Critical Phenomena in Linear Elliptic Problems

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Abstract

In this article we consider the existence and nonexistence of principal eigenvalues for the linear elliptic equation

$$-\Delta u = \lambda V(x)u \quad in \ \mathbb{R}^N \ and \ u \in \mathcal{D}^{1,2}(\mathbb{R}^N),$$

where $N \geq 3$, and $V \in L^1_{loc}(\mathbb{R}^N)$ with $V \overset{>}{\not\equiv} 0$ in $\mathbb{R}^N$.

We establish that when potential V has strong singularities phenomena which appear under critical exponent nonlinearities (Brezis and Nirenberg), are present in linear problems as well.

1 Introduction and statement of the main results

In this work we are interested in the existence of (positive) principal eigenvalues for linear elliptic equations of the form :

$$-\Delta u = \lambda V(x)u \quad in \ \mathbb{R}^N \ and \ u \in \mathcal{D}^{1,2}(\mathbb{R}^N), \tag{1.1}$$

where $N \geq 3$, $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is the space of functions find after completing the space of $C_0^\infty(\mathbb{R}^N)$ functions with the norm $(\int_{\mathbb{R}^N} |\nabla u|^2 \, dx)^{1/2}$, and $V \in L^1_{loc}(\mathbb{R}^N)$ with $V \overset{>}{\not\equiv} 0$ in $\mathbb{R}^N$. The type of potentials we have in mind will include potentials behaving like $\frac{1}{|x|^2}$, which appear in Hardy's inequality.

This type of problem arises in Population Genetics. In particular we arrive at (1.1) when we linearize nonlinear models such as those of Nagylaki [30, 31], or Slatkin [41]. It is then expected that branches of solutions to the nonlinear problem will bifurcate from the zero solution. This will happen at those values of the parameter λ which correspond to eigenvalues of our linear problem. The work that has been done in this direction [24, 42] supports it at least under certain conditions on the potential V and nonlinearities.

Another source is when you are dealing with supercritical semilinear elliptic equations in $\mathbb{R}^N$ (see Ni [22]), and especially when you linearize around the singular ground state, or around ground states.

Also, similar types of problems arise in quantum mechanics, where spectral properties of the Schrödinger operator in $\mathbb{R}^N$ play a fundamental role. This time the interest focuses on negative eigenvalues of operators of the form $A(V) = -\Delta - \lambda V$ in the $L^2(\mathbb{R}^N)$ space. We refer to S. Agmon [3], M. S. Birman-M. Z. Solomyak [9], B. Simon [40] and the references therein.

The existence of principal eigenvalues for similar type of problems has been studied previously by W. Allegretto [4], K. J. Brown, C. Cosner and J. Fleckinger [12], D. Daners [14], Z. Jin [23], M. Murata [29] and

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G. Rozenblioum-M. Solomyak [37]. In [4, 12, 37], the approach to the problem is by studying an appropriate self-adjoint operator in a suitable Hilbert space. The main step in this approach is to find suitable function spaces for V, that will ensure precompactness in the seminorm $(\int_{I\!R^N} Vu^2(x)\,dx)^{1/2}$. That is, for any bounded sequence u_k (in a suitable Hilbert space), there exists a subsequence u_{n_k} and u such that $\int_{I\!R^N} Vu_{n_k}^2\,dx \to \int_{I\!R^N} Vu^2\,dx \quad as\ k \to +\infty$. In [29, 33], the approach is by a very careful analysis of the asymptotics of the Green function of a suitable linear operator.

Our approach is by introducing the Rayleigh quotient $I[u] = \frac{\int_{I\!R^N} |\nabla u|^2\,dx}{\int_{I\!R^N} Vu^2\,dx}$, to study the minimization problem

$$\lambda_{cr} = inf\left\{ \frac{\int_{I\!R^N} |\nabla u|^2\,dx}{\int_{I\!R^N} Vu^2\,dx}\,,\ u \in \mathcal{A}\right\}\ with\ \mathcal{A} = \left\{u \in C_0^\infty(I\!R^N)\,,\ \int_{I\!R^N} Vu^2\,dx > 0\right\}. \tag{1.2}$$

Typically λ_{cr} will be an eigenvalue of (1.1) if the infimum λ_{cr}, is attained by some $u \in \mathcal{D}^{1,2}(I\!R^N)$. The main advantage of this approach is that we do not require precompactness in the seminorm $(\int_{I\!R^N} Vu^2\,dx)^{1/2}$, for all bounded sequences in $\mathcal{D}^{1,2}(I\!R^N)$.

We recall that over the last twenty years, there has been a tremendous success in dealing with problems which lack compactness see for example Aubin [5], A. Barhi and J.M. Coron [6], V. Benci and G. Cerami [7], H. Brezis [10], H. Brezis and L. Nirenberg [11], J.F. Escobar [16], D.G. Figueiredo and B. Ruf [19], M. Flucher [20], P.L.Lions [27] and references therein. One aspect, that has been influential and we believe connected to this work, is related to the model introduced by Brezis-Nirenberg [11]

$$-\Delta u = u^{\frac{N+2}{N-2}} + \lambda u \quad in\ \ \Omega\,,\quad u = 0 \quad on\ \ \partial\Omega\,, \tag{1.3}$$

where Ω is a bounded domain. Here one approach was to introduce the ratio $J_\lambda[u] = \frac{\int_\Omega |\nabla u|^2\,dx - \lambda\int_\Omega u^2\,dx}{(\int_\Omega |u|^{\frac{2N}{N-2}}\,dx)^{\frac{N-2}{N}}}$ and establish (as by Lieb), that if $S_\lambda = inf J_\lambda[u] < S_0 = inf J_0[u]$, then the infimum S_λ is attained. It was then proved that this was the case for $0 < \lambda < \lambda_1$ and $N \geq 4$, where λ_1 is the first eigenvalue of the Laplacian with Dirichlet boundary conditions. The case $N = 3$ was much more delicate ; there exists $0 < \lambda^* < \lambda_1$, such that $S_\lambda < S_0$ for $\lambda^* < \lambda < \lambda_1$ and $S_\lambda = S_0$ for $0 < \lambda < \lambda^*$. A similar delicate phenomenon appeared under nonlinear subcritical perturbations when $N = 3$ again. In the same work there was a so called "curious example" where the delicate phenomenon appeared for $N = 4$.

We would like to apply a similar technique to our problem but first someone has to identify the "asymptotic problem", which in the previous problem was the one with $\lambda = 0$. P. L. Lions [27, 26] , in dealing with (nonlinear) problems of best constants in unbounded domains, introduced the method of Concentration-Compactness to handle, among other things, problems with critical exponents. It is worth mentioning that P. L. Lions used various invariances (mainly dilations) to identify the "asymptotic problem". There, the nonlinearity prevented point concentrations into the minimizing sequence. Related to this is the fact that the following Hardy's best constant problem $(\frac{N-2}{2})^2 = inf\{\frac{\int_{I\!R^N} |\nabla u|^2\,dx}{\int_{I\!R^N} \frac{u^2}{|x|^2}\,dx}\ ,\ \ u \in \mathcal{D}^{1,2}(I\!R^N),\ u \not\equiv 0\}$, does not attain a minimizer.

We now describe the method we use to resolve the problem. Suppose $\{u_k\}_{k \in I\!N}$ is a minimizing sequence of (1.2) with the property $\int_{I\!R^N} Vu_k^2\,dx \ = 1, \quad k = 1, 2, \ldots$ and $\int_{I\!R^N} |\nabla u_k|^2\,dx \to \lambda_{cr}\ \ as \ \ k \to +\infty$. In order to avoid mass leaving at infinity, we introduce for $R > 0$, the following series of minimization problems $(B_R = B_R(0))$, $\mu_R = inf\, I[u]$ over the admissible set $\mathcal{A}_R = \{u \in C_0^\infty(I\!R^N - B_R)\,,\ \int_{I\!R^N - B_R} Vu^2\,dx > 0\,\}$, where we set $\mu_R = +\infty$, if set $\mathcal{A}_R$ is empty. It is then easy to see that for all $R > 0$, $\lambda_{cr} \leq \mu_R$ and that μ_R is a nondecreasing function of R. We then denote $\mathcal{C}^\infty := \lim_{R \to \infty} \mu_R$ and call $\mathcal{C}^\infty$, the concentration of the potential V at infinity (which may be $+\infty$). We then have :

Theorem 1.1 *Suppose $N \geq 3$, and that $0 < \lambda_{cr} < \mathcal{C}^\infty$. Then there exists a subsequence $\{u_{n_k}\}_{k \in I\!N}$ of the minimizing sequence, with the property: For each $\varepsilon > 0$, there exists $R_0 = R_0(\varepsilon)$ such that for all $R \geq R_0$,*

$$\int_{B_R} Vu_{n_k}^2(x)\,dx \geq 1 - \varepsilon, \qquad k = 1, 2, \ldots \tag{1.4}$$

In a similar way, to avoid loss of compactness at points, we consider for $x_0 \in I\!R^N$ and $r > 0$ the minimization problems:

$$\lambda_r(x_0) = inf \left\{ \frac{\int_{B_r(x_0)} |\nabla u|^2 \, dx}{\int_{B_r(x_0)} V u^2 \, dx} \ , \quad u \in C_0^\infty(B_r(x_0)) \ and \ \int_{B_r(x_0)} V u^2 \, dx > 0 \right\} . \tag{1.5}$$

Again we set $\lambda_r(x_0) = +\infty$, when there is no admissible function $u \in C_0^\infty(B_r(x_0))$ to satisfy $\int_{B_r(x_0)} V u^2 \, dx > 0$. We then define $\mathcal{C}(x_0) := \lim_{r\to 0^+} \lambda_r(x_0)$, and we will call $\mathcal{C}(x_0)$, the concentration of the potential V, at point x_0. Sometimes we will use the notation $\mathcal{C}(x_0; V)$ to avoid confusion.

Definition 1 *A point $x_0 \in I\!R^N$, is called critical singular point for the potential V ,when $0 < \mathcal{C}(x_0) < +\infty$, whereas it is called subcritical point for the potential V, when $\mathcal{C}(x_0) = +\infty$.*

Definition 2 *A potential V is called a critical potential iff $0 < min\{ \, inf_{x_0 \in I\!R^N} \mathcal{C}(x_0), \ \mathcal{C}^\infty \} < +\infty$, whereas is called a subcritical potential iff $\mathcal{C}(x_0) = +\infty$ for all $x_0 \in I\!R^N$, and $\mathcal{C}^\infty = +\infty$.*

Definition 3 *We say that a potential V, attains its infimum $\lambda_{cr}(V)$, when there exists $\phi \in \mathcal{D}^{1,2}(I\!R^N)$ such that $\lambda_{cr}(V) = \frac{\int_{I\!R^N} |\nabla \phi|^2 \, dx}{\int_{I\!R^N} V \phi^2 \, dx}$.*

Connected with the problem we are studying is the following:

<u>Problem :</u> *Suppose $N \geq 3$. Under what conditions a potential $V \in L^1_{loc}(I\!R^N)$ with $V \stackrel{>}{\neq} 0$ in $I\!R^N$ is a "Hardy potential", that is there exists a positive constant $c = c(V)$ such that*

$$\int_{I\!R^N} V u^2 \, dx \ \leq \ c \int_{I\!R^N} |\nabla u|^2 \, dx, \quad \forall \ u \ \in C_0^\infty(I\!R^N) \ ? \tag{1.6}$$

Concerning the previous problem and we refer to S.Y.A. Chang, J.M. Wilson and T.H. Wolff [13], C.L. Fefferman [18], E. Sawyer and R.L. Wheeden [38], M. Schechter [39] and the references therein. In this direction various spaces (Fefferman-Phong, Morrey and weak-L^p spaces) play an important role (see also section 3 for the weak -L^p space).

In this work however we are trying to avoid a specific choice of the function space. For this we are working with a very general potential V, where we suppose that a-priori V is a "Hardy" potential. We moreover assume throught this work, except in section 2, that the potential V satisfies (Notice we interpret $V \stackrel{>}{\neq} 0$ in $I\!R^N$, as $V \geqq 0$ a.e. in $I\!R^N$ and $meas\{ V > 0\} > 0$):

(H) $\quad$ *$V \in L^1_{loc}(I\!R^N)$ with $V \stackrel{>}{\neq} 0$ in $I\!R^N$, $\lambda_{cr}(V) > 0$ and for each $\delta > 0$ the set*

$$\Sigma_\delta \ = \ \{ x \ \in \ I\!R^N \ \mid \ \mathcal{C}(x) \ \leq \ \delta \}$$

is finite or empty.

Theorem 1.2 *Suppose $N \geq 3$, V satisfies (H) and in addition $0 < \lambda_{cr} < \mathcal{C}^\infty$, and $0 < \lambda_{cr} < \mathcal{C}(x)$, $\forall x \in I\!R^N$. Then V attains its infimum.*

It is then standard to see that λ_{cr} corresponds to the first eigenvalue of the problem (1.1). Also notice that the eigenfunction ϕ, might be taken of one sign, thus $\phi \geq 0$ in $I\!R^N$. Equation (1.1), throughout this work is understood in the distributional sense, that is $\int_{I\!R^N} \nabla u \, \nabla \eta \, dx \ = \ \lambda \int_{I\!R^N} V u \eta \, dx, \quad \forall \eta \in C_0^\infty(I\!R^N)$.

Let us consider now a critical potential V that does not attain its infimum. We are thinking of perturbing it by another potential g so that the new potential $V + g$ to attain its infimum. That is we want to follow a similar procedure to what Brezis - Nirenberg [11] did. However the problem we now face is to find the right space for the potential g to make the previous procedure working. To this we take a perturbation g, that leaves the concentration function of V invariant, that is

$$\mathcal{C}^{\infty}(V+g) \,=\, \mathcal{C}^{\infty}(V) \quad and \quad \mathcal{C}(x;V+g) \,=\, \mathcal{C}(x;V) \qquad \forall\, x \text{ in } I\!R^N\,. \tag{1.7}$$

Fortunately, this is the case if g is a subcritical potential! By doing this type of perturbations we obtain

Theorem 1.3 *Suppose $N \geq 3$, and consider a critical potential V that satisfies (H) and a subcritical g, with $g \in L^1_{loc}(I\!R^N)$ and $g \overset{>}{\neq} 0$ in $I\!R^N$. Then there exists $\varepsilon_0 \geq 0$, such that the critical potential $V + \varepsilon g$ attains its infimum for all $\varepsilon > \varepsilon_0$.*

The interest now is how small ε_0 can be. Can it be zero? Does a minimizer even exist for $\varepsilon = \varepsilon_0$? To answer some of the previous questions we consider the critical potential $V_0(x) = \frac{1}{1+|x|^2}$ and the subcritical potential $g \in C(I\!R^N)$ to satisfy $g \overset{>}{\neq} 0$ *in* $I\!R^N$, with compact support and define for $\varepsilon > 0$, $V_\varepsilon(x) = \frac{1}{1+|x|^2} + \varepsilon g(x)$, $x \in I\!R^N$. Then :

Theorem 1.4 *Suppose $N \geq 3$. Then there exists $\varepsilon_0 = \varepsilon_0(g) > 0$ such that V_ε does attain its infimum (and the corresponding eigenvalue problem has an eigenvalue) for all $\varepsilon > \varepsilon_0$, but does not for $0 \leq \varepsilon < \varepsilon_0$.*

Someone might see Theorem 1.4 (see also Theorem 4.6) as the analogue of the delicate phenomenon of Brezis-Nirenberg. What is worth noting is that in our problem this phenomenon occurs for any space dimension $N \geq 3$. The next natural question is whether Theorem 1.4 holds for all critical potentials, that do not attain their infimum (instead of $V_0(x) = \frac{1}{1+|x|^2}$). For this reason if we consider the critical potential $W_0(x) = \frac{\frac{2N}{N-2}+|x|^2}{(1+|x|^2)^2}$, and take $g \in L^1_{loc}(I\!R^N)$ any subcritical potential with $g \overset{>}{\neq} 0$ in $I\!R^N$, to define $W_1(x) = \frac{\frac{2N}{N-2}+|x|^2}{(1+|x|^2)^2} + g(x)$. Then :

Theorem 1.5 *Suppose $N \geq 3$. Then W_1 attains its infimum and the corresponding eigenvalue problem has a positive principal eigenvalue λ_{cr}.*

To see that if we do not require $u \in \mathcal{D}^{1,2}(I\!R^N)$ the eigenvalue problem that we consider might have lots of eigenvalues, we include a result proved in Murata [29], see also Theorem 5.1.

Proposition 1.6 *Suppose $N \geq 3$, then for all $\varepsilon \geq 0$, the eigenvalue problem*

$$-\Delta u \,=\, \lambda V_\varepsilon u \quad in\ I\!R^N, \quad \lim_{|x|\to+\infty} u(x) \,=\, 0, \tag{1.8}$$

has positive solutions for all $0 < \lambda \leq \lambda_{cr}(V_\varepsilon)$.

Finally we consider the critical potential $\frac{1}{|x|^2}$ that appears in Hardy's inequality . Then our result reads

Theorem 1.7 *Suppose $N \geq 3$, and let $g \in L^1_{loc}(I\!R^N)$ be any subcritical potential with $g \overset{>}{\neq} 0$ in $I\!R^N$. Then the critical potential $V(x) = \frac{1}{|x|^2} + g(x)$, attains its infimum in $\mathcal{D}^{1,2}(I\!R^N)$.*

It is worth noting that the scale invariance of the critical Sobolev exponent problem played an essential role in P.L. Lions [27] approach in order to avoid mass of escaping at infinity. In our case the scale invariance of the Hardy potential $\frac{1}{|x|^2}$ does not prevent mass of escaping at infinity(and at zero); it helps in obtaining the escaping level though.

When instead we perturb a critical potential by another critical one, then nearly everything is possible and we refer for this to section 4. For example we have two critical potentials that do not attain their infimum, but their sum does!

It is also worth mentioning that the solutions we obtain, when potential V is critical, may have singularities. It is already known that solutions might have singularities if V is not in the Kato class, see M. Aizenman and B. Simon [1] or B. Simon [40] (see also section 3).

An alternative approach suggested by the referee in the case the potential V is smooth and positive, is by considering the Friedrichs L^2 extension $\tilde{P}$ of the operator $P = -(V(x))^{-1}\Delta \ in \ I\!R^N$, and turning $I\!R^N$ into a Riemannian manifold M with metric $ds^2 = (V(x))^{-1}\sum_{i=1}^N dx_i^2$. In this case it is easy to establish that $\mathcal{C}^\infty = inf\sigma_{ess}(\tilde{P})$ and our criterion for existence of eigenvalues reads $0 < \lambda_{cr} < inf\sigma_{ess}(\tilde{P})$, a well known criterion in spectral theory (see S. Agmon [2]). Then by using results from M. Murata [29], Y. Pinchover [33] one could give an alternative proof to some of the results of this work (for example Theorems 1.3, 1.4).

With our (nonlinear) approach we hope to give some light in the connections between linear and nonlinear theory. One of the advantages of our approach is the flexibility of it. For example, the same approach works in a number of other differential operators with potentials having continuous strong singularities. We intend to proceed further this point in a subsequent work.

The paper is organized as follows: Section 2 contains the proof of Theorem 1.1. Existence of minimizers is established in section 3, where in section 4 we consider the problem of perturbing a critical potential; finally in the last section we have the proofs of Theorems 1.5 , 1.7 and Proposition 1.6.

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2 The Concentration at infinity

This section is devoted in establishing Theorem 1.1. The proof will be similar in the spirit to the work of P. L. Lions [27]. We make extensive use of Sobolev inequality at various places.

Proof of Theorem 1.1 : For the proof we will use the Concentration-Compactness Lemma of Lions, with concentration function $\varrho_k = |\nabla u_k|^2 + u_k^{\frac{2N}{N-2}} + Vu_k^2$. We also denote $L_k = \int_{I\!R^N} \varrho_k \, dx$, hence $L_k \geq \int_{I\!R^N} |\nabla u_k|^2 \, dx + \int_{I\!R^N} V u_k^2 \, dx$, which in turn implies that we may suppose $L_k \to L \geq \lambda_{cr} + 1$, as $k \to +\infty$.

Step 1: Vanishing does not occur. We will first prove that there exists a sequence $(y_k)_{k \in I\!N} \in I\!R^N$ such that $\varrho_{n_k}(. + y_k)$ is tight in $I\!R^N$. To this we suppose initially that vanishing occurs. That is for each $R > 0$, $\lim_{k \to +\infty} \left(\sup_{x \in I\!R^N} \int_{B_R(x)} \varrho_k(x) \, dx \right) = 0$.

Since $\lambda_{cr} < \mathcal{C}^\infty$, we may take R_0 large enough to satisfy $\lambda_{cr} < \mu_{R_0}$. We also take $\phi \in C_b^\infty(I\!R^N)$, satisfying $0 \leq \phi \leq 1 \ in \ I\!R^N$, with $\phi \equiv 0$ in B_1 and $\phi \equiv 1$ in $I\!R^N - B_2$. Hence $u_k \phi(\frac{\cdot}{R_0}) \in C_0^\infty(I\!R^N - B_{R_0})$ and in addition by using Sobolev inequality in $B_{2R_0} - B_{R_0}$ we have

$$\left| \int_{I\!R^N} |\nabla(u_k\phi)|^2 \, dx - \int_{I\!R^N} |\nabla u_k|^2 \phi^2 \, dx \right| \leq$$

$$\leq c \left(\int_{B_{2R_0} - B_{R_0}} |\nabla u_k|^2 \, dx \right)^{1/2} \left(\int_{B_{2R_0} - B_{R_0}} |u_k|^{\frac{2N}{N-2}} \, dx \right)^{\frac{N-2}{2N}} + c \left(\int_{B_{2R_0} - B_{R_0}} |u_k|^{\frac{2N}{N-2}} \, dx \right)^{\frac{N-2}{N}}$$

$$\leq c \left(\int_{B_{2R_0} - B_{R_0}} \varrho_k \, dx \right)^{\frac{N-2}{2N}} + c \left(\int_{B_{2R_0} - B_{R_0}} \varrho_k \, dx \right)^{\frac{N-2}{N}}, \tag{2.1}$$

where the constants c are independent of R_0 and k. Notice that we also have

$$\int_{I\!R^N} |\nabla u_k|^2 \, dx = \int_{I\!R^N} |\nabla(u_k\phi)|^2 \, dx + \int_{I\!R^N} |\nabla(u_k(1-\phi))|^2 \, dx + 2 \int_{I\!R^N} \nabla(u_k\phi) \, \nabla(u_k(1-\phi)) \, dx$$

$$\geq \mu_{R_0} \int_{\mathbb{R}^N} V u_k^2\, dx \,-\, c \int_{B_{R_0}} \varrho_k\, dx \,-\, c \left(\int_{B_{2R_0}} \varrho_k\, dx \right)^{\frac{N-2}{N}} \tag{2.2}$$

for some constant c independent of R_0, k. We now pass to the limit $k \to +\infty$, to conclude $\lambda_{cr} \;\geq\; \mu_{R_0}$, which is a contradiction.

<u>*Step 2: Dichotomy does not occur*</u>. In order to prove it we suppose that dichotomy does occur. We may therefore assume that there exists $\bar{a} \in (0, L)$ such that for every $\varepsilon > 0$ there exist $k_0 \geq 1$, $R_0 > 0$ and sequences $R_0 < R_k \to +\infty$ *as* $k \to +\infty$, $(y_k)_{k \in \mathbb{N}} \subset \mathbb{R}^N$ that satisfy for all $k \geq k_0$,

$$\int_{B_{R_k}(y_k) - B_{R_0}(y_k)} \varrho_{n_k}\, dx \,+\, \left| \int_{B_{R_0}(y_k)} \varrho_{n_k}\, dx \,-\, \bar{a} \right| \,+\, \left| \int_{\mathbb{R}^N - B_{R_k}(y_k)} \varrho_{n_k}\, dx \,-\, (L - \bar{a}) \right| \;\leq\; \varepsilon\,, \tag{2.3}$$

We then consider the functions ξ, η with $0 \leq \xi \leq 1$, $\xi \in C_0^\infty(\mathbb{R}^N)$ and $\xi \equiv 1$ if $|x| \leq 1$, $\xi \equiv 0$ if $|x| \geq 2$. Also, $0 \leq \eta \leq 1$, $\eta \in C_0^\infty(\mathbb{R}^N)$ with $\eta \equiv 1$ if $|x| \geq 1$ and $\eta \equiv 0$ if $|x| \leq 1/2$, and define $\xi_k(x) = \xi(\frac{x - y_k}{R_1}), \eta_k(x) = \eta(\frac{x - y_k}{R_k})$ for $k = 1, 2, \ldots$, where $R_1 \geq R_0$ will be determined later on. We also observe that

$$\begin{aligned}
&\left| \int_{\mathbb{R}^N} |\nabla u_k|^2\, dx \,-\, \int_{\mathbb{R}^N} |\nabla(\xi_k u_k)|^2\, dx \,-\, \int_{\mathbb{R}^N} |\nabla(\eta_k u_k)|^2\, dx \right| \;\leq \\
&\quad \leq \int_{B_{R_k}(y_k) - B_{R_1}(y_k)} |\nabla u_k|^2\, dx \,+\, \int_{B_{2R_1}(y_k) - B_{R_1}(y_k)} |\nabla(\xi_k u_k)|^2\, dx \,+\, \int_{B_{R_k}(y_k) - B_{\frac{1}{2}R_k}(y_k)} |\nabla(\eta_k u_k)|^2\, dx \\
&\quad \leq \varepsilon \,+\, c X_k^2\,,
\end{aligned} \tag{2.4}$$

provided k is large enough so that $4R_1 \leq R_k$, where we put

$$X_k^2 \;=\; \int_{B_{2R_1}(y_k) - B_{R_1}(y_k)} \Big(|\nabla \xi_k|^2 u_k^2 \,+\, \xi_k^2 |\nabla u_k|^2 \Big)\, dx + \int_{B_{R_k}(y_k) - B_{\frac{1}{2}R_k}(y_k)} \Big(|\nabla \eta_k|^2 u_k^2 \,+\, \eta_k^2 |\nabla u_k|^2 \Big)\, dx\,.$$

However by taking large k, $X_k^2 \;\leq\; \int_{R_0 \leq |x - y_k| \leq R_k} |\nabla u_k|^2\, dx + \; c\, (\int_{R_0 \leq |x - y_k| \leq R_k} |u_k|^{\frac{2N}{N-2}}\, dx)^{\frac{N-2}{N}}$, where $c = max\, \{ (\int_{\mathbb{R}^N} |\nabla \eta_k|^N\, dx)^{\frac{2}{N}}\,, (\int_{\mathbb{R}^N} |\nabla \xi_k|^N\, dx)^{\frac{2}{N}} \}$, is independent of k, R_k. We finally obtain $X_k^2 \,\leq \varepsilon + c\, \varepsilon^{\frac{N-2}{N}}$ for large k, which in turn implies that

$$\left| \int_{\mathbb{R}^N} |\nabla u_k|^2\, dx \,-\, \int_{\mathbb{R}^N} |\nabla(\xi_k u_k)|^2\, dx \,-\, \int_{\mathbb{R}^N} |\nabla(\eta_k u_k)|^2\, dx \right| \;\leq\; c\, (\varepsilon \,+\, \varepsilon^{\frac{N-2}{N}})\,. \tag{2.5}$$

We also have $|\int_{\mathbb{R}^N} V u_k^2\, dx \;-\; \int_{\mathbb{R}^N} V(\xi_k u_k)^2\, dx \;-\; \int_{\mathbb{R}^N} V(\eta_k u_k)^2\, dx\,| \;\leq\; \varepsilon$. If we denote $\alpha_k = \int_{\mathbb{R}^N} V \xi_k^2 u_k^2\, dx$, $\beta_k = \int_{\mathbb{R}^N} V \eta_k^2 u_k^2\, dx$, then without loss of generality, we may assume that $\alpha_k \to \alpha$ and $\beta_k \to \beta$ as $k \to +\infty$, where $0 \leq \alpha\,,\, \beta \leq 1$ with $|\, \alpha \,-\, (1 - \beta)\,| \;\leq\; \varepsilon$.

In a similar fashion we can also establish that

$$\left| \bar{a} \,-\, \int_{\mathbb{R}^N} \left\{ |\nabla(\xi_k u_k)|^2 \,+\, |\xi_k u_k|^{\frac{2N}{N-2}} \,+\, V |\xi_k u_k|^2 \right\} dx \right| \;\leq\; c\, (\varepsilon \,+\, \varepsilon^{\frac{N}{N-2}}) \tag{2.6}$$

and

$$\left| (L - \bar{a}) \,-\, \int_{\mathbb{R}^N} \left\{ |\nabla(\eta_k u_k)|^2 \,+\, |\eta_k u_k|^{\frac{2N}{N-2}} \,+\, V |\eta_k u_k|^2 \right\} dx \right| \;\leq\; c\, (\varepsilon \,+\, \varepsilon^{\frac{N}{N-2}}). \tag{2.7}$$

Therefore we may find $\gamma > 0$ such that $\int_{\mathbb{R}^N} |\nabla(\xi_k u_k)|^2\, dx \;\geq\; \gamma$, $\int_{\mathbb{R}^N} |\nabla(\eta_k u_k)|^2\, dx \;\geq\; \gamma$ for all $k \in \mathbb{N}$, $\varepsilon > 0$ small. Now, if for some sequence $\varepsilon_k \to 0^+$, we have $\alpha_k = \alpha(\varepsilon_k)$, and α_k either goes to zero or to one, we deduce from (2.5), that $\int_{\mathbb{R}^N} |\nabla u_k|^2\, dx \;\geq\; \gamma \,+\, \lambda_{cr}\, max\, \{\alpha_k,\, \beta_k\} \,-\, c\, (\varepsilon_k \,+\, \varepsilon_k^{\frac{N}{N-2}})$, and by passing to the limit, that $\lambda_{cr} \;\geq\; \gamma \,+\, \lambda_{cr}$, which is impossible. Therefore α and β stay away from zero as $\varepsilon \to 0^+$.

Our argument now depends on whether sequence $\{y_k\}_{k\in I\!N}$ is bounded or not. Suppose first that it is bounded. Then we may suppose without loss of generality that $y_k \to y_0$. Then by taking k large enough say $k \geq k_0$, we will also have $\int_{I\!R^N} |\nabla(\eta_k u_k)|^2\,dx \;\geq\; \mu_{R_0} \int_{I\!R^N} V(\eta_k u_k)^2\,dx$ and $\int_{I\!R^N} |\nabla(\xi_k u_k)|^2\,dx \;\geq\; \lambda_{cr} \int_{I\!R^N} V(\xi_k u_k)^2\,dx$ where μ_{R_0} is chosen to satisfy $\lambda_{cr} < \mu_{R_0}$. In connection with (2.5), we obtain $\int_{I\!R^N} |\nabla u_k|^2\,dx \geq \mu_{R_0}\alpha_k + \lambda_{cr}\beta_k - c\,(\varepsilon + \varepsilon^{\frac{N}{N-2}})$, and by passing to the limit that

$$\lambda_{cr} \;\geq\; \mu_{R_0}\alpha + \lambda_{cr}\beta - c\,(\varepsilon + \varepsilon^{\frac{N}{N-2}})\,. \tag{2.8}$$

We arrive in a similar situation when sequence $\{y_k\}_{k\in I\!N}$ is unbounded, the only difference will be that when $|y_k| \to +\infty$, we have $\int_{I\!R^N} |\nabla(\xi_k u_k)|^2\,dx \;\geq\; \mu_{R_0} \int_{I\!R^N} V(\xi_k u_k)|^2\,dx$, which in turn implies $\lambda_{cr} \;\geq\; \mu_{R_0}\beta + \lambda_{cr}\alpha - c\,(\varepsilon + \varepsilon^{\frac{N}{N-2}})$. In any case, by passing through a subsequence to the limit $\varepsilon = 0$, we may suppose the existence of $m,\, l \in (0,1)$ with $m + l = 1$ and $\alpha(\varepsilon_k) \to m$, $\beta(\varepsilon_k) \to l$ as $k \to +\infty$, and by passing to the limit in (2.8) (similar is the situation in the other case), that $\lambda_{cr} \;\geq\; \mu_{R_0} m + \lambda_{cr} l$, and finally arrive at the contradiction $\lambda_{cr} \;\geq\; \mu_{R_0}$. Thus dichotomy is avoided.

Step 3: Completion of the proof. We complete the proof of Theorem 1.1 by establishing, that the sequence $\{y_k\}_{k\in I\!N}$ is bounded. Suppose on the contrary that it is unbounded, without loss of generality we may take $|y_k| \to +\infty$. Then for each $\varepsilon > 0$, $\exists R$ such that $\int_{B_R(y_k)} \varrho_{n_k}\,dx \;\geq\; L - \varepsilon$, and by defining the function $\xi(\frac{x-y_k}{R})\,u_k$ (ξ as above), a computation like in (2.4) gives for large k,

$$\left|\int_{I\!R^N} V u_k^2\,dx - \int_{I\!R^N} V(\xi u_k)^2\,dx\right| + \left|\int_{I\!R^N} |\nabla u_k|^2\,dx - \int_{I\!R^N} |\nabla(\xi u_k)|^2\,dx\right| \leq c\,(\varepsilon + \varepsilon^{\frac{N}{N-2}})\,. \tag{2.9}$$

Therefore by taking k large enough, to obtain $supp\xi \subset I\!R^N - B_{R_0}$, thus

$$\int_{I\!R^N} |\nabla u_k|^2\,dx \;\geq\; \mu_{R_0} \int_{I\!R^N} V(\xi u_k)^2\,dx - c\,(\varepsilon + \varepsilon^{\frac{N}{N-2}}). \tag{2.10}$$

We may also suppose that for some $\alpha = \alpha(\varepsilon)$, $0 \leq \alpha \leq 1$, and $\alpha_k = \int_{I\!R^N} V(\xi u_k)^2\,dx \to \alpha$, as $k \to +\infty$, to conclude from (2.10) , that $|\,1 - \alpha\,| \;\leq\; \varepsilon$ and $\lambda_{cr} \;\geq\; \alpha\,\mu_{R_0} - c(\varepsilon + \varepsilon^{\frac{N}{N-2}})$. Then by passing to the limit $\varepsilon \to 0^+$ (through a subsequence), we finally arrive at a contradiction.

3 Existence of minimizers

As in the previous section we consider a minimizing sequence $\{u_k\}_{k\in I\!N} \subset \mathcal{A}$ that satisfy $I\,[u_k] \;\to\; \lambda_{cr}$, as $k \to +\infty$, and $\int_{I\!R^N} V u_k^2\,dx = 1$. Then from Theorem 1.1 we have that when $\lambda_{cr} \;<\; \mathcal{C}^\infty$, we may assume that the sequence u_k converges weakly in $\mathcal{D}^{1,2}$ and a.e. to some $u \in \mathcal{D}^{1,2}$. We may also suppose that the sequence $V u_k^2$ is a tight sequence, that is for each $\varepsilon > 0$, there exists $R_0 = R_0(\varepsilon)$ such that for $R \geq R_0$, and all k, $\int_{B_R} V u_k^2\,dx \;\geq\; 1 - \varepsilon$. Our main concern in this section is to find conditions under which we can pass to the limit. For this we may extend the ball B_R into a new larger one, also extend u_k to have common compact support in the new ball, but also $V u_k^2$ being tight and $u_k \rightharpoonup u$ in $\mathcal{D}^{1,2}$ and a.e. Hence we may suppose $u_k \to u$ *in* $L^2_{loc}(I\!R^N)$, $|\nabla u_k|^2 \;\rightharpoonup\; \mu$ *and* $V u_k^2 \;\rightharpoonup\; \nu$, *as* $k \to +\infty$, for some bounded nonnegative measures in $I\!R^N$. The main result in this section is to establish Theorem 1.2.

Throughout this section let $\Sigma = \{x_1, \ldots, x_k, \ldots\} \subset I\!R^N$, be the set of critical singular points of the potential V. Also, we set for $j = 1, 2, \ldots$ $\nu_j = \lim_{r\to 0^+} \nu(\,B(x_j\,,\,r)\,)$, $\mu_j = \lim_{r\to 0^+} \mu(\,B(x_j\,,\,r)\,)$. Then

Lemma 3.1 *Suppose $N \geq 3$, V satisfies (H) and that $0 < \lambda_{cr} < \mathcal{C}^\infty$and $0 < \lambda_{cr} < \mathcal{C}(x)$, $\forall x \in I\!R^N$. Then*

$$\mu \;\geq\; |\nabla u|^2 + \sum_{j=1}^{\infty} \mu_j \delta_{x_j} \;\; and \;\; \nu(I\!R^N) \;=\; \int_{I\!R^N} V u^2\,dx + \sum_{j=1}^{\infty} \nu_j\,. \tag{3.2}$$

A description of the "Hardy potentials" in terms of capacity is given in Maz'ja [28] . We recall that the capacity of a compact set E relative to Ω, where $E \subset \Omega$ *open* $\subset I\!R^N$, is defined by $Cap\,(\,E, \Omega\,) =$

$\inf\{ \int_\Omega |\nabla u|^2\,dx,\ u \in C_0^\infty(\Omega)\ with\ u \geq 1\ in\ E\}$, and we use the notation $Cap\,E$ when $\Omega = I\!R^N$. For some of the basic results concerning the capacity we refer to Maz'ja [28] . Then the result of Maz'ja [Theorem of page 112] reads as follows (see also Theorem 4.2 in [28]):

Proposition 3.2 *Suppose $N \geq 3$, and $V \in L^1_{loc}(I\!R^N)$ with $V \overset{>}{\not\equiv} 0$ in $I\!R^N$.*

1. *If there exists a positive constant β such that for all compact sets $E \subseteq I\!R^N$,*

$$\int_E V\,dx \;\leq\; \beta\, Cap E\,, \tag{3.3}$$

then there exists $c \leq 4\beta$ such that for all $u \in C_0^\infty(I\!R^N)$,

$$\int_{I\!R^N} Vu^2\,dx \;\leq\; c\int_{I\!R^N} |\nabla u|^2\,dx\,. \tag{3.4}$$

2. *If (3.4) is valid for some constant c and for all $u \in C_0^\infty(I\!R^N)$, then (3.3) holds for some constant β and for all compact sets E. In particular we may take β to satisfy $\beta \leq c$.*

We also recall some basic results concerning the weak $L^{\frac{N}{2}}$ space (or Marcinkiewicz space), which we denote by $L^{\frac{N}{2},\infty}$. A measurable function f defined on a domain $\Omega \subseteq I\!R^N$ is said to belong to $L^{\frac{N}{2},\infty}(\Omega)$ if and only if $\|f\|_{\frac{N}{2},\infty} = \sup_{s>0}\{\, s\,(\nu_f(s))^{\frac{N}{2}}\,\} < +\infty$, where the distribution function $\nu_f(s)$, is $\nu_f(s) = meas\{\, x \in \Omega,\ |f(x)| > s\,\}$. Another characterization of the space $L^{\frac{N}{2},\infty}(\Omega)$ is in the following way (C. Bennett and R.Sharpley [8]) f is in $L^{\frac{N}{2},\infty}(\Omega)$ if and only if $\|f\|^*_{\frac{N}{2},\infty} = \sup_E |E|^{-1+\frac{2}{N}} \int_E |f|\,dx < +\infty$, and $L^{\frac{N}{2},\infty}(\Omega)$ is a Banach space with norm $\|\ \|^*_{\frac{N}{2},\infty}$. Then by using the immediate inequality $(meas\,K)^{1-\frac{2}{N}} \;\leq\; c(N)\,Cap\,K$, it is then easy to see that if $V \in L^{\frac{N}{2},\infty}(I\!R^N)$ *with* $V \overset{>}{\not\equiv} 0$ *in* $I\!R^N$, then V is a "Hardy potential", that is $\lambda_{cr}(V) > 0$. On the other hand if $V \notin L^{\frac{N}{2},\infty}(I\!R^N)$ then it is easy to see that this might lead to the non-existence of eigenvalues for the problem

$$-\Delta u \;=\; \lambda V u \quad in \quad I\!R^N,\ \ u > 0 \quad in \quad I\!R^N. \tag{3.5}$$

In particular, we have the following result:

Lemma 3.3 *Suppose $N \geq 3$, V in $C^\alpha_{loc}(I\!R^N)$ with $V \overset{>}{\not\equiv} 0$ in $I\!R^N$. If in addition there exist E_k compact $\subset I\!R^N$, such that $\lim_{k\to+\infty} \frac{1}{Cap\,E_k}\int_{E_k} V\,dx = +\infty$, then $\lambda_{cr} \;=\; 0$. Moreover there is no $\lambda > 0$, so that problem (3.5) has a (distributional) solution.*

<u>Proof:</u> Suppose on the contrary that for some $\lambda > 0$, problem (3.5) has a solution u. Then under our hypotheses it is standard to see that the solution is a classical one. Also $\lambda_{cr} = 0$ follows from Proposition 3.2 part 2. From the definition of λ_{cr}, then follows the existence of $x_0 \in I\!R^N$, $R > 0$ such that λ_1, defined by

$$\lambda_1 \;=\; \inf\left\{ \frac{\int_{B(x_0,R)} |\nabla u|^2\,dx}{\int_{B(x_0,R)} Vu^2\,dx},\ u \in\ C_0^\infty(B(x_0,R))\,,\ \int_{B(x_0,R)} Vu^2\,dx > 0 \right\} \tag{3.6}$$

satisfies $0 < \lambda_1 < \lambda$. It is then standard the existence of a classical solution ϕ_1 to the problem $-\Delta\phi_1 = \lambda_1 V\phi_1$ *in* $B(x_0,R)$, $\phi_1 = 0$ *on* $\partial B(x_0,R)$. We arrive at a contradiction by establishing

$$\int_{B(x_0,R)} |\nabla\phi_1 \;-\; \frac{\phi_1}{u}\nabla u|^2\,dx \;+\; \frac{(\lambda-\lambda_1)}{\lambda_1}\int_{B(x_0,R)} |\nabla\phi_1|^2\,dx \;=\; 0. \tag{3.7}$$

We also have (Maz'ja [28] Theorem 2 on page 129):

Proposition 3.4 *Suppose $N \geq 3$, V in $L^1_{loc}(\mathbb{R}^N)$ with $V \gneq 0$ in $\mathbb{R}^N$, Ω open, bounded subset of $\mathbb{R}^N$ with smooth boundary.*

1. *Suppose for some small $\delta > 0$, there exists $K > 0$ such that for any E compact $\subseteq \Omega$ with $diam\,(E) \leq \delta$*
$$\int_E V\,dx \;\leq\; K\,Cap\,(E,\Omega)\,. \tag{3.8}$$
Then there exist $c = c(N)$ such that for all $u \in C_0^\infty(\Omega)$,
$$\int_\Omega Vu^2\,dx \;\leq\; c\,K\,\int_\Omega \Big\{|\nabla u|^2 \,+\, \delta^{-2}|u|^2\Big\}.\,dx \tag{3.9}$$

2. *Suppose (3.9) holds for some K, $\delta > 0$. Then there is a constant of the form $c\,K$, with $c = c(N)$ such that (3.8) holds as well.*

We now connect the previous results with our concentration function and establish:

Lemma 3.5 *Suppose $N \geq 3$ and Ω is an open and bounded subset of $\mathbb{R}^N$ with smooth boundary. If $\mathcal{C}(x) \geq M$ for all $x \in \bar{\Omega}$, then there exist constants $c = c(N)$, $\delta_0 = \delta_0(\Omega)$ such that:*

1. *For any E closed $\subseteq \Omega$ with $diam\,E \leq \delta_0$, $\int_E V\,dx \leq \frac{c}{M}\,Cap(E,\Omega)$.*
2. *For any $\phi \in C_0^\infty(\Omega)$ with $0 \leq \phi \leq 1$ in Ω, $|\int_\Omega \phi^2\,d\nu - \int_\Omega V\phi^2u^2\,dx| \leq \frac{c}{M^{1/2}}$.*

<u>Proof:</u> Let $\varepsilon > 0$, then for each $x \in \bar{\Omega}$, there exists $\delta_x > 0$ with the property $\lambda_r(x) \geq M - \varepsilon$, for $0 < r \leq 2\delta_x$. We then use the inclusion $\bar{\Omega} \subset \cup_{x\in\bar{\Omega}} B(x,\delta_x)$, in connection with Lebesque's lemma to obtain the existence of $\delta_0 > 0$, with the property : For any E closed $\subseteq \Omega$ with $diam\,E \leq \delta_0$, then there exists $x_j \in \bar{\Omega}$ such that $E \subset B(x_j,\delta_{x_j})$. Next we consider $\phi \in C_0^\infty(B(x_j,\delta_{x_j}) \cap \Omega)$, with $\phi \geq 1$ in E, and use the monotonicity of $\lambda_r(x_0)$ to conclude $M - \varepsilon \leq \frac{\int_{B(x_j,2\delta_{x_j})\cap\Omega} |\nabla\phi|^2\,dx}{\int_{B(x_j,2\delta_{x_j})\cap\Omega} V\phi^2\,dx}$, which in turn gives $(M - \varepsilon)\int_E V\,dx \leq Cap\,(E, B(x_j,2\delta_{x_j})\cap\Omega)$. Finally, Theorem 2 of Maz'ja [28] yields the existence of $c = c(N)$ with, $(M - \varepsilon)\int_E V\,dx \leq c\,Cap\,(E\,,\,\Omega)$. By the use of Proposition 3.4 we obtain
$$\int_\Omega V(u_k - u)^2\phi^2\,dx \;\leq\; \frac{c}{M-\varepsilon}\int_\Omega\Big\{|\nabla\phi(u_k-u)|^2 \,+\, \delta_0^{-2}\phi^2(u_k-u)^2\Big\}\,dx\;,\; k = 1,2,\ldots \tag{3.10}$$
for some constant $c = c(N)$. The result follows easily by passing to the limit in (3.10) and using the inequality
$$\left|\left(\int_\Omega Vu_k^2\phi^2\,dx\right)^{1/2} - \left(\int_\Omega Vu^2\phi^2\,dx\right)^{1/2}\right|^2 \;\leq\; \int_\Omega V(u_k-u)^2\phi^2\,dx\;.$$

<u>Proof of Lemma 3.1:</u> We start noting that $\sum_{j\in\mathbb{N}} \mu_j\delta_{x_j}$, is part of the singular part of the measure μ, hence $\mu \geq \sum_{j\in\mathbb{N}} \mu_j\delta_{x_j}$. Also by weak convergence $\mu \geq |\nabla u|^2$. Then the orthogonality gives the first part of (3.2). Working in a similar way, we have $\nu \geq Vu^2 + \sum_{j\in\mathbb{N}} \nu_j\delta_{x_j}$.

Without loss of generality we may suppose that the critical singular set Σ is defined in such a way to satisfy $\mathcal{C}(x_k) \leq \mathcal{C}(x_{k+1})$, $k = 1,2,\ldots$ Let $\varepsilon > 0$. Then for each $k \in \mathbb{N}$ there exist $R_k > 0$ such that $\nu(B_{R_j}(x_j)) \leq \nu_j + \frac{\varepsilon}{2^{1+j}}$, $j = 1,2,\ldots$ We also consider the set $\Sigma_{\varepsilon^2} = \{x \in \Sigma\,,\; \mathcal{C}(x) \leq \frac{1}{\varepsilon^2}\}$,and let us assume that for some $m \in \mathbb{N}$, $\Sigma_{\varepsilon^2} = \{x_1,\ldots,x_m\}$. From Theorem 1.1 there exists a subsequence (which we keep the same notation) and R_0, such that for all $R \geq R_0$, $\int_{B_R} Vu_k^2dx \geq 1 - \varepsilon$, $k = 1,2,\ldots$. We also choose R large enough to satisfy $\Sigma_{\varepsilon^2} \subset B_R$ and define $\Omega_\varepsilon = B_{2R} - \bigcup_{i=1}^m \bar{B}_{\frac{1}{2}R_i}(x_i)$. Let us also consider $\phi \in C_0^\infty(\mathbb{R}^N)$, with $0 \leq \phi \leq 1$, $\phi \equiv 1$ in B_R, $\phi \equiv 0$ in $\mathbb{R}^N - B_{2R}$ and $\eta \in C_b^\infty(\mathbb{R}^N)$ with $0 \leq \eta \leq 1$ in $\mathbb{R}^N$, $\eta \equiv 1$ in $\mathbb{R}^N - \bigcup_{i=1}^m B_{R_i}(x_i)$ and $\eta \equiv 0$ in $\bigcup_{i=1}^m B_{\frac{1}{2}R_i(x_i)}$. Then $1 - \varepsilon \leq \int_{B_R} Vu_k^2dx \leq \int_{B_{2R}} Vu_k^2\phi^2dx$, and by passing to the limit $1-\varepsilon \le \int_{B_{2R}} \phi^2 d\nu$. But $\int_{B_{2R}} \phi^2 d\nu \overset{\le}{=} \int_{B_{2R}-\bigcup_{i=1}^m \bar{B}_{\frac{1}{2}R_i(x_i)}} \phi^2\eta^2 d\nu + \sum_{i=1}^m \int_{B_{R_i}(x_i)} \phi^2 d\nu$. We then use Lemma 3.5 to conclude $\int_{\Omega_\varepsilon} \phi^2\eta^2 d\nu \le \int_{I\!R^N} Vu^2 dx + c\cdot\varepsilon$, for some constant $c = c(N)$. Hence

$$\begin{aligned}\int_{B_{2R}} \phi^2 d\nu &\le \int_{I\!R^N} Vu^2 dx + c\cdot\varepsilon + \sum_{i=1}^m \int_{B_{R_i}(x_i)} \phi^2 d\nu \\ &\le \int_{I\!R^N} Vu^2 dx + \sum_{j=1}^\infty \nu_j + (c+\frac{1}{2})\,\varepsilon. \end{aligned} \tag{3.11}$$

We then pass to the limit $\varepsilon \to 0^+$ in (3.11), to obtain $1 \le \int_{I\!R^N} Vu^2 dx + \sum_{j=1}^\infty \nu_j$, and the result then follows.

Proof of Theorem 1.2: We first claim that $\mathcal{C}(x_j)\nu_j \le \mu_j \quad for \;\; j = 1, 2, \ldots$.

To prove this we consider a test function $\phi \in C_0^\infty(I\!R^N)$ with $0 \le \phi \le 1$ in $I\!R^N$, $\phi \equiv 1$ in B_1 and $\phi \equiv 0$ in $I\!R^N - B_2$, and set $u_k\,\phi(\frac{x-x_j}{R})$ for $k, j \in I\!N$, to obtain for $j \in I\!N$ and $R > 0$,

$$\lambda_{2R}(x_j)\int_{2B_R(x_j)} V\left(u_k\phi\left(\frac{x-x_j}{R}\right)\right)^2 dx \le \int_{B_{2R}(x_j)} \left|\nabla\left(u_k\phi\left(\frac{x-x_j}{R}\right)\right)\right|^2 dx\,. \tag{3.12}$$

Therefore,

$$\begin{aligned}\lambda_{2R}(x_j)\int_{B_R(x_j)} Vu_k^2 dx &\le \int_{B_{2R}(x_j)} |\nabla u_k|^2 dx + \int_{B_{2R}(x_j)-B_R(x_j)} u_k^2|\nabla\phi|^2 dx \\ &+ \;2\int_{B_{2R}(x_j)} (u_k-u)\,\phi\nabla u_k\cdot\nabla\phi dx + 2\int_{B_{2R}(x_j)} u\phi\nabla u_k\cdot\nabla\phi dx. \end{aligned} \tag{3.13}$$

We now pass to the limit $k\to\infty$ to conclude,

$$\lambda_{2R}(x_j)\,\nu\,(B_R(x_j)) \le \mu\,(B_{3R}(x_j)) + \int_{B_{2R}(x_j)-B_R(x_j)} u^2|\nabla\phi|^2 dx + 2\int_{B_{2R}(x_j)} u\,\phi\,\nabla u\cdot\nabla\phi dx. \tag{3.14}$$

But since $\int_{B_{2R}(x_j)} u^2|\nabla\phi|^2 dx \le c\,(\int_{B_{2R}(x_j)-B_R(x_j)} u^{\frac{2N}{N-2}}\,dx\,)^{\frac{N-2}{N}}$, where the constant c depends only on N, we could pass to the limit $R \to 0^+$ in (3.14) and the result follows.

Next we complete the proof of the Theorem. We have from Lemma 3.1 $\int_{I\!R^N} |\nabla u|^2\,dx + \sum_{j\in I\!N}\mu_j \le \lambda_{cr}(\int_{I\!R^N} Vu^2\,dx + \sum_{j\in I\!N}\nu_j)$, so by using our previous claim, we deduce that

$$\int_{I\!R^N} |\nabla u|^2\,dx - \lambda_{cr}\int_{I\!R^N} Vu^2\,dx \le \sum_{j\in I\!N}(\lambda_{cr} - \mathcal{C}(x_j))\,\nu_j\,. \tag{3.15}$$

Since $\int_{I\!R^N} |\nabla u|^2\,dx - \lambda_{cr}\int_{I\!R^N} Vu^2\,dx \ge 0$, it follows $\nu_j \equiv 0\,,\quad j = 1, 2, \ldots$, which completes the proof.

Corollary 3.6 *Suppose $N \ge 3$, $V \in L^1_{loc}(I\!R^N)$ with $V \overset{>}{\neq} 0$ in $I\!R^N$ is a subcritical potential, that is $\mathcal{C}^\infty = +\infty$ and $\mathcal{C}(x) = +\infty \quad for\; all\; x \in I\!R^N$. Then V attains its infimum and the eigenvalue problem (1.1) has a positive principal eigenvalue.*

Proof: The result follows from Theorem 1.2, once we prove that $\lambda_{cr}(V) > 0$. This is easily achieved by a partition of unity argument combined with Gagliardo- Nirenberg- Sobolev inequality.

Remark 3.1 *It is worth mentioning that for $N \ge 3$, with $V \overset{>}{\neq} 0$ in $I\!R^N$, then V is a subcritical potential in any of the following cases: 1) $V \in L^{N/2}(I\!R^N)$, 2) $2p > N$ and $\int_{I\!R^N} |x|^{2p-N}V^p(x)\,dx < \infty$ or 3) $V \in L^{\frac{N}{2},\infty}(I\!R^N)$ with $\nu_V(s) = o(s^{-N/2})$, as $s \to 0$ and as $s \to \infty$. Hence a positive principal eigenvalue to (1.1) exists in these cases.*

In the theory of Schrödinger operators the Kato class of functions play an important role. We recall that for a measurable function f it belongs to the Kato class $K^N_{loc}(I\!\!R^N)$ $(N \geq 3)$ if for all $R > 0$,

$$\lim_{r\to 0} sup_{y\in B_R} \int_{B_r(y)} \frac{|f(x)|}{|x-y|^{N-2}}\, dx \;=\; 0. \tag{3.16}$$

The importance of the Kato class is due to the fact that when $V \in K^N_{loc}(I\!\!R^N)$, nonnegative distributional solutions of (1.1) are in fact continuous functions (they even satisfy a Harnack inequality), see M. Aizenman-B. Simon [40]. The connection of the Kato class with our results comes in a straightforward manner from Lemma (1.1) of E. Fabes, N. Garofalo and F.-H. Lin [17]. In particular, it is the easy to see that if $V \in K^N_{loc}(I\!\!R^N)$ with $V \overset{>}{\neq} 0$ in $I\!\!R^N$, then $\mathcal{C}(x;\, V) \;=\; +\infty$ for all x in $I\!\!R^N$. Hence we have

Corollary 3.7 *Suppose $N \geq 3$ and let $V \overset{>}{\neq} 0$ in $I\!\!R^N$, $V \in K^N_{loc}(I\!\!R^N)$ with $\lambda_{cr}(V) > 0$. If in addition $\lambda_{cr}(V) < \mathcal{C}^\infty$, then V attains its infimum and the eigenvalue problem (1.1) has a positive principal eigenvalue with the corresponding eigenfunction being continuous.*

This is in particular the case when $V \in L^{\frac{N}{2},\infty}(I\!\!R^N) \cap K^N_{loc}(I\!\!R^N)$ and in addition $\lambda_{cr}(V) < \mathcal{C}^\infty$.

4 Perturbing critical potentials

In this section we perturb critical potentials by subcritical ones and see how this effect the existence of minimizers to the corresponding minimization problems. Before doing this, we first study how the concentration function of a potential changes when we perturb it; that is we establish that when we perturb a critical potential by a subcritical one then the concentration function of the critical potential does not change. Notice that this is not the case when we perturb a critical potential by another critical potential. We start with the following lemma:

Lemma 4.1 *Suppose $N \geq 3$, and V satisfies (H). If g is a subcritical potential with $g \overset{>}{\neq} 0$ in $I\!\!R^N$, then $\mathcal{C}(x_0\,; V+g) = \mathcal{C}(x_0; V)$ $\forall x_0 \in I\!\!R^N$ and $\mathcal{C}^\infty(V+g) = \mathcal{C}^\infty(V)$. That is the concentration of a potential V, does not change under subcritical nonnegative perturbations of it.*

Proof: We will only prove for $x_0 \in I\!\!R^N$, since the other case is similar. We assume that for $r > 0$ small enough, $\lambda_r(x_0; V) < +\infty$ *and* $\lambda_r(x_0; g) < +\infty$, since otherwise the proof is much easier. Then from the definition we have for $u \in C_0^\infty(I\!\!R^N)$, $\int_{B_r(x_0)} (V+g)u^2\, dx \;\leq\; (\frac{1}{\lambda_r(x_0;V)} + \frac{1}{\lambda_r(x_0;g)}) \int_{B_r(x_0)} |\nabla u|^2\, dx$, which suggests that $\lambda_r(x_0\,; V+g) \;\geq\; \frac{1}{\frac{1}{\lambda_r(x_0;V)}+\frac{1}{\lambda_r(x_0;g)}}$, and by passing to the limit $r \to 0^+$, conclude that $\mathcal{C}(x_0\,; V+g) \;\geq\; \mathcal{C}(x_0; V)$. Since the reverse inequality is obvious the result then follows.

Remark 4.1 *A similar statement to Lemma 4.1 holds when potential g is not of one sign, but instead $|g|$ is a subcritical potential and $V+g \geq 0\, in\, I\!\!R^N$.*

Lemma 4.2 *Suppose $N \geq 3$, and consider the potentials V, g that satisfy (H). We also suppose g to be a subcritical potential. Then there exists $\varepsilon_1 \geq 0$, such that $\lambda_{cr}(V+\varepsilon g) < \lambda_{cr}(V)$, for $\varepsilon > \varepsilon_1$. Moreover, we may take as $\varepsilon_1 = \frac{\lambda_{cr}(g)}{\lambda_{cr}(V)}$.*

Proof: We observe initially that for $\varepsilon > 0$, $\lambda_{cr}(\varepsilon g) = \frac{1}{\varepsilon}\lambda_{cr}(g)$ and $\lambda_{cr}(V+\varepsilon g) \leq \lambda_{cr}(\varepsilon g)$. Hence if $\phi \in \mathcal{D}^{1,2}(I\!\!R^N)$ is the corresponding minimizer for the potential g, we have for $\varepsilon > \varepsilon_0 = \frac{\lambda_{cr}(g)}{\lambda_{cr}(V)}$,

$$\lambda_{cr}(V+\varepsilon g) \;\leq\; \frac{\int_{I\!\!R^N} |\nabla\phi|^2\, dx}{\int_{I\!\!R^N}(V+\varepsilon g)\phi^2\, dx} \;<\; \frac{\int_{I\!\!R^N} |\nabla\phi|^2\, dx}{\varepsilon\int_{I\!\!R^N} g\phi^2\, dx} \;<\; \lambda_{cr}(V), \tag{4.2}$$

which concludes the result.

Lemma 4.3 *Suppose $N \geq 3$,V satisfies (H) and $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ is a solution of*

$$-\Delta u = V(x)u \quad in\ \mathbb{R}^N, \quad u \geq 0 \quad in\ \mathbb{R}^N. \tag{4.3}$$

Then either $u \equiv 0$ or $u > 0$ in $\mathbb{R}^N$.

<u>Proof:</u>(Suggested by C. Stuart). Notice that we can apply Theorem 8.19 of Gilbarg and Trudinger [21], with $z = -u \leq 0$ *in* $\mathbb{R}^N$, since $\Delta z \geq 0$ in $\mathbb{R}^N$, and $\mathcal{D}^{1,2}(\mathbb{R}^N) \subset W^{1,2}_{loc}(\mathbb{R}^N)$.

Corollary 4.4 (Strict monotonicity) *Suppose $N \geq 3$ and V that satisfies (H) and attains its infimum. Then for any potential g that also satisfies (H), we have $\lambda_{cr}(V+g) < \lambda_{cr}(V)$.*

The proof follows from Lemma 4.3. As a consequence of the previous Lemmas, we also obtain the proof of Theorem 1.3.

In critical exponent problems the S.Pohozaev [35] identity plays an essential role. The same is true for the linear problem (F. Rellich [35]). We have in particular the following standard by now result (see also H. Brezis [10]).

Proposition 4.5 *Suppose $N \geq 3$, and for some $x_0 \in \mathbb{R}^N$, $V \in C^1_{loc}(\mathbb{R}^N - \{x_0\})$. If the eigenvalue problem (1.1) has a solution u, for some $\lambda > 0$, then*

$$\int_{\mathbb{R}^N} [2V(x) + (x - x_0) \cdot \nabla V(x)]u^2(x)\, dx = 0. \tag{4.4}$$

It is then easy to see that when $2V(x) + (x - x_0) \cdot \nabla V(x)$ has a specific sign, this leads to nonexistence of eigenvalues. However, in connection with Theorem 1.3, a much stronger result holds. For this we give the following

Definition 4 *A critical potential $V \overset{>}{\not\equiv} 0$, is called a strong Hardy type potential (SH), if there exists $x_0 \in \mathbb{R}^N$ with $V \in C^1_{loc}(\mathbb{R}^N - \{x_0\})$, which in addition satisfies:*

$$2V(x) + (x - x_0)\nabla V(x) > 0 \quad in\ \mathbb{R}^N - \{x_0\}. \tag{4.5}$$

Theorem 4.6 *Suppose $N \geq 3$ and $V \in C^1_{loc}(\mathbb{R}^N - \{x_0\})$ is a strong Hardy type potential with $\lambda_{cr}(V) > 0$. Then for each $g \in C(\mathbb{R}^N)$ with compact support and $g \overset{>}{\not\equiv} 0$ in $\mathbb{R}^N$, there exists $\varepsilon_0 = \varepsilon_0(g) > 0$ such that, the potential $V_\varepsilon = V + \varepsilon g$ does attain a minimizer for every $\varepsilon > \varepsilon_0$, but does not for $0 \leq \varepsilon < \varepsilon_0$.*

In particular V_ε has a principal eigenvalue for all $\varepsilon > \varepsilon_0$, but has none for $0 \leq \varepsilon < \varepsilon_0$.

<u>Proof :</u> We first establish the result when $g \in C^1(\mathbb{R}^N)$ with $g \overset{>}{\not\equiv} 0$ in $\mathbb{R}^N$ and having compact support. In this case, it is easy to see, that there exists $\varepsilon_1 > 0$ such that the critical potential $V + \varepsilon g$, is a strong Hardy type potential for $0 \leq \varepsilon < \varepsilon_1$. Next we define the set $S = \{\varepsilon > 0\,,\ V + \varepsilon g$ attains a minimizer $\}$. Then by Theorem 1.3 in connection with Corollary 4.4, we obtain that S is an unbounded interval. In addition we have $\varepsilon_0 = \inf S \geq \varepsilon_1$. It is then easy to conclude that for $0 \leq \varepsilon < \varepsilon_0$ the critical potential $V + \varepsilon g$ does not attain a minimizer and the eigenvalue problem has no principal eigenvalue.

Next we consider the general case when $g \in C(\mathbb{R}^N)$. Following the argument for the previous case, the result will follow once we prove the existence of $\varepsilon_1 > 0$, with the property the critical potential $V + \varepsilon g$, does not attain a minimizer for $0 \leq \varepsilon < \varepsilon_1$. To this we consider any potential $G \in C^1(\mathbb{R}^N)$, that additionally satisfies $G \overset{>}{\not\equiv} 0\ in \mathbb{R}^N$ with compact support, and $Suppg \subset \{\, x \in \mathbb{R}^N \mid G(x) > 0 \,\}$. For the potential G there exists $\varepsilon_2 > 0$, such that the critical potential $V + \varepsilon G$ does not attain a minimizer for $0 \leq \varepsilon \leq \varepsilon_2$. Also, from continuity there exists $\varepsilon_3 > 0$, with the property $\varepsilon_3 g \leq \varepsilon_2 G\ in \mathbb{R}^N$. But then the critical potential $V + \varepsilon_3 g$ does not attain a minimizer. This in turn implies that the potential $V + \varepsilon g$ does not attain a minimizer for $0 \leq \varepsilon \leq \varepsilon_3$.

<u>Proof of Theorem 1.4 :</u> We observe that the critical potential $V_0(x) = \frac{1}{1+|x|^2}$ is a strong Hardy type potential, since $2V_0(x) + x \cdot \nabla V_0(x) = \frac{2}{(1+|x|^2)^2}$, $x \in \mathbb{R}^N$, and then the result follows from Theorem 4.6.

Remark 4.2 *As easily seen, similar results to Theorem 4.6 and Proposition 4.5 hold if the critical potential* V *satisfies the reverse inequality* $2\,V(x) + (x - x_0)\,\nabla V(x) < 0$ *in* $\mathbb{R}^N - \{x_0\}$*, instead of (4.5).*

Remark 4.3 *We could improve Theorem 4.6 by allowing perturbations* g *to the critical (strong Hardy type) potential* V *, which do not have compact support, but then we should place some decaying assumptions (at infinity) on* g *, the nature of which will depend upon the asymptotics of the critical potential* V *at infinity.*

Next we will see that if we perturb a critical potential by another critical one, nearly everything is possible. We start with two critical potentials that do not attain their infimum and contrary to what someone might expect the perturbed potential sometimes does attain its infimum.

Lemma 4.7 *For* $N \geq 3$*, we consider for* $a \in \mathbb{R}^+$ *the critical potentials* $V(x) = \frac{1}{|x|^2(1+|x|^2)}$ *and* $W_a(x) = \frac{a+|x|^2}{(1+|x|^2)^2}$*,* $x \in \mathbb{R}^N - \{0\}$*. Then the potentials* V*,* W_a *do not attain their infimum for* $0 \leq a \leq \frac{2N}{N-2}$*, whereas the potential* $V + W_a$ *does not attain its infimum when* $0 \leq a \leq 1$*, but it does when* $1 < a \leq \frac{2N}{N-2}$ *.*

We will give the proof of Lemma 4.7 in section 5.

Similarly let V be a strong Hardy type potential and $g \in C(\mathbb{R}^N)$, $g \gneqq 0$ *in* $\mathbb{R}^N$ with compact support and $\varepsilon > 0$. We define the critical potential $V_\varepsilon = V + \varepsilon\, g$. From Theorem 4.6 then follows that there exists $\varepsilon_0 = \varepsilon_0(g) > 0$ with the property V_ε attains its infimum when $\varepsilon > \varepsilon_0$ and does not attain it when $0 \leq \varepsilon < \varepsilon_0$. We will use the potentials V and V_ε for $\varepsilon > \varepsilon_0$, to see that when we perturb a critical potential, that does not attain its infimum (V), by another critical potential, that does attain its infimum (V_ε for $\varepsilon > \varepsilon_0$), then $V + V_\varepsilon$ might and might not attain its infimum.

Lemma 4.8 *Suppose* $N \geq 3$ *and let* V *be a strong Hardy type potential with* $\lambda_{cr} > 0$*. Then the critical potential* $V + V_\varepsilon$ *does not attain its infimum for* $\varepsilon_0 < \varepsilon < 2\varepsilon_0$*, but it does for* $\varepsilon > 2\varepsilon_0$ *.*

The proof of Lemma 4.8 follows easily from Theorem 4.6 by observing that $V + V_\varepsilon = 2\{V + \frac{\varepsilon}{2} g\}$, *in* $\mathbb{R}^N$.

<u>**Notation**</u> For a critical potential V, we denote $\Sigma(V)$ the set of critical singular points of V, including $+\infty$ in case $\mathcal{C}^\infty(V) < +\infty$ (that is $\Sigma(V) \subset \mathbb{R}^N \cup \{+\infty\}$).

Lemma 4.8 suggests that perturbing a critical potential by another one is very sensitive in attaining its infimum, when the perturbed potentials have common critical singular points. For this reason we consider the case of adding two critical potentials that have disjoint critical sets. Our first result then is:

Theorem 4.9 *Suppose* $N \geq 3$ *and* V_1, V_2 *are two critical potentials that satisfy* (H)*, and* V_1 *attains its infimum. If in addition we have* $\Sigma(V_1) \cap \Sigma(V_2) = \emptyset$ *and* $\lambda_{cr}(V_1) \leq \lambda_{cr}(V_2)$*, then the critical potential* $V_1 + V_2$ *attains its infimum.*

<u>Proof:</u> We first notice that by using $\Sigma(V_1) \cap \Sigma(V_2) = \emptyset$, we could easily establish, as in Lemma 4.1 that $\mathcal{C}(x; V_1 + V_2) = \min\{\,\mathcal{C}(x; V_1)\,, \mathcal{C}(x; V_2)\,\}\ \ \forall\, x \in \mathbb{R}^N$ and $\mathcal{C}^\infty(V_1 + V_2) = \min\{\,\mathcal{C}^\infty(V_1)\,, \mathcal{C}^\infty(V_2)\,\}$. It is then a simple consequence of the strict monotonicity (Corollary 4.4) to conclude $\lambda_{cr}(V_1 + V_2) < \lambda_{cr}(V_1) \leq \lambda_{cr}(V_2)$. Thus arriving at $\lambda_{cr}(V_1 + V_2) < \mathcal{C}^\infty(V_1 + V_2)$, and $\lambda_{cr}(V_1 + V_2) < \mathcal{C}(x; V_1 + V_2)$, $\forall\, x \in \mathbb{R}^N$. It is also fairly easy to see that $V_1 + V_2$ satisfies (H) as well. The result then follows from the Theorem 1.2.

In a similar fashion, we could establish:

Theorem 4.10 *Let* $N \geq 3$ *and* V_1 *,* V_2 *be two critical potentials that attain their infimum. If in addition* $\Sigma(V_1) \cap \Sigma(V_2) = \emptyset$*, then the critical potential* $V_1 + V_2$ *attains its infimum.*

5 Boundary Critical potentials

We start in this section by establishing a more general result than Proposition 1.6 (for another proof of Proposition 1.6, see Theorem 5.7 in Murata [29]). We believe that the existence of solutions should be well known to the specialists, but we have not find a reference for it. Instead we sketch a proof in the spirit of Z. Jin [23], see also Y. Pinchover [34]. What we expect to be new is that under the assumptions of our next theorem, all solutions tend to zero at infinity.

Theorem 5.1 *Suppose $N \geq 3$, $\alpha_1, \ldots, \alpha_k \in I\!R^N$, V is a locally Hölder continuous in $I\!R^N - \{\alpha_1, \ldots, \alpha_k\}$ with $\mathcal{C}(\alpha_i) > 0$, $i = 1, 2, \ldots, k$ and there exists $c > 0$ with $0 \overset{<}{\neq} |x|^2 V(x) \leq c$ for large $|x|$. Then $\lambda_{cr} > 0$ and for every $0 < \lambda \leq \lambda_{cr}$, the equation*

$$-\Delta u = \lambda V(x) u \quad in \ I\!R^N - \{\alpha_1, \ldots, \alpha_k\}, \tag{5.2}$$

has a positive solution $u \in C^2(I\!R^N - \{\alpha_1, \ldots, \alpha_k\})$.

If in addition for some R_0, $\{\alpha_1, \ldots, \alpha_k\} \subset B_{R_0}$, potential V satisfies $\int_{I\!R^N - B_{R_0}} \frac{V(x)}{|x|^{N-2}}\, dx = +\infty$, then $\lim_{|x| \to +\infty} u(x) = 0$.

Proof: We give the proof for $k = 1$. Let $0 < \lambda \leq \lambda_{cr}$. For $R > 2$ we first solve the problem

$$-\Delta w = \lambda V(x) w \quad in \ B_R - \bar{B}_{1/R}, \quad w = 1 \quad on \ \partial B_{1/R}, \qquad w = 0 \quad on \ \partial B_R. \tag{5.3}$$

This is possible, since the first eigenvalue λ_1^R of the problem $-\Delta \phi_1 = \lambda_1^R V(x) \phi_1$ in $B_R - \bar{B}_{1/R}$, $\phi_1 = 0$ on $\partial B_R \cup \partial B_{1/R}$, satisfies, $\lambda_1^R > \lambda_{cr}$. We then normalize the classical solution w of (5.3), by $w(e) = 1$ where $e \in I\!R^N$, $|e| = 1$. Next we apply the Harnack inequality to each domain $B_{R_0} - \bar{B}_{1/R_0}$ $(R_0 > 2)$ and establish the existence of C_{R_0} such that for $R \geq R_0$, $w^R \leq C_{R_0}$, *in* $B_{\frac{3}{4}R_0} - \bar{B}_{3/2R_0}$. Then by interior estimates, we obtain a new constant C_{R_0}, with $\|w^R\|_{C^{2,a}(B_{1/2R_0} - \bar{B}_{2/R_0})} \leq C_{R_0}$ for $R \geq R_0$. The result then follows by a diagonalization procedure.

We also use the Harnack inequality (Lemma 6.3 in Y. Pinchover [34]) to conclude the existence of a constant c independent of R) with

$$u(x) \leq c\, u(y) \qquad for \ all \ |x| = |y| = R \geq R_0. \tag{5.4}$$

If we define $\bar{u}(r) = \frac{1}{|\partial B_r|} \int_{\partial B_r} u(x)\, dS = \fint_{\partial B_r} u\, dS$, we then have $(r^{N-1}\, \bar{u}'(r))' + r^{N-1} \fint_{\partial B_r} \lambda V u\, dS = 0$. It is also not difficult to see that the monotonicity of $\bar{u}$ and (5.4) suggests that if the result does not hold, there exists $a > 0$ such that $u(x) \geq a$, for large $|x|$. Then an easy calculation gives for large r, say for $r \geq 1$, $r\bar{u}''(r) + (N-1)\bar{u}'(r) + \lambda a r \fint_{\partial B_r} V\, dS \leq 0$, and by integration, we obtain for $r > 1$,

$$r\bar{u}'(r) + (N-2)\bar{u}(r) - \{\bar{u}'(1) + (N-2)\bar{u}(1)\} + \frac{\lambda a}{c_N} \int_{B_r - B_1} \frac{V(x)}{|x|^{N-2}}\, dx \leq 0 \quad . \tag{5.5}$$

We may therefore suppose that there exists $A > 0$ with, $r\bar{u}'(r) + (N-2)\bar{u}(r) \leq -A$ for $r \geq r_0$, which in turn implies that for some $c \in I\!R$ and r large, $r^{N-2}\bar{u}(r) \leq -\frac{A}{N-2} r^{N-2} + c$. This in turn implies $\lim_{r \to +\infty} \sup \bar{u}(r) \leq -\frac{A}{N-2} < 0$, and the result follows.

Proof of Theorem 1.5 : It is easy to see that $u(x) = (1 + |x|^2)^{-\frac{N-2}{4}}$ is a positive solution of $-\Delta u = (\frac{N-2}{2})^2 W_0 u$ *in* $I\!R^N$. From Theorem 5.1 then follows $\lambda_{cr}(W_0) = (\frac{N-2}{2})^2$. W_0 does not attain its infimum since function u is not in $\mathcal{D}^{1,2}(I\!R^N)$. Next we establish the result when g has compact support and is locally Hölder continuous. To this it is enough to establish $\lambda_{cr}(W_0 + g) < (\frac{N-2}{2})^2$. We suppose the contrary. Then also by Theorem 5.1, the problem

$$-\Delta u = \left(\frac{N-2}{2}\right)^2 (W_0 + g) u \quad in \ I\!R^N, \quad \lim_{|x| \to +\infty} u(x) = 0, \tag{5.6}$$

has a positive classical solution u. We then define $f(r) = (1 + r^2)^{\frac{N-2}{4}} \fint_{\partial B_r} u\, dS$ and $h(r) = (\frac{N-2}{2})^2 \fint_{\partial B_r} g u\, dS$ for $r > 0$. Therefore

$$f(r) = f(0) - \int_0^r \frac{(1+t^2)^{\frac{N-2}{2}}}{t^{N-1}} \int_0^t \frac{s^{N-1}}{(1+s^2)^{\frac{N-2}{4}}} h(s)\, ds\, dt \quad for \ r > 0. \tag{5.7}$$

Since g has compact support, it follows that $h(r) = 0$ for large r. This in turn implies that for some $c > 0$, and r large enough $f(r) \leq -c \ln r$, hence getting a contradiction.

We finally consider the case of a general subcritical potential g. In this case we approximate from below function $g \in L^1_{loc}(I\!R^N)$, by a non-negative Hölder continuous function $\hat{g}$ with compact support that in addition satisfies $0 \overset{<}{\neq} \hat{g} \leq g$ in $I\!R^N$. We then apply the previous step to conclude that $\lambda_{cr}(W_0 + \hat{g}) < (\frac{N-2}{2})^2$, and the result then follows.

Proof of Theorem 1.7 : The result will follow from Theorem 1.2 once we prove $\lambda_{cr}(\frac{1}{|x|^2} + g) < (\frac{N-2}{2})^2$. We may suppose on the contrary that for some g Hölder continuous with compact support, $\lambda_{cr}(\frac{1}{|x|^2} + g) = (\frac{N-2}{2})^2$. Then by applying Theorem 5.1 we obtain a positive solution $u \in C^2(I\!R^N - \{0\})$ that satisfies $-\Delta u = (\frac{N-2}{2})^2 \, (\frac{1}{|x|^2} + g) \, u$ in $I\!R^N - \{0\}$ and $\lim_{|x| \to +\infty} u(x) = 0$. Then set $f(r) = r^{\frac{N-2}{2}} \fint_{\partial B_r} u \, dS$, and $h(r) = (\frac{N-2}{2})^2 \fint_{\partial B_r} gu \, dS$. It is then easy to see that there exist $\alpha, \beta \in I\!R$ such that $f(r) = \alpha + \beta \ln r - \int_1^r \frac{1}{t} \int_1^t s^{N/2} h(s) \, ds \, dt, \;\; r > 0$. Then using the positivity of f close to zero, we obtain $\beta \leq 0$, which combined with the fact that h has compact support, implies the existence of $k > 0$, $f(r) \leq -k \ln r$ for r large enough. Hence getting a contradiction.

Proof of Lemma 4.7 : By using the inequality $\frac{1}{|x|^2(1+|x|^2)} \leq \frac{1}{|x|^2}$, $x \in I\!R^N - \{0\}$ follows $\lambda_{cr}(V) = (\frac{N-2}{2})^2$. In a similar fashion from Theorem 1.5 we obtain $\lambda_{cr}(W_a) = (\frac{N-2}{2})^2$ for $0 \leq a \leq \frac{2N}{N-2}$ and $\lambda_{cr}(W_a) < (\frac{N-2}{2})^2$, for $a > \frac{2N}{N-2}$. We have in addition that W_a does not attain its infimum for $0 \leq a \leq \frac{2N}{N-2}$ and it does for $a > \frac{2N}{N-2}$. The later follows by using the splitting $\frac{a+|x|^2}{(1+|x|^2)^2} = \frac{\frac{2N}{N-2}+|x|^2}{(1+|x|^2)^2} + \frac{a - \frac{2N}{N-2}}{(1+|x|^2)^2}$. V does not attain its infimum since (see also remark 4.3), $2\,V(x) + x \cdot \nabla V(x) = -\frac{2}{(1+|x|^2)^2} < 0$ *in* $I\!R^N - \{0\}$. We now consider the potential $V + W_a$ and have $V(x) + W_a(x) = \frac{1}{|x|^2} + \frac{a-1}{(1+|x|^2)^2}$. The result then follows from Theorem 1.7 for $1 < a \leq \frac{2N}{N-2}$. Part 1 of the Lemma follows by contradiction. That is if $V + W_a$ attains its infimum for some $0 \leq a < 1$, then it follows from the previous splitting that the potential $\frac{1}{|x|^2}$ should attain its infimum, which is impossible.

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