambda \in (0, \lambda_0)$.

Proof: Since $c_\lambda = \inf_{u\in H^1(\Omega)} \max_{t\geq 0} \mathcal{J}_\lambda(tu)$, we take $u \equiv 1$ and we will show that $\max_{t\geq 0} \mathcal{J}_\lambda(t\cdot 1) < S_\infty$. We compute

$$\mathcal{J}_\lambda(t) = f(t) := \frac{a}{2}t^2 - \frac{bN(N-2)}{2^*}t^{2^*} - \frac{c}{q}t^q,$$

with

$$a = \lambda|\Omega| > 0, \qquad b = \int_\Omega Q(x)dx > 0, \qquad c = \int_{\partial\Omega} P(x)dx.$$

Differentiating once we find

$$f'(t) = -t(bN(N-2)t^{\frac{4}{N-2}} + ct^{\frac{2}{N-2}} - a).$$

The quantity inside the parentheses is quadratic in $t^{\frac{2}{N-2}}$ and has a unique positive root t_0 given by

$$t_0^{\frac{2}{N-2}} = \frac{-c + \sqrt{c^2 + 4abN(N-2)}}{2bN(N-2)}.$$

It is easy to check that $f(t)$ has a unique (global) maximum at t_0, hence $\max_{t\geq 0} f(t) = f(t_0)$. Also,

$$f(t_0) = f(t_0) - \frac{t_0}{q} f'(t_0) = \frac{a}{2(N-1)} t_0^2 + \frac{b(N-2)^2}{2(N-1)} t_0^{\frac{2N}{N-2}},$$

We note that $t_0(a)$ and $f(a, t_0(a))$ are both increasing functions of a (and therefore of λ).

Assume that $c \geq 0$. Then $\lambda \to 0$, implies $t_0 \to 0$ and $f(t_0) \to 0$, and therefore for small λ (3.28) is true.

Assume now that $c < 0$. Then $\lambda \to 0$, implies that

$$t_0^{\frac{2}{N-2}} \to \frac{|c|}{bN(N-2)},$$

and therefore

$$f(t_0) \to \frac{b(N-2)^2}{2(N-1)} \left(\frac{|c|}{bN(N-2)} \right)^N.$$

The constant in the right hand side is smaller than S_∞ iff (3.27) holds. ///

4 Existence for $\lambda = 0$

In this case problem (1.1) takes the form

$$\begin{cases} -\Delta u = N(N-2)Q(x)|u|^{2^*-2}u & \text{in } \Omega, \\ u > 0 \text{ on } \bar{\Omega}, \quad \frac{\partial u}{\partial \nu} = P(x)|u|^{q-2}u & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

It is easy to check that if P and Q are both positive, then problem (4.1) does not have a positive solution. Indeed, assuming that a positive solution u exists, by Green's theorem we have

$$\begin{aligned} \int_{\partial\Omega} P(x)\, dS_x &= \int_{\partial\Omega} u^{-(q-1)} \frac{\partial u}{\partial \nu}\, dS_x = \int_\Omega u^{-(q-1)} \Delta u\, dx - (q-1) \int_\Omega |\nabla u|^2 u^{-q}\, dx \\ &= -N(N-2) \int_\Omega Q(x) u^{2^*-q}\, dx - (q-1) \int_\Omega |\nabla u|^2 u^{-q}\, dx < 0. \end{aligned}$$

Therefore the inequality $\int_\Omega P(x)\, dS_x < 0$ is a necessary condition for the existence of a solution. In the sequel we will some find sufficient conditions.

We first establish some preliminary estimates. We recall that the first eigenvalue of

$$\begin{cases} -\Delta u = \mu u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.2}$$

is equal to 0 and the corresponding eigenfunctions are constant. We decompose $H^1(\Omega) = \mathbf{R} \oplus V$, where

$$V = \{v \in H^1(\Omega);\ \int_\Omega v(x)\, dx = 0\}.$$

The subspace V of $H^1(\Omega)$ is continuously embedded into $L^{2^*}(\Omega)$ and $L^q(\partial\Omega)$. We introduce an equivalent norm in $H^1(\Omega)$

$$\|u\|_V^2 = t^2 + \int_\Omega |\nabla v|^2\, dx,$$

if $u = t + v$.

We then have

Lemma 4.1 *Let $\int_{\partial\Omega} P(x)\, dS_x < 0$. Then there exists a constant $\eta > 0$ such that for every $t \in \mathbf{R}$ and $v \in V$ the inequality*

$$\left(\int_\Omega |\nabla v|^2\, dS_x\right)^{\frac{1}{2}} \leq \eta |t|,$$

implies

$$\int_{\partial\Omega} P(x)|t + v(x)|^q\, dS_x \leq \frac{|t|^q}{2} \int_{\partial\Omega} P(x)\, dS_x.$$

This is a consequence of the continuity of the embedding of V into $L^q(\partial\Omega)$ (see also [8]).

Proposition 4.2 *Let $\int_{\partial\Omega} P(x)\, dS_x < 0$. Then there exist constants $\rho > 0$ and $\beta > 0$ such that $\mathcal{J}_0(u) \geq \beta$ for every u satisfying $\|u\|_V = \rho$.*

Proof: Let $\eta > 0$ be the constant from Lemma 4.1. We distinguish two cases: (i) $\|\nabla v\|_2 \leq \eta|t|$ and (ii) $\|\nabla v\|_2 > \eta|t|$.

(i) If $\|\nabla v\|_2 \leq \eta|t|$ and $\|\nabla v\|_2^2 + t^2 = \rho^2$, then $t^2 \geq \frac{\rho^2}{1+\eta^2}$. It follows from Lemma 4.1 that

$$\int_{\partial\Omega} P(x)|t + v(x)|^q\, dS_x \leq -|t|^q \alpha,$$

with $\alpha = -\frac{1}{2}\int_{\partial\Omega} P(x)\, dS_x > 0$. Using this and the Sobolev inequality in V we estimate $\mathcal{J}_0$ from below

$$\begin{aligned} \mathcal{J}_0(u) &\geq -C(\|\nabla v\|_2^{2^*} + |t|^{2^*}) + \frac{|t|^q}{q}\alpha \\ &\geq -C_1\rho^{2^*} + \frac{\alpha\rho^q}{q(1+\eta^2)^{\frac{q}{2}}} \geq \frac{\rho^q \alpha}{2q(1+\eta^2)^{\frac{q}{2}}}, \end{aligned}$$

for $\rho > 0$ sufficiently small, say $\rho \leq \rho_\circ$ and some constants $C > 0$ and $C_1 > 0$.

In case (ii) we have $\|u\|_V \leq \|\nabla v\|_2(1+\frac{1}{\eta^2})^{\frac{1}{2}}$. By the Sobolev inequalities we get

$$\int_\Omega Q(x)|u|^{2^*}\,dx \leq C_2\|u\|_V^{2^*} \leq C_2(1+\frac{1}{\eta^2})^{\frac{2^*}{2}}\|\nabla v\|_2^{2^*},$$

and

$$\Big|\int_{\partial\Omega} P(x)|u|^q\,dS_x\Big| \leq C_3(1+\frac{1}{\eta^2})^{\frac{q}{2}}\|\nabla v\|_2^q,$$

where $C_2>0$ and $C_3>0$ are constants. Thus

$$\mathcal{J}_0(u) \geq \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - C_2(1+\frac{1}{\eta^2})^{\frac{2^*}{2}}\|\nabla v\|_2^{2^*} - C_3(1+\frac{1}{\eta^2})^{\frac{q}{2}}\|\nabla v\|_2^q.$$

Taking $\|\nabla v\|_2 \leq \rho$ sufficiently small, we derive from the above inequality that

$$\mathcal{J}_0(u) \geq \frac{1}{4}\|\nabla v\|_2^2.$$

Finally, we observe that if $\|u\|_V = \rho$, then $\rho \leq \|\nabla v\|_2\frac{(1+\eta^2)^{\frac{1}{2}}}{\eta}$. Therefore

$$\mathcal{J}_0(u) \geq \frac{\eta^2\rho^2}{4(1+\eta^2)}.$$

We choose

$$\beta = \min\Big(\frac{\rho^q\alpha}{2q(1+\eta^2)^{\frac{q}{2}}}, \frac{\eta^2\rho^2}{4(1+\eta^2)}\Big),$$

and the result follows. ///

We are now in position to prove the analogue of Theorem 2.4.

Theorem 4.3 *Let $\int_{\partial\Omega} P(x)\,dS_x < 0$. If*

$$c_0 = \inf_{u\in\mathcal{M}_0} \mathcal{J}_0(u) < S_\infty,$$

then c_0 is achieved and problem (4.1) has a solution.

Proof: The proof is quite similar to the proof of Theorem 2.4 except from step 1 which we present in detail.

The positivity of c_0 is a consequence of Proposition 4.2 since it follows easily that $\max_{t\geq 0}\mathcal{J}_0(tu) \geq \beta$, and therefore

$$c_0 = \inf_{u\in\mathcal{M}_0}\mathcal{J}_0(u) = \inf_{u\in H^1(\Omega)}\max_{t\geq 0}\mathcal{J}_0(tu) \geq \beta > 0.$$

Let $\{u_m\}$ be a minimizing sequence for c_0. Working as in the derivation of (2.25) we get that (as $m\to\infty$)

$$\frac{1}{2(N-1)}\int |\nabla u_m|^2 dx + \frac{(N-2)^2}{2(N-1)}\int_\Omega Q|u_m|^{2^*}dx \leq c_\lambda + o(1)\,\|u_m\|_{H^1} + o(1),$$

for $\lambda = 0$, and therefore

$$\int_\Omega |\nabla u_m|^2 \, dx \le C + o(1) \, \|u_m\|_{H^1}, \tag{4.3}$$

for some constant $C > 0$. Also, by Young's inequality,

$$\int_\Omega u_m^2 \, dx \le \frac{2}{2^*} \int_\Omega |u_m|^{2^*} \, dx + \frac{2^* - 2}{2^*} |\Omega| \le \frac{2}{2^* \min_{x \in \bar{\Omega}} Q(x)} \int_\Omega Q(x) |u_m|^{2^*} \, dx + \frac{2^* - 2}{2^*} |\Omega|,$$

and therefore

$$\int_\Omega u_m^2 \, dx \le C + o(1) \, \|u_m\|_{H^1}. \tag{4.4}$$

Inequalities (4.3) and (4.4) yield the boundedness of $\{u_m\}$ in $H^1(\Omega)$. The rest of the proof is quite similar to the proof of Theorem 2.4 and is omitted. ///

We finally state the following existence result.

Theorem 4.4 *Let $\int_{\partial\Omega} P(x) \, dS_x < 0$.*

(a) Suppose that (3.1) holds and let $\mathcal{C}(x_0) = \min_{x \in \partial\Omega} \mathcal{C}(x)$. We assume that both functions Q and P are differentiable at the point x_0 and we denote by $\frac{\partial Q}{\partial \nu}(x_0)$ the outward normal derivative and by $H(x_0)$ the mean curvature of $\partial\Omega$ at x_0. Then problem (4.1) has a solution, provided that:
(i) in case $N = 3$, $H(x_0) > 0$,
(ii) in case $N \ge 4$, $H(x_0) \ge 0$, $\frac{\partial Q}{\partial \nu}(x_0) \le 0$ and $H(x_0) - \frac{\partial Q}{\partial \nu}(x_0) > 0$.

(b) Suppose that (3.26) holds and moreover

$$\frac{\left| \int_{\partial\Omega} P(x) dS_x \right|^N Q_M^{\frac{N-2}{2}}}{\left(\int_\Omega Q(x) dx \right)^{N-1}} < 2(N-1) N^{\frac{N}{2}} (N-2)^{\frac{N-2}{2}} S^{\frac{N}{2}}. \tag{4.5}$$

Then problem (4.1) has a solution.

Proof: The proof of part (a) is the same as the proof of Theorem 3.1 (the positivity of λ played no role there). Part (b) is an immediate consequence of Theorems 3.2 and 4.3.

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