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Source: *Mathematics of Computation*, Vol. 57, No. 196 (Oct., 1991), pp. 799-815
Published by: American Mathematical Society
Stable URL: http://www.jstor.org/stable/2938718
Accessed: 22/11/2009 14:24

SOLVING A SPECIFIC THUE-MAHLER EQUATION

N. TZANAKIS AND B. M. M. DE WEGER

ABSTRACT. The diophantine equation $x^3 - 3xy^2 - y^3 = \pm 3^{n_0} 17^{n_1} 19^{n_2}$ is completely solved as follows. First, a large upper bound for the variables is obtained from the theory of linear forms in p-adic and real logarithms of algebraic numbers. Then this bound is reduced to a manageable size by p-adic and real computational diophantine approximation, based on the L^3-algorithm. Finally the complete list of solutions is found in a sieving process. The method is in principle applicable to any Thue-Mahler equation, as the authors will show in a forthcoming paper.

1. INTRODUCTION

Let $f(X, Y) \in \mathbb{Z}[X, Y]$ be a binary form with at least three distinct linear factors over $\mathbb{C}$. Let $p_1, \ldots, p_s$ be fixed prime numbers. The diophantine equation

$$f(x, y) = \pm \prod_{i=1}^{s} p_i^{n_i}$$

in the variables $x, y \in \mathbb{Z}$ and $n_1, \ldots, n_s \in \mathbb{Z}_{\geq 0}$ with $(x, y) = 1$ is known as a 'Thue-Mahler equation.' It is well known that this equation has only finitely many solutions (Mahler), and that they can, at least in principle, be determined effectively, since an effectively computable upper bound for the variables can be derived from the p-adic theory of linear forms in logarithms (Baker, Coates, and Sprindžuk). For the history of the Thue-Mahler equation we refer to [3, Chapter 7].

In this paper we work out a paradigmatic example of determining all the solutions of a specific Thue-Mahler equation. In this way we hope to convince the reader that it is possible to solve any Thue-Mahler equation, not only in principle, but also in practice. We use the (real and p-adic) theory of linear forms in logarithms of algebraic numbers, for obtaining explicit but very large upper bounds for the unknowns. Then we reduce these bounds considerably by a combination of real and p-adic computational diophantine approximation

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techniques based on the L^3-algorithm for reducing bases of lattices. For a survey of such techniques and examples, see [7], and for an outline of the general method for solving Thue-Mahler equations, see [6]. In a forthcoming publication we intend to give a general and detailed treatment, such as we did for the Thue equation (cf. [4]). Since the present paper is more or less a sequel to [4], we keep the same numbering of the constants $c_1, c_2, \ldots$.

From now on we concentrate on the following example. Let $f(x, y) = x^3 - 3xy^2 - y^3$. We study the equation

$$(1) \qquad\qquad f(x, y) = \pm 3^{n_0} 17^{n_1} 19^{n_2}$$

in $x, y \in \mathbb{Z}$, $n_0, n_1, n_2 \in \mathbb{Z}_{\geq 0}$ with $(x, y) = 1$. Note that 3, 17, and 19 are the only primes below 20 that occur as prime factors of $f(x, y)$ for $x, y \in \mathbb{Z}$ with $(x, y) = 1$. We were motivated to choose this Thue-Mahler equation as a paradigmatic example by the following facts: the cubic field generated by the roots of $f(x, 1) = 0$ is totally real, so that there are two fundamental units; this field is a Galois field and has class number 1, which saves us from some trouble; the number of primes p_i is larger than one; and there are interestingly large solutions (the only other example of a Thue-Mahler equation that has been treated before by (a variant of) the method of this paper, involved only one fundamental unit and one prime (cf. [1])).

Consider the transformation

$$\tau(x, y) = (y, -(x + y)).$$

It is straightforward to check that

$$\tau^2(x, y) = (-(x + y), x), \qquad \tau^3 = \mathrm{id},$$

$$f(x, y) = f(\tau(x, y)).$$

Thus, every solution of (1) belongs to a class of six solutions:

$$\{\pm(x, y), \pm\tau(x, y), \pm\tau^2(x, y)\}.$$

These six solutions are distinct, and there is just one among them, (x, y) say, with $x > 0$, $y \geq 0$.

2. THE CUBIC FIELD

We work in the field $K = \mathbb{Q}(\vartheta)$, where ϑ is defined by $f(\vartheta, 1) = 0$. K is a Galois field with class number 1 and discriminant $81 = 3^4$. The conjugates of ϑ are $\sigma(\vartheta) = 2 - \vartheta^2$ and $\sigma^2(\vartheta) = -2 - \vartheta + \vartheta^2$, where σ generates the Galois group.

The mappings τ and σ are related as follows. Put $\tau(x, y) = (x_\tau, y_\tau)$. Then, as can easily be checked,

$$(2) \qquad\qquad \vartheta \cdot \sigma(x - y\vartheta) = x_\tau - y_\tau \vartheta.$$

Note that $\mathscr{O}_K = \mathbb{Z}[\vartheta]$ is the ring of integers of K, and that a system of fundamental units of $\mathscr{O}_K$ is given by $\{\vartheta, \eta\}$ with $\eta = 1 + \vartheta$. Investigating how 3,

17, and 19 factorize into prime ideals of $\mathscr{O}_K$, we find that 3 is totally ramified, and that 17 and 19 split completely. In fact, we have:

$\cdot$	$\sigma(\cdot)$	$\sigma^2(\cdot)$	$N(\cdot)$
ϑ	$-\vartheta^{-1}\eta = 2 - \vartheta^2$	$-\eta^{-1} = -2 - \vartheta + \vartheta^2$	1
$\eta = 1 + \vartheta$	$-\vartheta^{-1} = 3 - \vartheta^2$	$\vartheta\eta^{-1} = -1 - \vartheta + \vartheta^2$	-1
$\pi = 3 - \vartheta$	$1 + \vartheta^2$	$5 + \vartheta - \vartheta^2$	17
$\rho = 3 + \vartheta$	$5 - \vartheta^2$	$1 - \vartheta + \vartheta^2$	19
$\psi = -1 + \vartheta$	$-\eta\psi = 1 - \vartheta^2$	$-\vartheta^{-1}\eta\psi = -3 - \vartheta + \vartheta^2$	$\vartheta^{-1}\eta^2\psi^3 = 3$

3. FACTORING THE EQUATION IN THE FIELD

Put $\beta = x - y\vartheta$, where x, y is any solution of (1). Then

$$N(\beta) = \pm 3^{n_0} 17^{n_1} 19^{n_2}.$$

Suppose that $\alpha \mid (\beta, \sigma(\beta))$ for a prime $\alpha \in \mathscr{O}_K$. Then $\alpha \mid (\beta - \sigma(\beta)) = -y(\vartheta - \sigma(\vartheta))$. If $\alpha \mid y$ then, by $y \in \mathbb{Z}$, also $N(\alpha) \mid y$. Also $N(\alpha) \mid N(\beta) = f(x, y)$. Hence $N(\alpha) \mid x$, and this contradicts $(x, y) = 1$. So it follows that $\alpha \nmid y$, and thus $\alpha \mid (\vartheta - \sigma(\vartheta)) = \vartheta^{-1}\eta^2\psi^2$. Therefore we may take $\alpha = \psi$. If $\psi^2 \mid \beta$, then $9 = N(\psi^2) \mid N(\beta) = f(x, y)$. By considering $f(x, y) \pmod 9$, we find that $9 \mid f(x, y)$ if and only if $3 \mid x$ and $3 \mid y$. By $(x, y) = 1$ it follows that $\psi^2 \nmid \beta$. Hence $(\beta, \sigma(\beta)) = (\psi)^{n_0}$ for $n_0 \in \{0, 1\}$, and other values for n_0 in equation (1) are impossible.

In view of the above discussion, equation (1) now yields

$$(3) \qquad \beta = \varepsilon\psi^{n_0}\sigma^i(\pi)^{n_1}\sigma^j(\rho)^{n_2},$$

where $\varepsilon \in \mathscr{O}_K$ is a unit, and $i, j \in \{0, 1, 2\}$. We claim that in (3) we may assume without loss of generality that $i = 0$, i.e., $\sigma^i(\pi) = \pi$. Applying σ to (2) yields

$$(4) \qquad -\eta \cdot \sigma^2(x - y\vartheta) = x_{\tau^2} - y_{\tau^2}\vartheta.$$

We apply σ once to (3) if $i = 2$, and twice if $i = 1$. Then (2) or (4) state that we find again an equation of type (3), with $i = 0$, with a β corresponding to a solution (x, y) that is in the same class of six solutions generated by $\pm\tau$, and with a somewhat different unit ε. This proves our claim.

For $\alpha \in K$ and $h \in \{0, 1, 2\}$ we will sometimes write α_h instead of $\sigma^h(\alpha)$. With this convention it suffices to solve

$$(5) \qquad \beta = x - y\vartheta = \pm\psi^{n_0}\pi^{n_1}\rho_h^{n_2}\vartheta^{\alpha_1}\eta^{\alpha_2},$$

where $n_0 \in \{0, 1\}$ and $h \in \{0, 1, 2\}$, and where the variables are $x, y \in \mathbb{Z}$, $n_1, n_2 \in \mathbb{Z}_{\geq 0}$, and $a_1, a_2 \in \mathbb{Z}$.

4. NOTATION

For a rational prime p (in our case $p = 17$ or 19) we denote by $\mathbb{Z}_p$ the ring of p-adic integers, by $\mathbb{Q}_p$ the field of the p-adic numbers (the completion of $\mathbb{Q}$ under the p-adic metric), and by $\mathbb{C}_p$ the completion (under the p-adic metric) of the algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. In general, if $[\mathbb{Q}(\vartheta) : \mathbb{Q}] = n$ and $F(X) \in \mathbb{Q}[X]$ is the minimal polynomial of ϑ over $\mathbb{Q}$, there are n isomorphic embeddings φ_i $(i = 1, \dots, n)$ of $\mathbb{Q}(\vartheta)$ into $\mathbb{C}_p$. For every φ_i we have $\varphi_i(a) = a$ if $a \in \mathbb{Q}$ and $F(\varphi_i(\vartheta)) = 0$. For simplicity in our notation we put $\varphi_i(\alpha) = \alpha^{(i)}$ for every $\alpha \in \mathbb{Q}(\vartheta)$. If we work with more than one prime p, we will use the same notation $\alpha^{(i)}$, and it will be clear from the context with which prime p we are working. The above remarks remain valid for the 'infinite prime' $p = \infty$. For this 'prime' the p-adic metric is the usual absolute value, and $\mathbb{Q}_p = \mathbb{R}$, $\mathbb{C}_p = \mathbb{C}$. Finally, if $\xi \in \mathbb{Q}_p$ and ξ has the p-adic expansion

$$\xi = u_{-m}p^{-m} + \cdots + u_{-1}p^{-1} + u_0 + u_1 p + u_2 p^2 + \cdots$$
$$(0 \leq u_j \leq p - 1, \; m \geq 0),$$

we write $\xi = u_{-m} \cdots u_{-1}.u_0 u_1 u_2 \cdots$. In particular, if $\xi \in \mathbb{Z}_p$, we write $\xi = 0.u_0 u_1 u_2 \cdots$.

We make the following convention for numbering the conjugates of ϑ:

	in $\mathbb{R}$	in $\mathbb{Q}_{17}$	in $\mathbb{Q}_{19}$
$\vartheta^{(1)}$	$1.879385241\ldots$	$0.3\ 12\ 14\ 14\ldots$	$0.16\ 3\ 5\ 10\ldots$
$\vartheta^{(2)}$	$-1.532088886\ldots$	$0.10\ 12\ 5\ 8\ldots$	$0.12\ 4\ 15\ 1\ldots$
$\vartheta^{(3)}$	$-0.3472963554\ldots$	$0.4\ 9\ 13\ 10\ldots$	$0.10\ 10\ 17\ 6\ldots$

5. THE S-UNIT EQUATION

We now return to equation (1), or rather to (5). Consider the three conjugate relations $\sigma^i(\beta) = x - y\sigma^i(\vartheta)$. Eliminate x and y to obtain

$$(6) \qquad \qquad \delta \frac{\sigma^2(\beta)}{\sigma(\beta)} - 1 = \delta' \frac{\beta}{\sigma(\beta)},$$

where

$$\delta = \frac{\vartheta - \sigma(\vartheta)}{\vartheta - \sigma^2(\vartheta)} = \vartheta^{-1}\eta, \qquad \delta' = \frac{\sigma^2(\vartheta) - \sigma(\vartheta)}{\vartheta - \sigma^2(\vartheta)} = \vartheta^{-1}.$$

We want to substitute (5) into (6). Put

$$\xi = \frac{\sigma^2(\pi)}{\sigma(\pi)} = \frac{75 + 10\vartheta - 23\vartheta^2}{17}, \qquad \xi^* = \frac{\pi}{\sigma(\pi)} = \frac{1}{\sigma^2(\xi)} = \frac{52 - \vartheta - 13\vartheta^2}{17},$$

$$\chi = \frac{\sigma^2(\rho)}{\sigma(\rho)} = \frac{3 - 4\vartheta + 11\vartheta^2}{19}, \qquad \chi^* = \frac{\rho}{\sigma(\rho)} = \frac{1}{\sigma^2(\chi)} = \frac{14 + 13\vartheta + 7\vartheta^2}{19}.$$

Using our knowledge on the action of σ and σ^2 on ϑ, η, and ψ, we obtain from (6) the following so-called 'S-unit equation':

$$(7) \qquad (-1)^{a_2} \xi^{n_1} \chi_h^{n_2} \vartheta^{a_1'} \eta^{a_2'} - 1 = (-1)^{a_1 + a_2 + n_0} \xi^{*n_1} \chi_h^{*n_2} \vartheta^{2a_1 + a_2 - 1} \eta^{-a_1 + a_2 - n_0},$$

where

$$a_1' = a_1 + 2a_2 - n_0 - 1, \qquad a_2' = -2a_1 - a_2 + 1.$$

Note that ψ has disappeared, but n_0 still occurs.

6. BOUNDING THE p-ADIC ORDERS

Now we work $\wp$-adically with a conveniently chosen prime ideal $\wp$ over p, where $p \in \{17, 19\}$. For $p = 17$ we take $\wp = \wp_{17} = (\pi)$, and for $p = 19$ we take $\wp = \wp_{19} = (\rho)$. Note that for both p's we have $\mathrm{ord}_\wp(\cdot) = \mathrm{ord}_p(\cdot)$. From the definition of ξ^* and χ^* it is clear that $\mathrm{ord}_{\wp_{17}}(\xi^*) = 1$, $\mathrm{ord}_{\wp_{17}}(\chi^*) = 0$, and, since ϑ and η are units in $\mathscr{O}_K$, also $\mathrm{ord}_{\wp_{17}}(\vartheta) = \mathrm{ord}_{\wp_{17}}(\eta) = 0$. Therefore, (7) immediately implies the relation

$$(8) \qquad n_1 = \mathrm{ord}_{\wp_{17}}((-1)^{a_2} \xi^{n_1} \chi_h^{n_2} \vartheta^{a_1'} \eta^{a_2'} - 1).$$

When $p = 19$, we apply to (7) the automorphism σ^{-h} and get

$$(-1)^{a_2} \xi_{-h}^{n_1} \chi^{n_2} \vartheta_{-h}^{a_1'} \eta_{-h}^{a_2'} - 1 = (-1)^{a_1 + a_2 + n_0} \xi_{-h}^{*n_1} \chi^{*n_2} \vartheta_{-h}^{2a_1 + a_2 - 1} \eta_{-h}^{-a_1 + a_2 - n_0}.$$

Note that $\mathrm{ord}_{\wp_{19}}(\chi^*) = 1$, while all other factors in the right-hand side have $\wp_{19}$-adic order equal to zero. Therefore,

$$(9) \qquad n_2 = \mathrm{ord}_{\wp_{19}}((-1)^{a_2} \xi_{-h}^{n_1} \chi^{n_2} \vartheta_{-h}^{a_1'} \eta_{-h}^{a_2'} - 1).$$

We put

$$N = \max\{n_1, n_2\}, \quad A = \max\{|a_1'|, |a_2'|\}, \quad H = \max\{A, N\}.$$

Then by

$$a_1 = \tfrac{1}{3}(-a_1' - 2a_2' - n_0 + 1), \qquad a_2 = \tfrac{1}{3}(2a_1' + a_2' + 2n_0 + 1),$$

it follows that

$$|a_1| \le A, \qquad |a_2| \le A + 1,$$

with $|a_2| = A + 1$ only if $a_1' = a_2' = A$ and $n_0 = 1$. We now can apply K. R. Yu's theorem (cf. [9]) to the right-hand sides of (8) and (9) in order to find upper bounds for them, which are of the form $c_{13}(\log H + c_{14})$ (for the

precise formulation of Yu's theorem see the Appendix). We have calculated (see the Appendix) that in both cases we can take

$$c_{13} = 6.190047 \cdot 10^{24}, \qquad c_{14} = 4.28.$$

Thus, from (8) and (9) it follows that

(10) $$N \leq c_{13}(\log H + c_{14}).$$

7. UPPER BOUNDS FOR ALL THE VARIABLES

Inequality (10) will play an important role in computing an upper bound for H, a task which we undertake in this section. First we need the following lemma.

Lemma 1. *For the unit* $\varepsilon = \vartheta^{a_1'} \eta^{a_2'}$ *(appearing in the right-hand side of (7)) there exists an index* $k \in \{2, 3\}$ *such that*

$$either \quad |\varepsilon^{(k)}| \geq e^{c_{15}A} \quad or \quad |\varepsilon^{(k)}| \leq e^{-c_{15}A},$$

where, as before, $A = \max\{|a_1'|, |a_2'|\}$, *and* $c_{15} = 0.572215$.

Proof. We have

$$\begin{pmatrix} \log|\varepsilon^{(2)}| \\ \log|\varepsilon^{(3)}| \end{pmatrix} = \begin{pmatrix} \log|\vartheta^{(2)}| & \log|\eta^{(2)}| \\ \log|\vartheta^{(3)}| & \log|\eta^{(3)}| \end{pmatrix} \begin{pmatrix} a_1' \\ a_2' \end{pmatrix},$$

from which we derive

$$\begin{pmatrix} a_1' \\ a_2' \end{pmatrix} = \frac{-1}{R} \begin{pmatrix} \log|\eta^{(3)}| & -\log|\eta^{(2)}| \\ -\log|\vartheta^{(3)}| & \log|\vartheta^{(2)}| \end{pmatrix} \begin{pmatrix} \log|\varepsilon^{(2)}| \\ \log|\varepsilon^{(3)}| \end{pmatrix},$$

where R (≈ 0.849287450) is the regulator of the field K. Computing approximately the elements of the matrix in the right-hand side, we obtain

$$|a_1'| \leq 0.502341215|\log|\varepsilon^{(2)}|| + 0.742910686|\log|\varepsilon^{(3)}||,$$

$$|a_2'| \leq 1.245251901|\log|\varepsilon^{(2)}|| + 0.502341215|\log|\varepsilon^{(3)}||.$$

Let $k \in \{2, 3\}$ be such that $|\log|\varepsilon^{(k)}|| = \max\{|\log|\varepsilon^{(2)}||, |\log|\varepsilon^{(3)}||\}$. From the above inequalities it follows that

$$A \leq 1.747593116|\log|\varepsilon^{(k)}||,$$

i.e.,

$$|\log|\varepsilon^{(k)}|| \geq c_{15}A,$$

from which the result follows. □

We now take a positive constant $c_{16} < \frac{1}{2}c_{15}$ (e.g., $c_{16} = 0.085$), and for our further study we distinguish three cases. In the first two of them, k will be the index defined by Lemma 1.

Case 1. $\min_{1 \leq i \leq 3} |\beta^{(i)}| > e^{-c_{16}A}$ and $|\varepsilon^{(k)}| \geq e^{c_{15}A}$. We have $|\beta^{(k)}| \prod_{i \neq k} |\beta^{(i)}| = 3^{n_0} 17^{n_1} 19^{n_2}$ and then, in view of the first inequality above,

(11) $$|\beta^{(k)}| < 3^{n_0} 17^{n_1} 19^{n_2} e^{2c_{16}A}.$$

Then

$$e^{c_{15}A} \leq |\varepsilon^{(k)}| = \frac{|\beta^{(k)}|}{|\psi^{(k)}|^{n_0}|\pi^{(k)}|^{n_1}|\rho_h^{(k)}|^{n_2}} \leq \left|\frac{3}{\psi^{(k)}}\right|^{n_0}\left|\frac{17}{\pi^{(k)}}\right|^{n_1}\left|\frac{19}{\rho_h^{(k)}}\right|^{n_2} e^{2c_{16}A}$$

$$< 3.2266816 \cdot 5.0787257^{n_1} \cdot 12.94356302^{n_2} e^{2c_{16}A}$$

$$< e^{c_{18}+c_{17}N+2c_{16}A},$$

where $c_{17} = 4.185659$ and $c_{18} = 1.1715$. It follows that

(12) $$(c_{15} - 2c_{16})A < c_{18} + c_{17}N.$$

Case 2. $\min_{1 \leq i \leq 3}|\beta^{(i)}| > e^{-c_{16}A}$ and $|\varepsilon^{(k)}| \leq e^{-c_{15}A}$. Now

$$e^{-c_{15}A} \geq |\varepsilon^{(k)}| = \frac{|\beta^{(k)}|}{|\psi^{(k)}|^{n_0}|\pi^{(k)}|^{n_1}|\rho_h^{(k)}|^{n_2}} > \frac{e^{-c_{16}A}}{\overline{|\psi|}\,\overline{|\pi|}^{n_1}\,\overline{|\rho|}^{n_2}}$$

$$\geq \frac{e^{-c_{16}A}}{\overline{|\psi|}(\overline{|\pi|}\,\overline{|\rho|})^N} > e^{-(c'_{18}+c'_{17}N+c_{16}A)},$$

where $c'_{17} = 3.0962022$ and $c'_{18} = 0.93$. Then

(13) $$(c_{15} - c_{16})A < c'_{18} + c'_{17}N.$$

Before we proceed with the third case, we show how in Cases 1 and 2 an upper bound for H follows. Combining (12) with (10) we get for Case 1:

$$A < c_{19} + c_{20}\log H,$$

where

$$c_{19} = \frac{c_{18} + c_{13}c_{14}c_{17}}{c_{15} - 2c_{16}}, \qquad c_{20} = \frac{c_{13}c_{17}}{c_{15} - 2c_{16}}.$$

On the other hand, by (13) and (10), on noting that $c'_{17} < c_{17}$ and $c'_{18} < c_{18}$, we get the same for Case 2. Finally, (10) implies that $N < c_{19} + c_{20}\log H$, and therefore in both Case 1 and Case 2 we have by the definition of H,

$$H < c_{19} + c_{20}\log H,$$

from which we easily get the following upper bound for H:

(14) $$H < 4.38 \cdot 10^{27}.$$

Case 3. $\min_{1 \leq i \leq 3}|\beta^{(i)}| \leq e^{-c_{16}A}$. We put $|\beta^{(i_0)}| = \min_{1 \leq i \leq 3}|\beta^{(i)}|$. Then for any $j \neq i_0$ ($j \in \{1, 2, 3\}$) we have $|y||\vartheta^{(j)} - \vartheta^{(i_0)}| = |\beta^{(j)} - \beta^{(i_0)}| \leq 2|\beta^{(j)}|$, from which

$$|\beta^{(j)}| \geq \tfrac{1}{2}\min_{l \neq i}|\vartheta^{(l)} - \vartheta^{(i)}|.$$

Put $(\sigma(\beta))^{(i_0)} = \beta^{(j)}$ and $(\sigma^2(\beta))^{(i_0)} = \beta^{(k)}$ for $i_0 \neq j \neq k \neq i_0$. We have $|\delta'^{(i_0)}\beta^{(i_0)}/\beta^{(j)}| < c_{21}e^{-c_{16}A}$, where

$$c_{21} = 1.7057371 > \tfrac{1}{2}\min_{l \neq i}|\vartheta^{(l)} - \vartheta^{(i)}| \cdot \overline{|\delta'|}.$$

Then, in view of (6),

$$(15) \qquad \left| \delta^{(i_0)} \frac{\beta^{(k)}}{\beta^{(j)}} - 1 \right| < c_{21} e^{-c_{16} H} < 0.72542132,$$

provided that $H \geq 3$. In particular, $\delta^{(i_0)} \beta^{(k)} / \beta^{(j)} > 0$, and therefore

$$\delta^{(i_0)} \frac{\beta^{(k)}}{\beta^{(j)}} = \left| \delta^{(i_0)} \frac{\beta^{(k)}}{\beta^{(j)}} \right| = \left| \left\{ \delta \frac{\sigma^2(\beta)}{\sigma(\beta)} \right\}^{(i_0)} \right| = |\xi^{(i_0)}|^{n_1} |\chi_h^{(i_0)}|^{n_2} |\vartheta^{(i_0)}|^{a_1'} |\eta^{(i_0)}|^{a_2'} = e^{\Lambda_0},$$

where

$$(16) \qquad \Lambda_0 = n_1 \log |\xi^{(i_0)}| + n_2 \log |\chi_h^{(i_0)}| + a_1' \log |\vartheta^{(i_0)}| + a_2' \log |\eta^{(i_0)}|.$$

It is clear from (6) that $\Lambda_0 \neq 0$. Also, from (15) we have $|e^{\Lambda_0} - 1| < 0.72542132$, from which we easily see that $|\Lambda_0| < 1.782 |e^{\Lambda_0} - 1|$ (namely, if $|e^x - 1| < \delta < 1$, then $|x| < -(\log(1-\delta)/\delta)|e^x - 1|)$. Thus, we finally get

$$(17) \qquad 0 < |\Lambda_0| < 1.782 c_{21} e^{-c_{16} H}.$$

By Waldschmidt's theorem (cf. [5]) positive constants c_7 and c_8 can be computed such that

$$(18) \qquad |\Lambda_0| > e^{-c_7 (\log H + c_8)}.$$

Using the version of this theorem as it is formulated in Appendix II of [4], we have computed the following values:

$$c_7 = 2.6467 \cdot 10^{29}, \qquad c_8 = 2.442325.$$

On combining (17) with (18) we get

$$H < \frac{c_7 c_8 + \log(1.782 c_{21})}{c_{16}} + \frac{c_7}{c_{16}} \log H,$$

which implies the following upper bound for H in Case 3:

$$(19) \qquad H < 5.76 \cdot 10^{32}.$$

Comparing this upper bound with that for Cases 1 and 2 (cf. (14)), we see that in any case (19) is true.

8. REAL AND p-ADIC LINEAR FORMS IN LOGARITHMS

In the forthcoming sections we show how the upper bound for H given in (19) can be reduced. First we transform equations (8) and (9) into equations involving linear forms in p-adic logarithms of numbers that are algebraic over $\mathbb{Q}_p$, with $p = 17$ and 19, respectively. We need a simple general lemma.

Lemma 2. *If $z \in \mathbb{Z}_p$, $\mathrm{ord}_p(z - 1) \geq 1$, and $k \in \mathbb{N}$ with $p \nmid k$, then*

$$\mathrm{ord}_p(z^k - 1) = \mathrm{ord}_p(z - 1).$$

The proof is an easy exercise, which we leave to the reader. We will also make use of the following

Lemma 3. *Let* $z_1, \ldots, z_m \in \mathbb{C}_p$ *with* $\mathrm{ord}_p(z_i - 1) > 1/(p-1)$ $(1 \leq i \leq m)$, *and* $n_1, \ldots, n_m \in \mathbb{C}_p$ *with* $\mathrm{ord}_p(n_i) \geq 0$ $(1 \leq i \leq m)$. *Then the p-adic logarithms* $\log_p z_i$ *are defined for* $1 \leq i \leq m$, *and*

$$\mathrm{ord}_p(z_1^{n_1} \cdots z_m^{n_m} - 1) = \mathrm{ord}_p(n_1 \log_p z_1 + \cdots + n_m \log_p z_m).$$

For a proof of this lemma, see [8, equation (1.2)].

Now we put

$$\Lambda_j = n_1 \lambda_{j1} + n_2 \lambda_{j2} + a_1' \mu_{j1} + a_2' \mu_{j2} \qquad (j = 1, 2, 0),$$

where

$$\lambda_{11} = \log_{17}(\xi^{(1)^{16}}), \qquad \lambda_{12} = \log_{17}(\chi_h^{(1)^{16}}),$$

$$\mu_{11} = \log_{17}(\vartheta^{(1)^{16}}), \qquad \mu_{12} = \log_{17}(\eta^{(1)^{16}}),$$

all in $\mathbb{Q}_{17}$,

$$\lambda_{21} = \log_{19}(\xi_{-h}^{(1)^{18}}), \qquad \lambda_{22} = \log_{19}(\chi^{(1)^{18}}),$$

$$\mu_{21} = \log_{19}(\vartheta_{-h}^{(1)^{18}}), \qquad \mu_{22} = \log_{19}(\eta_{-h}^{(1)^{18}}),$$

all in $\mathbb{Q}_{19}$, and

$$\lambda_{01} = \log|\xi^{(i_0)}|, \quad \lambda_{02} = \log|\chi_h^{(i_0)}|, \quad \mu_{01} = \log|\vartheta^{(i_0)}|, \quad \mu_{02} = \log|\eta^{(i_0)}|,$$

all in $\mathbb{R}$.

We now go back to (8) and apply Lemma 2 to

$$z = (-1)^{a_2} \xi^{(1)^{n_1}} \chi_h^{(1)^{n_2}} \vartheta^{(1)^{a_1'}} \eta^{(1)^{a_2'}} \in \mathbb{Q}_{17}.$$

Since $\mathrm{ord}_{\wp_{17}}(\cdot) = \mathrm{ord}_{17}(\cdot)$ and $\mathrm{ord}_{17}(\alpha^{16} - 1) \geq 1$ for any $\alpha \in K$ with $\mathrm{ord}_\pi(\alpha) = 0$, it follows from the above two lemmas that

$$(20) \qquad\qquad n_1 = \mathrm{ord}_{17}(\Lambda_1).$$

Analogously, working in $\mathbb{Q}_{19}$, we get from (9)

$$(21) \qquad\qquad n_2 = \mathrm{ord}_{19}(\Lambda_2).$$

Summarizing, we now have the following situation:

- first

$$n_1, n_2 \in \mathbb{Z}_{\geq 0}, \qquad N = \max\{n_1, n_2\},$$
$$a_1', a_2' \in \mathbb{Z}, \qquad A = \max\{|a_1'|, |a_2'|\},$$

$$(22) \qquad\qquad H = \max\{A, N\} < K_0 = 5.76 \cdot 10^{32}$$

(cf. (19));

- either

(23) $A < K_1 + K_2 N$,   where $K_1 = 2.913$ and $K_2 = 10.4066$

(cf. (12) and (13)),

or

(24) $0 < |\Lambda_0| < K_3 e^{-K_4 H}$,   where $K_3 = 3.04 > 1.782 \cdot c_{22}$
$\qquad\qquad\qquad\qquad\qquad\qquad$ and $K_4 = 0.085 = c_{16}$

(cf. (17));

- and finally

(25) $\operatorname{ord}_{17}(\Lambda_1) = n_1$ and $\operatorname{ord}_{19}(\Lambda_2) = n_2$.

9. THE p-ADIC REDUCTION STEP

We start working in $\mathbb{Q}_{17}$. An easy computation shows that $\operatorname{ord}_{17}(\lambda_{1j}) = 1$ and $\operatorname{ord}_{17}(\mu_{1j}) = 1$ $(j = 1, 2)$. We write

$$-\frac{\lambda_{11}}{\lambda_{12}} = \beta_1 \quad \text{and} \quad -\frac{\mu_{1j}}{\lambda_{12}} = \alpha_j \qquad (j = 1, 2),$$

so that $\beta_1, \alpha_1, \alpha_2 \in \mathbb{Z}_{17}$.

We choose a positive integer m such that 17^m is of the size of K_0^4, and m should be large enough. We denote by $\beta_1^{(m)}$ and $\alpha_j^{(m)}$ the unique rational integers in the interval $[0, 17^m - 1]$ which are congruent $(\operatorname{mod} 17^m)$ to β_1 and α_j, respectively $(j = 1, 2)$. Next, we consider the lattice $\Gamma_m \subset \mathbb{Z}^4$ generated by the column vectors of the matrix

$$\mathscr{A}_m = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \alpha_1^{(m)} & \alpha_2^{(m)} & \beta_1^{(m)} & 17^m \end{pmatrix}.$$

Using the L^3-algorithm (cf. [2; 7, §3.5] for an integer version), we computed a reduced basis $\{\mathbf{b}_1, \dots, \mathbf{b}_4\}$ of Γ_m. Then every nonzero vector of Γ_m has a length which is greater than or equal to $2^{-3/2}|\mathbf{b}_1|$ (cf. [2, Proposition (1.11)]), and we expect that this be of size $\det(\Gamma_m)^{1/4} = \det(\mathscr{A}_m)^{1/4} = 17^{m/4}$, which is of the size of K_0.

Proposition 4. *If* $|\mathbf{b}_1| > 2^{5/2} K_0$, *then* $n_1 \leq m$.

Proof. Suppose that

(26) $n_1 \geq m + 1$.

We claim that

(27) $\begin{pmatrix} a_1' \\ a_2' \\ n_1 \\ n_2 \end{pmatrix} \in \Gamma_m.$

Indeed,

$$\begin{pmatrix} a'_1 \\ a'_2 \\ n_1 \\ n_2 \end{pmatrix} = \mathscr{A}_m \begin{pmatrix} a'_1 \\ a'_2 \\ n_1 \\ \lambda \end{pmatrix},$$

where

$$\lambda = \frac{-a'_1 \alpha_1^{(m)} - a'_2 \alpha_2^{(m)} - n_1 \beta_1^{(m)} + n_2}{17^m}.$$

Then it suffices to show that $\lambda \in \mathbb{Z}$, or, equivalently, that

(28) $\qquad \operatorname{ord}_{17}(-a'_1 \alpha_1^{(m)} - a'_2 \alpha_2^{(m)} - n_1 \beta_1^{(m)} + n_2) \geq m.$

By hypothesis,

$$\operatorname{ord}_{17}(\alpha_j^{(m)} - \alpha_j) \geq m \quad (j = 1, 2) \quad \text{and} \quad \operatorname{ord}_{17}(\beta_1^{(m)} - \beta_1) \geq m.$$

Therefore, in order that (28) be true, it suffices that

(29) $\qquad \operatorname{ord}_{17}(-a'_1 \alpha_1 - a'_2 \alpha_2 - n_1 \beta_1 + n_2) \geq m.$

Note that $-a'_1 \alpha_1 - a'_2 \alpha_2 - n_1 \beta_1 + n_2 = \Lambda_1/\lambda_{12}$, which means that (29) is equivalent to $\operatorname{ord}_{17}(\Lambda_1) \geq m + \operatorname{ord}_{17}(\lambda_{12}) = m + 1$. But this is true, since in view of (26) and (25) $\operatorname{ord}_{17}(\Lambda_1) = n_1 \geq m + 1$.

Now, (27) is proved, therefore we must have

$$a_1'^2 + a_2'^2 + n_1^2 + n_2^2 \geq 2^{-3}|\mathbf{b}_1|^2,$$

and a fortiori $4K_0^2 \geq 2^{-3}|\mathbf{b}_1|^2$, i.e., $|\mathbf{b}_1| \leq 2^{5/2}K_0$, which contradicts the assumption of the proposition. $\square$

In our application of the above proposition we have chosen $m = 110$ or 111, so that 17^m is somewhat larger than K_0^4, and we computed a reduced basis for the lattice Γ_m (there are three possibilities for Γ_m, corresponding to $h = 0, 1, 2$; cf. (20)). Then we checked that $|\mathbf{b}_1| > 2^{5/2}K_0$, from which we concluded that $n_1 \leq 111$.

In an analogous way we worked with the prime $p = 19$ in place of $p = 17$, and we concluded that $n_2 \leq 108$. Thus we have found a new, small upper bound $N_0 = 111$ for $N = \max\{n_1, n_2\}$, instead of the large bound $K_0 = 5.76 \cdot 10^{32}$. This completes the 'p-adic reduction step.'

10. THE REAL REDUCTION STEP

Now, in view of (23) and (24) we have either

(30) $\qquad\qquad A < K_1 + K_2 N_0 < 1159$

or

$$0 < |\Lambda_0| < K_3 e^{-K_4 H}.$$

If the second alternative holds, then we apply the 'real reduction step,' as explained below. Note that there are nine possibilities for Λ_0, corresponding to the nine possibilities for $(i_0, h) \in \{1, 2, 3\} \times \{0, 1, 2\}$.

We choose a 'big' constant C of the size of $K_0^2 N_0^2$ (in our application we took $C = 10^{75}$ or 10^{76}), and we put

$$\beta_j = [C\lambda_{0j}], \quad \alpha_j = [C\mu_{0j}] \qquad (j = 1, 2).$$

Here, $[x] = \lfloor x \rfloor$ if $x \geq 0$, and $[x] = \lceil x \rceil$ if $x < 0$. Next, we consider the lattice $\Gamma \subset \mathbb{Z}^4$ generated by the column vectors of the matrix

$$\mathscr{A} = \begin{pmatrix} [K_0/N_0] & 0 & 0 & 0 \\ 0 & [K_0/N_0] & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta_1 & \beta_2 & \alpha_1 & \alpha_2 \end{pmatrix},$$

and we find by the L^3-algorithm a reduced basis $\{\mathbf{b}_1, \ldots, \mathbf{b}_4\}$. Then every nonzero vector of Γ has a length greater than or equal to $2^{-3/2}|\mathbf{b}_1|$, and one expects that this be of the size of $\det(\Gamma)^{1/4} = \det(\mathscr{A})^{1/4} \approx K_0^{1/2} N_0^{-1/2} C^{1/4}$, which is of the size of K_0.

Proposition 5. *If* $|\mathbf{b}_1| > 2^{3/2}\sqrt{4(K_0 + N_0)^2 + 3K_0^2}$, *then*

$$H < \frac{1}{K_4}\{\log K_3 + \log C - \log(\sqrt{2^{-3}|\mathbf{b}_1|^2 - 3K_0^2} - 2(K_0 + N_0))\}.$$

Proof. Consider the lattice point

$$\mathbf{x} = \mathscr{A}\begin{pmatrix} n_1 \\ n_2 \\ a_1' \\ a_2' \end{pmatrix} = \begin{pmatrix} [K_0/N_0]n_1 \\ [K_0/N_0]n_2 \\ a_1' \\ \lambda \end{pmatrix} \in \Gamma,$$

where $\lambda = n_1\beta_1 + n_2\beta_2 + a_1'\alpha_1 + a_2'\alpha_2$. It is obvious that $|\lambda - C\Lambda_0| < 2N_0 + 2K_0$, and therefore, also in view of (24),

(31) $$|\lambda| < CK_3 e^{-K_4 H} + 2(N_0 + K_0).$$

On the other hand, we must have $|\mathbf{x}| \geq 2^{-3/2}|\mathbf{b}_1|$, which is equivalent to

$$[K_0/N_0]^2(n_1^2 + n_2^2) + a_1'^2 + \lambda^2 \geq 2^{-3}|\mathbf{b}_1|^2.$$

Since $n_1^2, n_2^2 \leq N_0^2$ and $|a_1'| \leq K_0$, it then follows that $3K_0^2 + \lambda^2 \geq 2^{-3}|\mathbf{b}_1|^2$, i.e.,

$$|\lambda| > \sqrt{2^{-3}|\mathbf{b}_1|^2 - 3K_0^2}$$

(note that by hypothesis the argument of the square root is positive). Combine now the last inequality with (31) to get

$$CK_3 e^{-K_4 H} > \sqrt{2^{-3}|\mathbf{b}_1|^2 - 3K_0^2} - 2(N_0 + K_0),$$

and the right-hand side is positive by hypothesis. Taking logarithms, we obtain the desired upper bound for H. □

Remark. In practice we estimate the reduced upper bound for H as follows: let $\delta_1 \in \mathbb{R}$ satisfy $\delta_1 < |\mathbf{b}_1|/K_0$. Then, if the hypothesis of Proposition 5 is true, we have

$$H < \frac{1}{K_4}(\log K_3 + \log C - \log K_0 - \log \delta_2),$$

where $\delta_2 < \sqrt{\delta_1^2/8 - 3} - 2(1 + N_0/K_0)$.

In our application with the chosen values for C, the hypothesis of Proposition 5 was checked to be true, and this implied $H \leq 1169$. Compare with $N \leq 111$ and (30) to get the new reduced upper bound $H \leq 1169$. This is extremely small indeed, compared to the initial upper bound (cf. (22)), but it is still quite large for practical application. Therefore, we applied once more the whole reduction process as described above, but now with $K_0 = 1169$ and in the p-adic reduction step $m = 14$ for both $p = 17$ and $p = 19$. The p-adic reduction step gave

(32) $N \leq 14$,

and either (23) holds, in which case $A < K_1 + 14K_2 < 149$, or (24) holds, in which case we can apply the real reduction step with $C = 10^{13}$ or 10^{14}, which yielded $H \leq 296$. Thus, $\max\{|a_1'|, |a_2'|\} \leq 296$, and consequently

(33) $\max\{|a_1|, |a_2|\} \leq 297$.

Numerical details of all our computations can be obtained from the authors on request.

11. THE FINAL SIEVE

It remains to check for which values $(n_0, n_1, n_2, a_1, a_2)$ satisfying (32) and (33) the coefficient of ϑ^2 in the right-hand side of (5) is 0. We have worked as follows. We have chosen a 'large' number such that the orders of ϑ, η, π, and ρ modulo this number are 'small.' Such a number is $2701 = 37 \cdot 73$, and the orders of ϑ, η, π, and ρ $(\mathrm{mod}\, 2701)$ are all equal to 72. Fix $n_0 \in \{0, 1\}$ and $h \in \{0, 1, 2\}$ in (5), and check (by computer, of course) for which quadruples

$$(n_1, n_2, a_1, a_2) \in \{0, \ldots, 14\}^2 \times \{0, \ldots, 71\}^2$$

the coefficient of ϑ^2 in the right-hand side of (5) is 0 $(\mathrm{mod}\, 2701)$. The number of such quadruples turned out to be between 450 and 500 (depending on n_0 and h). Now, in the range determined by (32) and (33) we have only about $4 \cdot 10^4$ quadruples to check, instead of the initial $15^2(2 \cdot 297 + 1)^2 \approx 7.96 \cdot 10^7$. Choose a prime $p \neq 37, 73$, and of the so selected quadruples keep only those for which the coefficient of ϑ^2 in the right-hand side of (5) is 0 $(\mathrm{mod}\, p)$. We did so for $p = 601$ and $p = 541$ (chosen at random), and were left with about 14 quadruples (the exact number depending on n_0 and h), which then could be checked further by straightforward computations.

Note that this sieving process has to be applied six times: for $n_0 = 0, 1$ and for $h = 0, 1, 2$.

12. CONCLUSION: THE COMPLETE LIST OF SOLUTIONS

Now we have a complete proof of the following theorem, listing all the solutions of equation (1).

Theorem A. *The Thue-Mahler equation*

$$x^3 - 3xy^2 - y^3 = \pm 3^{n_0} 17^{n_1} 19^{n_2}$$

in $x, y \in \mathbb{Z}$ *and* $n_0, n_1, n_2 \in \mathbb{Z}_{\geq 0}$ *with* $(x, y) = 1$ *has exactly the* 156 *solutions presented in Table* 1.

TABLE 1

The solutions of equation (1)

n_0 n_1 n_2	$\pm(x, y)$		
0 0 0	(1, 0)	(0, −1)	(1, −1)
0 0 0	(2, 1)	(1, −3)	(3, −2)
0 0 1	(1, 2)	(2, −3)	(3, −1)
0 0 2	(4, 5)	(5, −9)	(9, −4)
0 1 0	(3, 1)	(1, −4)	(4, −3)
0 1 0	(3, 2)	(2, −5)	(5, −3)
0 1 0	(15, 8)	(8, −23)	(23, −15)
0 1 1	(1, 6)	(6, −7)	(7, −1)
0 1 1	(3, 5)	(5, −8)	(8, −3)
0 1 1	(28, 15)	(15, −43)	(43, −28)
0 2 1	(59, 31)	(31, −90)	(90, −59)
0 2 1	(31, 15)	(15, −46)	(46, −31)
0 2 1	(18, 13)	(13, −31)	(31, −18)
0 2 2	(206, 109)	(109, −315)	(315, −206)
0 2 5	(896, 37)	(37, −933)	(933, −896)
0 3 1	(97, 54)	(54, −151)	(151, −97)
1 0 0	(1, 1)	(1, −2)	(2, −1)
1 0 1	(7, 4)	(4, −11)	(11, −7)
1 0 1	(13, 7)	(7, −20)	(20, −13)
1 0 1	(5, 2)	(2, −7)	(7, −5)
1 1 0	(4, 1)	(1, −5)	(5, −4)
1 1 1	(10, 1)	(1, −11)	(11, −10)
1 1 2	(29, 20)	(20, −49)	(49, −29)
1 1 3	(73, 13)	(13, −86)	(86, −73)
1 2 0	(4, 7)	(7, −11)	(11, −4)
1 2 2	(712, 379)	(379, −1091)	(1091, −712)

In Table 1, one line represents a class of six solutions (cf. §1). In the column for $\pm(x, y)$ the three possible values for (x, y) with $x \geq 0$ are printed.

APPENDIX

The following theorem is a corollary to a recent theorem of K. R. Yu (cf. [9]).

Theorem. *Let* $\alpha_1, \ldots, \alpha_n$ $(n \geq 2)$ *be nonzero algebraic numbers,* $K = \mathbb{Q}(\alpha_1, \ldots, \alpha_n)$, $[K : \mathbb{Q}] = d$, *and let* p *be a rational prime number. As usual, let* i *satisfy* $i^2 = -1$, *and put* $\omega = (-1 + i\sqrt{3})/2$. *Put*

$$(D, \delta) = \begin{cases} (d, 1) & \text{if } (p > 2 \text{ and } i \in K) \text{ or } (p = 2 \text{ and } \omega \in K), \\ (2d, 2) & \text{otherwise.} \end{cases}$$

Let $\wp$ *be a prime ideal of the ring of integers of* K *lying above* p, *and let* $f_\wp$ *be its residue class degree. For* $\alpha \in K$, $\alpha \neq 0$, *we denote by* $\operatorname{ord}_\wp(\alpha)$ *the order to which* $\wp$ *divides the fractional ideal* (α) *generated by* α, *and we set* $\operatorname{ord}_\wp(0) = \infty$. *Also, for* $\alpha \neq 0$ *we denote by* $h(\alpha)$ *the absolute logarithmic height of* α *(see, e.g., [5, §2] or [4, Appendix II]), and we put* $\log \alpha = \log|\alpha| + i \operatorname{Arg} \alpha$, *with* $-\pi < \operatorname{Arg} \alpha \leq \pi$. *For any* $j \in \{1, \ldots, n\}$ *we put*

$$V_j \geq \max\left\{ h(\alpha_j), \frac{|\log \alpha_j|}{2\pi D}, \frac{f_\wp \log p}{d} \right\},$$

and $V = \max_{1 \leq j \leq n} V_j$. *Let* $b_1, \ldots, b_n \in \mathbb{Z}$, *and* $B = \max\{|b_1|, \ldots, |b_n|\}$. *If* $\operatorname{ord}_\wp(\alpha_j) = 0$ *for all* $j = 1, \ldots, n$ *and* $\alpha_1^{b_1} \cdots \alpha_n^{b_n} \neq 1$, *then*

$$\operatorname{ord}_\wp(\alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1) < C_1 C_2 C_3 (\log B + C_4),$$

where

$$C_1 = \begin{cases} 35009 \cdot (45/2)^n & \text{if } p \equiv 1 \pmod{4}, \\ 30760 \cdot 25^n & \text{if } p \equiv 3 \pmod{4}, \\ 197142 \cdot 36^n & \text{if } p = 2, \end{cases}$$

$$C_2 = (n+1)^{2n+4} p^{\delta f_\wp} (f_\wp \log p)^{-(n+1)} D^{n+2} V_1 \cdots V_n,$$

$$C_3 = \begin{cases} \log(2^{11}(n+1)^2 D^2 V) & \text{if } p > 2, \\ \log(3 \cdot 2^{10}(n+1)^2 D^2 V) & \text{if } p = 2, \end{cases}$$

$$C_4 = 2 \log D.$$

We can apply this theorem to the right-hand side of (8) as follows. Suppose that $n_1 \geq 1$. Then

$$\operatorname{ord}_{\wp_{17}}((-1)^{a_2} \xi^{n_1} \chi_h^{n_2} \vartheta^{a_1'} \eta^{a_2'} + 1) = 0,$$

and consequently

$$n_1 = \operatorname{ord}_{\wp_{17}}(\xi^{2n_1}\chi_h^{2n_2}\vartheta^{2a_1'}\eta^{2a_2'} - 1).$$

Analogously, we can do this with (9). Then we may apply the theorem with $B = 2H$, $D = 6$, $\delta = 2$, and $n = 4$. Thus, upper bounds for n_1 and n_2 can easily be found by means of this theorem as follows:

$$\frac{f_\wp \log p}{d} < \begin{cases} 0.944405 & \text{if } p = 17, \\ 0.981480 & \text{if } p = 19, \end{cases}$$

$$h(\xi) < 1.410173, \; \frac{|\log \xi|}{2\pi D} < 0.0371, \quad \text{hence } V_\xi = 1.410173,$$

$$h(\chi) < 1.381873, \; \frac{|\log \chi|}{2\pi D} < 0.032, \quad \text{hence } V_\chi = 1.381873,$$

$$h(\vartheta) < 0.352526, \; \frac{|\log \vartheta|}{2\pi D} < 0.088, \quad \text{hence } V_\vartheta = \begin{cases} 0.944405 & \text{if } p = 17, \\ 0.981480 & \text{if } p = 19, \end{cases}$$

$$h(\eta) < 0.352526, \; \frac{|\log \eta|}{2\pi D} < 0.085, \quad \text{hence } V_\eta = \begin{cases} 0.944405 & \text{if } p = 17, \\ 0.981480 & \text{if } p = 19. \end{cases}$$

Here, of course, V_{α_i} stands for V_i. From these values we computed

$$C_1 = \begin{cases} 8.972424 \cdot 10^9 & \text{if } p = 17, \\ 1.2015625 \cdot 10^{10} & \text{if } p = 19, \end{cases} \qquad C_2 = \begin{cases} 3.134053 \cdot 10^{13} & \text{if } p = 17, \\ 3.487752 \cdot 10^{13} & \text{if } p = 19, \end{cases}$$

$$C_3 = 14.77073, \qquad C_4 = 2\log 6.$$

It follows that we can take for c_{13} and c_{14} (cf. (10)):

$$c_{13} = 6.190047 \cdot 10^{24} > C_1 C_2 C_3, \qquad c_{14} = 4.28 > 2\log 6 + \log 2.$$

Note that (10) is true also for $N = 0$.

Addendum. After we had completed the present work, the following important papers appeared:

J. Blass, A. M. W. Glass, D. K. Manski, D. B. Meronk, and R. P. Steiner, *Constants for lower bounds for linear forms in the logarithms of algebraic numbers II. The homogeneous rational case*, Acta Arith. **55** (1990), 15–22.

Kunrui Yu, *Linear forms in p-adic logarithms*. II, Compositio Math. **74** (1990), 15–113 (this is the paper announced in [9]).

The results included in these papers improve on the results for c_7 and c_{13}, respectively, and thus would have given an upper bound for H that is considerably better than (19), although it still would have been very large. This would not have implied anything for our method, but it would have considerably reduced the computation time.

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