Notes on isotropic convex bodies

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Contents

Notation

We work on $\mathbb{R}^n$, which is equipped with a Euclidean structure $\langle \cdot, \cdot \rangle$. We denote by $|\cdot|$ the corresponding Euclidean norm, and write B_2^n for the Euclidean unit ball and S^{n-1} for the unit sphere. We fix a coordinate system defined by an orthonormal basis $\{e_1, \dots, e_n\}$. Volume (n-dimensional Lebesgue measure) and the cardinality of a finite set are also denoted by $|\cdot|$. We write ω_n for the volume of B_2^n.

We write $L(\mathbb{R}^n)$ for the family of all linear transformations $T : \mathbb{R}^n \to \mathbb{R}^n$. The class of invertible $T \in L(\mathbb{R}^n)$ is denoted by $GL(n)$, and $SL(n)$ denotes the subclass of volume preserving transformations.

In these notes, convex body is a compact convex subset K of $\mathbb{R}^n$ with $0 \in \mathrm{int}(K)$. A convex body K is called symmetric if $x \in K \Rightarrow -x \in K$. We say that K has center of mass at the origin if

$$\int_K \langle x, \theta \rangle dx = 0 \tag{1}$$

for every $\theta \in S^{n-1}$.

The radial function $\rho_K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^+$ of K is defined by

$$\rho_K(x) = \max\{\lambda > 0 : \lambda x \in K\}. \tag{2}$$

The support function $h_K : \mathbb{R}^n \to \mathbb{R}$ of K is defined by

$$h_K(x) = \max\{\langle x, y \rangle : y \in K\}. \tag{3}$$

The width of K in the direction of $\theta \in S^{n-1}$ is the quantity $w(K, \theta) = h_K(\theta) + h_K(-\theta)$, and the mean width of K is defined by

$$w(K) = \frac{1}{2} \int_{S^{n-1}} w(K, \theta) \sigma(d\theta) = \int_{S^{n-1}} h_K(\theta) \sigma(d\theta), \tag{4}$$

where σ is the rotationally invariant probability measure on S^{n-1}. We write μ for the Haar probability measure on $O(n)$. With $G_{n,k}$ we denote the Grassmann manifold of k-dimensional subspaces of $\mathbb{R}^n$. Then, $O(n)$ equips $G_{n,k}$ with a Haar probability measure $\nu_{n,k}$.

The circumradius of K is the quantity

$$R(K) = \max\{|x| : x \in K\}. \tag{5}$$

The polar body K° of K is

$$K^\circ := \{y \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } x \in K\}. \tag{6}$$

The Brunn-Minkowski inequality describes the effect of Minkowski addition to volume: If A and B are two non empty compact subsets of $\mathbb{R}^n$, then

$$|A+B|^{1/n} \geq |A|^{1/n} + |B|^{1/n}, \tag{7}$$

where $A+B := \{a+b \mid a \in A, b \in B\}$. It follows that, for every $\lambda \in (0,1)$

$$|\lambda A + (1-\lambda)B|^{1/n} \geq \lambda |A|^{1/n} + (1-\lambda)|B|^{1/n}, \tag{8}$$

and, by the arithmetic-geometric means inequality,

$$|\lambda A + (1-\lambda)B| \geq |A|^{\lambda} |B|^{1-\lambda}. \tag{9}$$

Let K be a symmetric convex body in $\mathbb{R}^n$. The function

$$\|x\|_K = \min\{\lambda \geq 0 : x \in \lambda K\} \tag{10}$$

is a norm on $\mathbb{R}^n$. The normed space $(\mathbb{R}^n, \|\cdot\|_K)$ will be denoted by X_K. Conversely, if $X = (\mathbb{R}^n, \|\cdot\|)$ is a normed space, then the unit ball $K_X = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ of X is a symmetric convex body in $\mathbb{R}^n$.

The dual norm $\|\cdot\|_*$ of $\|\cdot\|$ is defined by $\|y\|_* = \max\{|\langle x, y\rangle| : \|x\| \leq 1\}$. From the definition it is clear that $|\langle x, y\rangle| \leq \|y\|_* \|x\|$ for all $x, y \in \mathbb{R}^n$. If $X^* = (\mathbb{R}^n, \|\cdot\|_*)$ is the dual space of X, then $K_{X^*} = K_X^{\circ}$. Note that if $\theta \in S^{n-1}$ then $\rho_K(\theta) = 1/\|\theta\|_K$ and $h_K(\theta) = \|\theta\|_{K^\circ}$.

Whenever we write $a \simeq b$, we mean that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leq b \leq c_2 a$ (also, $b \ll a$ means that a exceeds Cb for some (large) absolute constant $C > 1$). The letters c, c', C, c_1, c_2 etc. denote absolute positive constants which may change from line to line.

Basic references on classical and asymptotic convex geometry are the books [17], [45], [51] and [53].

Chapter 1

Isotropic position and the slicing problem

1.1 Isotropic position: existence and uniqueness

A convex body K in $\mathbb{R}^n$ is called *isotropic* if it has volume $|K| = 1$, center of mass at the origin, and there is a constant $\alpha > 0$ such that

$$\int_K \langle x, y\rangle^2 dx = \alpha^2 |y|^2 \tag{1.1.1}$$

for all $y \in \mathbb{R}^n$.

Remark 1: (i) It is not hard to check that the *isotropic condition* (1.1.1) is equivalent to each one of the following statements:
1. For every $i, j = 1, \ldots, n$,

$$\int_K x_i x_j dx = \alpha^2 \delta_{ij}, \tag{1.1.2}$$

where $x_1, \ldots, x_n$ are the coordinates of x with respect to some orthonormal basis.
2. For every $T \in L(\mathbb{R}^n)$,

$$\int_K \langle x, Tx\rangle dx = \alpha^2 (\mathrm{tr} T). \tag{1.1.3}$$

(ii) Note that if K satisfies the isotropic condition (1.1.1) then

$$\int_K |x|^2 dx = n\alpha^2. \tag{1.1.4}$$

(ii) If K is an isotropic convex body in $\mathbb{R}^n$ and $U \in O(n)$, then $U(K)$ is also isotropic.

Our first result proves the existence of an isotropic body in every linear class.

Proposition 1.1.1. *Let K be a convex body in $\mathbb{R}^n$ with center of mass at the origin. There exists $T \in GL(n)$ such that $T(K)$ is isotropic.*

Proof. The operator $M \in L(\mathbb{R}^n)$ defined by $M(y) = \int_K \langle x, y\rangle x dx$ has a symmetric and positive square root S. Consider the linear image $\tilde{K} = S^{-1}(K)$ of K. Then, for every $y \in \mathbb{R}^n$ we have

$$\begin{aligned}\int_{\tilde{K}} \langle x, y\rangle^2 dx &= |\mathrm{det}S|^{-1} \int_K \langle S^{-1}x, y\rangle^2 dx \\ &= |\mathrm{det}S|^{-1} \int_K \langle x, S^{-1}y\rangle^2 dx \\ &= |\mathrm{det}S|^{-1} \Big\langle \int_K \langle x, S^{-1}y\rangle x dx, S^{-1}y\Big\rangle \\ &= |\mathrm{det}S|^{-1} \langle MS^{-1}y, S^{-1}y\rangle = |\mathrm{det}S|^{-1}|y|^2.\end{aligned}$$

Normalizing volume, we get the result. □

Let K be a convex body in $\mathbb{R}^n$ with volume 1 and center of mass at the origin. We say that $\tilde{K}$ is a *position* of K if $\tilde{K} = TK$ for some $T \in SL(n)$. The next Theorem shows that the isotropic position of a convex body is uniquely determined (if we ignore orthogonal transformations) and arises as a solution of a minimization problem.

Theorem 1.1.2. *Let K be a convex body in $\mathbb{R}^n$ with volume 1 and center of mass at the origin. Define*

$$B(K) = \inf\left\{ \int_{TK} |x|^2 dx : T \in SL(n)\right\}. \tag{1.1.5}$$

Then, a position K_1 of K is isotropic if and only if

$$\int_{K_1} |x|^2 dx = B(K). \tag{1.1.6}$$

If K_1 and K_2 are isotropic positions of K, there exists $U \in O(n)$ such that $K_2 = U(K_1)$.

Proof. (1) Fix an isotropic position K_1 of K. Remark 1 shows that there exists $\alpha > 0$ such that

$$\int_{K_1} \langle x, Tx\rangle dx = \alpha^2 (\mathrm{tr}T) \tag{1.1.7}$$

for every $T \in L(\mathbb{R}^n)$. Then, for every $T \in SL(n)$ we have

$$\begin{aligned}\int_{TK_1} |x|^2 dx &= \int_{K_1} |Tx|^2 dx = \int_{K_1} \langle x, T^*Tx\rangle dx \\ &= \alpha^2 \mathrm{tr}(T^*T) \geq n\alpha^2 = \int_{K_1} |x|^2 dx,\end{aligned}$$

where we have used the arithmetic-geometric means inequality in the form

$$\mathrm{tr}(T^*T) \geq n[\mathrm{det}(T^*T)]^{1/n}. \tag{1.1.8}$$

This shows that K_1 satisfies (1.1.6). In particular, the infimum in (1.1.5) is a minimum.

(2) Note also that if we have equality in (1.1.8) then $T^*T = I$, that is $T \in O(n)$. This shows that any other position $\tilde{K}$ of K which satisfies (1.1.6) is an orthogonal image of K_1, therefore it is isotropic.

(3) Finally, if K_2 is some other isotropic position of K then the first part of the proof shows that K_2 satisfies (1.1.6). By (2) we must have $K_2 = U(K_1)$ for some $U \in O(n)$. $\square$

Remark 2: An alternative way to see that if K is a solution of the minimization problem above then K is isotropic, is using the following simple variational argument. Let $T \in L(\mathbb{R}^n)$. For small $\varepsilon > 0$, $I + \varepsilon T$ is invertible, and hence $(I + \varepsilon T)/[\det(I + \varepsilon T)]^{1/n}$ preserves volume. Consequently,

$$\int_K |x|^2 dx \leq \int_K \frac{|x + \varepsilon Tx|^2}{[\det(I + \varepsilon T)]^{2/n}} dx. \tag{1.1.9}$$

Noting that $|x + \varepsilon Tx|^2 = |x|^2 + 2\varepsilon\langle x, Tx\rangle + O(\varepsilon^2)$ and $[\det(I + \varepsilon T)]^{2/n} = 1 + 2\varepsilon\frac{\mathrm{tr}T}{n} + O(\varepsilon^2)$ and letting $\varepsilon \to 0^+$, we see that

$$\frac{\mathrm{tr}T}{n} \int_K |x|^2 dx \leq \int_K \langle x, Tx\rangle dx. \tag{1.1.10}$$

Since T was arbitrary, a similar inequality holds for $-T$, therefore

$$\frac{\mathrm{tr}T}{n} \int_K |x|^2 dx = \int_K \langle x, Tx\rangle dx \tag{1.1.11}$$

for all $T \in L(\mathbb{R}^n)$. By Remark 1(i), this condition implies that K is isotropic.

1.2 Isotropic constant

The preceding discussion shows that, for every convex body K in $\mathbb{R}^n$ with center of mass at the origin, the constant

$$L_K^2 = \frac{1}{n} \min\left\{ \frac{1}{|TK|^{1+\frac{2}{n}}} \int_{TK} |x|^2 dx \;\Big|\; T \in GL(n) \right\} \tag{1.2.1}$$

is well-defined and depends only on the linear class of K. Also, if K is isotropic then for all $\theta \in S^{n-1}$ we have

$$\int_K \langle x, \theta\rangle^2 dx = L_K^2. \tag{1.2.2}$$

The constant L_K is called *isotropic constant* of K.

Conjecture 1 (isotropic constant): *There exists an absolute constant $C > 0$ such that*

$$L_K \leq C \tag{1.2.3}$$

for every convex body K with center of mass at the origin. Equivalently, if K is an isotropic convex body in $\mathbb{R}^n$, then

$$\int_K \langle x, \theta\rangle^2 dx \leq C^2 \tag{1.2.4}$$

for every $\theta \in S^{n-1}$.

A reverse estimate is available and quite simple. Therefore, what the conjecture states is that the isotropic constants of all convex bodies are (uniformly with respect to n) of the order of 1.

Proposition 1.2.1. *For every isotropic convex body* K *in* $\mathbb{R}^n$,

$$L_K \geq L_{B_2^n} \geq c, \tag{1.2.5}$$

where $c > 0$ *is an absolute constant.*

Proof. If $r_n = \omega_n^{-1/n}$, then $|r_n B_2^n| = 1$ and $r_n B_2^n$ is isotropic. Let K be an isotropic convex body. Observe that $|x| > r_n$ on $K \setminus r_n B_2^n$ and $|x| \leq r_n$ on $r_n B_2^n \setminus K$. It follows that

$$\begin{aligned} nL_K^2 &= \int_K |x|^2 dx \\ &= \int_{K \cap r_n B_2^n} |x|^2 dx + \int_{K \setminus r_n B_2^n} |x|^2 dx \\ &\geq \int_{K \cap r_n B_2^n} |x|^2 dx + \int_{r_n B_2^n \setminus K} |x|^2 dx \\ &= \int_{r_n B_2^n} |x|^2 dx = nL_{B_2^n}^2. \end{aligned}$$

A simple computation shows that

$$L_{B_2^n}^2 = \frac{1}{n} \int_{r_n B_2^n} |x|^2 dx = \frac{1}{n} \frac{n\omega_n}{n+2} r_n^{n+2} = \frac{\omega_n^{-2/n}}{n+2} \geq c^2, \tag{1.2.6}$$

where $c > 0$ is an absolute constant, therefore $L_K \geq L_{B_2^n} \geq c$. □

In the symmetric case, a first upper bound for the isotropic constant follows from John's theorem.

Proposition 1.2.2. *Let* K *be a symmetric convex body in* $\mathbb{R}^n$. *Then,* $L_K \leq c\sqrt{n}$.

Proof. There exist $r > 0$ and $T \in GL(n)$ such that rB_2^n is the ellipsoid of maximal volume inscribed in TK and $|TK| = 1$. Since $rB_2^n \subseteq TK$ we have $r \leq c\sqrt{n}$, and Theorem 1.1.2 shows that

$$nL_K^2 \leq \int_{TK} |x|^2 dx \leq R^2, \tag{1.2.7}$$

where $R = R(K)$ is the circumradius of TK. Therefore,

$$L_K \leq \frac{R}{r} \cdot \frac{r}{\sqrt{n}} \leq c\frac{R}{r}. \tag{1.2.8}$$

Finally, from John's theorem (see [29]) we have $R \leq \sqrt{n} r$. □

Let us also note the following consequence of the reverse Santaló inequality [15].

Proposition 1.2.3. *Let K be a symmetric convex body in $\mathbb{R}^n$. Then,*

$$L_K^2 L_{K^\circ}^2 \leq cn\phi(K), \tag{1.2.9}$$

where

$$\phi(K) = \frac{1}{|K|\cdot|K^\circ|}\int_K\int_{K^\circ}\langle x,y\rangle^2 dydx. \tag{1.2.10}$$

In particular, $L_K L_{K^\circ} \leq c_1\sqrt{n}$.

Proof. Observe that $\phi(K) = \phi(TK)$ for every $T \in GL(n)$. So, we may assume that K° is isotropic. Then, Theorem 1.1.2 shows that

(1.2.11)

$$\frac{1}{|K|^{1+2/n}\cdot|K^\circ|^{1+2/n}}\int_K\int_{K^\circ}\langle x,y\rangle^2 dydx = L_{K^\circ}^2\frac{1}{|K|^{1+2/n}}\int_K |x|^2dx \geq nL_K^2L_{K^\circ}^2.$$

Therefore,

$$nL_K^2L_{K^\circ}^2 \leq \big(|K|\cdot|K^\circ|\big)^{-2/n}\phi(K), \tag{1.2.12}$$

and (1.2.9) follows from the reverse Santaló inequality:

$$\big(|K|\cdot|K^\circ|\big)^{1/n} \geq c_2\omega_n^{2/n} \geq c_3/n, \tag{1.2.13}$$

where $c_3 > 0$ is an absolute constant. Since $\phi(K)$ is trivially bounded by 1, the proof is complete. □

Finally, we give an estimate for the diameter of an isotropic convex body K in terms of the isotropic constant L_K (this proof is from [33]).

Theorem 1.2.4. *Let K be an isotropic convex body in $\mathbb{R}^n$. Then, $R(K) \leq (n+1)L_K$.*

Proof. Let $x \in K$. We define $h : S^{n-1} \to \mathbb{R}$ by

$$h(u) = \max\{t > 0 : x + tu \in K\}. \tag{1.2.14}$$

We can express the volume of K as

$$\begin{aligned} 1 = |K| &= n\omega_n\int_{S^{n-1}}\int_0^{h(u)} t^{n-1}dt\sigma(du) \\ &= \omega_n\int_{S^{n-1}} h^n(u)\sigma(du). \end{aligned}$$

Since K is isotropic, for every $\theta \in S^{n-1}$ we may write

$$\begin{aligned} L_K^2 &= \int_K \langle y,\theta\rangle^2 dy \\ &= n\omega_n\int_{S^{n-1}}\int_0^{h(u)} t^{n-1}\langle x+tu,\theta\rangle^2 dt\sigma(du) \\ &= n\omega_n\int_{S^{n-1}}\int_0^{h(u)} \left(t^{n-1}\langle x,\theta\rangle^2 + 2t^n\langle x,\theta\rangle\langle u,\theta\rangle + t^{n+1}\langle u,\theta\rangle^2\right) dt\sigma(du) \\ &= n\omega_n\int_{S^{n-1}}\left(\frac{h^n(u)}{n}\langle x,\theta\rangle^2 + 2\frac{h^{n+1}(u)}{n+1}\langle x,\theta\rangle\langle u,\theta\rangle + \frac{h^{n+2}(u)}{n+2}\langle u,\theta\rangle^2\right)\sigma(du) \\ &\geq n\omega_n\int_{S^{n-1}}\frac{h^n(u)}{n(n+1)^2}\langle x,\theta\rangle^2\sigma(du) \\ &= \frac{\langle x,\theta\rangle^2}{(n+1)^2}\omega_n\int_{S^{n-1}} h^n(u)\sigma(du) = \frac{\langle x,\theta\rangle^2}{(n+1)^2}. \end{aligned}$$

This means that for every $x \in K$ and $\theta \in S^{n-1}$,

$$|\langle x, \theta\rangle| \le (n+1)L_K. \tag{1.2.15}$$

Therefore,

$$|x| = \max_{\theta\in S^{n-1}} |\langle x, \theta\rangle| \le (n+1)L_K. \tag{1.2.16}$$

Since $x \in K$ was arbitrary, the proof is complete. □

1.3 Connection with Sylvester's problem

Let K be a convex body of volume 1 in $\mathbb{R}^n$. We choose $(n+1)$ points $x_1, \dots, x_{n+1}$ independently and uniformly from K. Their convex hull $\mathrm{co}(x_1, \dots, x_{n+1})$ is a random simplex in K. For every $p > 0$ we define

$$m_p(K) = \left(\int_K \cdots \int_K |\mathrm{co}(x_1, \dots, x_{n+1})|^p dx_{n+1} \dots dx_1\right)^{1/p}. \tag{1.3.1}$$

If we do not want to assume that $|K| = 1$, the correct normalization is

$$m_p(K) = \left(\frac{1}{|K|^{n+p+1}} \int_K \cdots \int_K |\mathrm{co}(x_1, \dots, x_{n+1})|^p dx_{n+1} \dots dx_1\right)^{1/p}. \tag{1.3.2}$$

Then, $m_p(K)$ is invariant under non-degenerate affine transformations: If $T \in GL(n)$ and $u \in \mathbb{R}^n$, then $m_p(K) = m_p(TK + u)$ for all $p > 0$. The quantity $m_1(K)$ is the expectation of the normalized volume of a random simplex inside K.

Sylvester's problem is the following question: describe the affine classes of convex bodies for which $m_p(K)$ is minimized or maximized. It is known that, for every $p > 0$,

$$m_p(K) \ge m_p(B_2^n) \tag{1.3.3}$$

with equality if and only if K is an ellipsoid. In the other direction, the problem is open if $n \ge 3$.

Conjecture 2 (simplex conjecture): *For every convex body K in $\mathbb{R}^n$,*

$$m_1(K) \le m_1(S_n), \tag{1.3.4}$$

where S_n is a simplex in $\mathbb{R}^n$.

The conjecture is correct when $n = 2$. In this Section we will see that Sylvester's problem is connected with the isotropic constant problem: If the simplex conjecture is correct, then $L_K \le C$ for every convex body K.

To see this, we define the variant of $m_p(K)$

$$S_p(K) = \left(\frac{1}{|K|^{n+p}} \int_K \cdots \int_K |\mathrm{co}(0, x_1, \dots, x_n)|^p dx_n \dots dx_1\right)^{1/p}. \tag{1.3.5}$$

Proposition 1.3.1. *Let K be a convex body in $\mathbb{R}^n$ with volume 1 and center of mass at the origin. Then, for every $p \geq 1$ we have*

$$S_p(K) \leq m_p(K) \leq (n+1)S_p(K). \tag{1.3.6}$$

Proof. For every $x \in K$ we define

$$S_p(K;x) = \left(\int_K \cdots \int_K |\mathrm{co}(x, x_1, \ldots, x_n)|^p dx_n \ldots dx_1\right)^{1/p}. \tag{1.3.7}$$

We know that $|\mathrm{co}(x, x_1, \ldots, x_n)| = |\det(\tilde{x}, \tilde{x_1}, \ldots, \tilde{x_n})|/n!$ where $\tilde{z} = (z, 1) \in \mathbb{R}^{n+1}$ if $z \in \mathbb{R}^n$, and this determinant is an affine function of x. It follows that $|\mathrm{co}(x, x_1, \ldots, x_n)|^p$ is a convex function on K. Integrating with respect to $x_1, \ldots, x_n$ we see that $S_p^p(K;x)$ is also convex. Since 0 is the center of mass of K and $|K| = 1$, we conclude that

$$S_p^p(K;0) \leq \int_K S_p^p(K;x)dx, \tag{1.3.8}$$

which gives

$$S_p^p(K) \leq m_p^p(K). \tag{1.3.9}$$

This proves the left hand side inequality. For the right hand side inequality we observe that if $x_1, \ldots, x_{n+1} \in K$, then

$$|\mathrm{co}(x_1, \ldots, x_{n+1})| = \frac{1}{n!}|\det(\tilde{x}_1 \ldots, \tilde{x}_{n+1})| \tag{1.3.10}$$

where $\tilde{x}_j = (x_j, 1) \in \mathbb{R}^{n+1}$, and if we develop the determinant in the column $(1, \ldots, 1)$ and apply the triangle inequality we get

$$|\mathrm{co}(x_1, \ldots, x_{n+1})| \leq \sum_{j=1}^{n+1} |\mathrm{co}(0, x_i : i \neq j)|. \tag{1.3.11}$$

It follows that

$$\begin{aligned}
m_p(K) &= \left(\int_K \cdots \int_K |\mathrm{co}(x_1, \ldots, x_{n+1})|^p dx_{n+1} \ldots dx_1\right)^{1/p} \\
&\leq \left(\int_K \cdots \int_K \left(\sum_{j=1}^{n+1} |\mathrm{co}(0, x_i : i \neq j)|\right)^p dx_{n+1} \ldots dx_1\right)^{1/p} \\
&\leq \sum_{j=1}^{n+1} \left(\int_K \cdots \int_K |\mathrm{co}(0, x_i : i \neq j)|^p dx_{n+1} \ldots dx_1\right)^{1/p} \\
&= (n+1)S_p(K).
\end{aligned}$$

□

Remark 1: The function $f_i : K \to \mathbb{R}$ defined by $x_i \mapsto \det(x_1, \ldots, x_n)$ for fixed $x_j, j \neq i$ in K, is a linear functional. This leads to the following Proposition (we postpone its proof, which is a simple consequence of the reverse Hölder inequalities of §2.1).

Proposition 1.3.2. *Let K be a convex body in $\mathbb{R}^n$ with volume 1 and center of mass at the origin. Then,*

$$S_2(K) \leq c^n S_1(K), \tag{1.3.12}$$

where $c > 0$ is an absolute constant. □

Fix an orthonormal basis in $\mathbb{R}^n$ and write $M(K)$ for the matrix with entries

$$[M(K)]_{ij} = \int_K x_i x_j dx. \tag{1.3.13}$$

This is the *matrix of inertia* of K. The connection of $m_p(K)$ and $S_p(K)$ with the isotropic constant of K becomes clear by the next Proposition.

Proposition 1.3.3. *Let K be a convex body in $\mathbb{R}^n$ with volume 1 and center of mass at the origin. Then,*

$$S_2^2(K) = \frac{\det(M(K))}{n!}. \tag{1.3.14}$$

Proof. By definition,

$$S_2^2(K) = \int_K \cdots \int_K |\mathrm{co}(0, x_1, \ldots, x_n)|^2 dx_n \ldots dx_1. \tag{1.3.15}$$

We write $x_i = (x_{ij}), j = 1, \ldots, n$. Then,

$$(n!)^2 S_2^2(K) = \int_K \cdots \int_K |\det(x_1, \ldots, x_n)|^2 dx_n \ldots dx_1, \tag{1.3.16}$$

and expanding the determinant we get

$$\begin{aligned}
(n!)^2 S_2^2(K) &= \int_K \cdots \int_K \left(\sum_\sigma \epsilon_\sigma \prod_{i=1}^n x_{i,\sigma(i)}\right)\left(\sum_\tau \epsilon_\tau \prod_{i=1}^n x_{i,\tau(i)}\right) dx_n \ldots dx_1 \\
&= \int_K \cdots \int_K \left(\sum_{\sigma,\tau} \epsilon_\sigma \epsilon_\tau \prod_{i=1}^n x_{i,\sigma(i)} x_{i,\tau(i)}\right) dx_n \ldots dx_1 \\
&= \int_K \cdots \int_K \left(\sum_{\sigma,\varphi} \epsilon_\varphi \prod_{i=1}^n x_{i,\sigma(i)} x_{i,\varphi(\sigma(i))}\right) dx_n \ldots dx_1 \\
&= \sum_{\sigma,\varphi} \epsilon_\varphi \prod_{i=1}^n \left(\int_K x_i x_{\varphi(i)} dx\right) \\
&= n! \det(M(K)).
\end{aligned}$$

□

Remark 2: Let K be a convex body in $\mathbb{R}^n$ with volume 1 and center of mass at the origin. Since $S_2(K)$ is invariant under invertible linear transformations, the identity (1.3.14) shows that if $T \in SL(n)$ then

$$\det(M(TK)) = \det(M(K)). \tag{1.3.17}$$

If we choose T so that TK will be isotropic, then $M(TK) = L_K^2 I$. So,

$$\det(M(K)) = L_K^{2n}. \tag{1.3.18}$$

Thus, we have proved the following.

Theorem 1.3.4. *Let K be a convex body in $\mathbb{R}^n$ with volume 1 and center of mass at the origin. Then,*

$$L_K^{2n} = n!\, S_2^2(K). \tag{1.3.19}$$

Corollary 1.3.5. *$L_K \leq c\sqrt{n}$ for every convex body K in $\mathbb{R}^n$ with center of mass at the origin.*

Proof. We may assume that K is isotropic. Since $S_2(K)$ is obviously bounded by 1, we get

$$L_K \leq \sqrt[2n]{n!}, \tag{1.3.20}$$

and $\sqrt[2n]{n!} \leq c\sqrt{n}$ for some absolute constant $c > 0$. □

Corollary 1.3.6. *If the simplex conjecture is correct, then*

$$L_K \leq C \tag{1.3.21}$$

for every convex body K in $\mathbb{R}^n$ with center of mass at the origin.

Proof. Consider the simplex

$$S_n = \left\{ x \in \mathbb{R}^n : -\frac{1}{n+1} \leq x_i \leq \frac{n}{n+1}\ ,\ \sum_{i=1}^n x_i \leq \frac{1}{n+1} \right\}. \tag{1.3.22}$$

Then, $S_n' = (n!)^{1/n} S_n$ has volume 1 and center of mass at the origin. A simple computation shows that

$$\int_{S_n'} x_i^2 dx < \frac{(n!)^{1+\frac{2}{n}}}{(n+2)!}, \tag{1.3.23}$$

and since $M(K)$ is symmetric and positive definite, Hadamard's inequality gives

$$S_2^2(S_n') = \frac{\det(M(S_n'))}{n!} \leq \frac{1}{n!} \left(\frac{(n!)^{1+\frac{2}{n}}}{(n+2)!} \right)^n \leq \frac{1}{n!}.$$

Now, if K is an isotropic convex body in $\mathbb{R}^n$ we have $m_1(K) \leq m_1(S_n')$, and combining Propositions 1.3.1, 1.3.2 and Theorem 1.3.4 we obtain

$$\begin{aligned}
L_K^n &= \sqrt{n!}\, S_2(K) \leq \sqrt{n!}\, c^n S_1(K) \\
&\leq \sqrt{n!}\, c^n m_1(K) \leq \sqrt{n!}\, c^n m_1(S_n') \\
&\leq \sqrt{n!}\, c^n (n+1) S_1(S_n') \leq \sqrt{n!}\, c^n (n+1) S_2(S_n') \\
&\leq (n+1)c^n.
\end{aligned}$$

It follows that $L_K \leq 2c$. □

1.4 Binet ellipsoid of inertia

Let K be a convex body in $\mathbb{R}^n$ with volume 1 and center of mass at the origin. The Binet ellipsoid $E_B(K)$ of K is defined by

$$\|y\|_{E_B(K)}^2 = \int_K \langle x,y\rangle^2 dx = \langle My, y\rangle. \tag{1.4.1}$$

Therefore, K is isotropic if and only if $E_B(K) = L_K^{-1}B_2^n$. In particular, if K is an isotropic convex body in $\mathbb{R}^n$, we have

$$|E_B(K)| = \omega_n L_K^{-n}. \tag{1.4.2}$$

Lemma 1.4.1. *Let K be a convex body in $\mathbb{R}^n$ with volume 1 and center of mass at the origin. Then,*

$$|E_B(K)| = \omega_n L_K^{-n}. \tag{1.4.3}$$

Proof. We have already seen that $|\det(M(K)| = |\det M(TK)|$ for every $T \in SL(n)$. This shows that

$$|E_B(TK)| = \omega_n |\det M(TK)|^{-1/2} = \omega_n |\det M(K)|^{-1/2} = |E_B(K)| \tag{1.4.4}$$

for every $T \in SL(n)$. We now choose $T \in SL(n)$ so that $T(K)$ is isotropic and recall (1.4.2). $\square$

Corollary 1.4.2. *Let K be a convex body in $\mathbb{R}^n$ with volume 1 and center of mass at the origin. There exists $\theta \in S^{n-1}$ such that*

$$\int_K \langle x,\theta\rangle^2 dx \leq L_K^2. \tag{1.4.5}$$

Proof. Note that

$$L_K^{-n} = |E_B(K)|/\omega_n = \int_{S^{n-1}} \|\theta\|_{E_B(K)}^{-n} \sigma(d\theta). \tag{1.4.6}$$

It follows that $\min_{\theta\in S^{n-1}} \|\theta\|_{E_B(K)} \leq L_K$. $\square$

1.5 Sections of an isotropic convex body

Let K be a convex body in $\mathbb{R}^n$ and fix a direction $\theta \in S^{n-1}$. We define a function $f = f_{K,\theta} : \mathbb{R} \to \mathbb{R}^+$ by

$$f(t) = |K \cap (\theta^\perp + t\theta)|. \tag{1.5.1}$$

Here, $|\cdot|$ denotes $(n-1)$-dimensional volume. So, $f(t)$ gives the area of the section of K with the hyperplane which is perpendicular to θ, at a distance t from $\theta^\perp$.

Theorem 1.5.1. *Let K be a convex body in $\mathbb{R}^n$, $\theta \in S^{n-1}$, and $f(t) = |K \cap (\theta^\perp + t\theta)|$. Then, $f^{\frac{1}{n-1}}$ is concave on its support.*

Proof. We may assume that $\theta = e_n$, and identify $\theta^\perp$ with $\mathbb{R}^{n-1}$. For every $t \in \mathbb{R}$ we set

$$K(t) = \{x \in \mathbb{R}^{n-1} : (x,t) \in K\}. \tag{1.5.2}$$

Let $I = \{t : K(t) \neq \emptyset\}$. Then, $K(t)$ is convex for every $t \in I$, and if $t, s \in I$, $\lambda \in (0,1)$, then

$$\lambda K(t) + (1-\lambda)K(s) \subseteq K(\lambda t + (1-\lambda)s). \tag{1.5.3}$$

Applying the Brunn-Minkowski inequality on $\mathbb{R}^{n-1}$, we get

$$|K(\lambda t + (1-\lambda)s)|^{\frac{1}{n-1}} \geq \lambda |K(t)|^{\frac{1}{n-1}} + (1-\lambda)|K(s)|^{\frac{1}{n-1}}. \tag{1.5.4}$$

Since $f(t) = |K \cap (\theta^\perp + t\theta)| = |K(t)|$, the proof is complete. □

Corollary 1.5.2. *In the notation of Theorem* 1.5.1*, f is a log-concave function.* □

Corollary 1.5.3. *If K is a symmetric convex body, then $\|f\|_\infty = f(0)$.* □

If we do not assume the symmetry of K, we still have that the hyperplane section passing through the center of mass of K is comparable to the maximal section in this direction.

Proposition 1.5.4. *Let K be a convex body in $\mathbb{R}^n$ with volume* 1 *and center of mass at the origin. If $\theta \in S^{n-1}$ and $f(t) = |K \cap (\theta^\perp + t\theta)|$, then*

$$\|f\|_\infty \leq \left(\frac{n+1}{n}\right)^{n-1} f(0) \leq e|K \cap \theta^\perp|. \tag{1.5.5}$$

Proof. Let $[-a,b]$ be the support of f, where $a, b > 0$, and assume that $\|f\|_\infty = f(t_0)$. We will show that

$$f(0) \geq \left(\frac{n}{n+1}\right)^{n-1} f(t_0) \geq \frac{1}{e} f(t_0). \tag{1.5.6}$$

Replacing θ by $-\theta$ if needed, we may assume that $0 < t_0 \leq b$ (we can also assume that $f(0) < f(t_0)$; otherwise, there is nothing to prove). Since K has its center of mass at the origin, we have

$$\int_{-a}^{b} t f(t)dt = \int_K \langle x, \theta \rangle dx = 0. \tag{1.5.7}$$

Therefore,

$$\int_{-a}^{0} (-t) f(t)dt = \int_0^b t f(t)dt \geq \int_0^{t_0} t f(t)dt. \tag{1.5.8}$$

We set $h = f^{\frac{1}{n-1}}$ and consider the linear function g defined by the equations $g(0) = h(0)$ and $g(t_0) = h(t_0)$. By the Brunn-Minkowski inequality, h is concave, which implies that $g \geq h$ on $[-a, 0]$ and $g \leq h$ on $[0, t_0]$. Since $g(0) < g(t_0)$, we see that g is strictly increasing and $g(-\gamma)=0$ for some $-\gamma\leq -a$. In other words, $g(t)=c(t+\gamma)$ for some $c>0$ and $\gamma\geq\alpha$. Then,

$$\begin{aligned}\int_{-\gamma}^{0}(-t)[g(t)]^{n-1}dt &\geq \int_{-a}^{0}(-t)[g(t)]^{n-1}dt\geq\int_{-a}^{0}(-t)[h(t)]^{n-1}dt\\ &\geq \int_{0}^{t_0}t[h(t)]^{n-1}dt\geq\int_{0}^{t_0}t[g(t)]^{n-1}dt.\end{aligned}$$

This means that

$$\begin{aligned}0 &\geq \int_{-\gamma}^{t_0}t[g(t)]^{n-1}dt=c^{n-1}\int_{-\gamma}^{t_0}t(t+\gamma)^{n-1}dt\\ &= c^{n-1}\int_{0}^{\gamma+t_0}t^{n-1}(t-\gamma)dt=c^{n-1}\left(\frac{(\gamma+t_0)^{n+1}}{n+1}-\gamma\frac{(\gamma+t_0)^n}{n}\right),\end{aligned}$$

which implies $t_0\leq\frac{\gamma}{n}$. Therefore,

$$\frac{f(0)}{f(t_0)}=\left(\frac{g(0)}{g(t_0)}\right)^{n-1}=\left(\frac{\gamma}{\gamma+t_0}\right)^{n-1}\geq\left(\frac{n}{n+1}\right)^{n-1}. \tag{1.5.9}$$

□

Our next observation is that the volume of the $(n-1)$-dimensional section $|K\cap\theta^{\perp}|$ is closely related to integrals of the form

$$I_2(K,\theta):=\left(\int_K\langle x,\theta\rangle^2dx\right)^{1/2}. \tag{1.5.10}$$

This connection becomes clear if we write $I_2(K,\theta)$ in the form

$$\int_K\langle x,\theta\rangle^2dx=\int_{\mathbb{R}}t^2f_{K,\theta}(t)dt, \tag{1.5.11}$$

and will provide a link between the volume of sections and the isotropic position.

Proposition 1.5.5. *Let K be a convex body in $\mathbb{R}^n$ with volume 1 and center of mass at the origin. For every $\theta\in S^{n-1}$,*

$$\left(\int_K\langle x,\theta\rangle^2dx\right)^{1/2}\geq\frac{1}{2\sqrt{3}e}\frac{1}{|K\cap\theta^{\perp}|}. \tag{1.5.12}$$

Proof. Let $f:=f_{K,\theta}$. We set $B=\int_0^{+\infty}f(t)dt$ and define

$$g(t)=\|f\|_{\infty}\chi_{[0,B/\|f\|_{\infty}]}(t). \tag{1.5.13}$$

Since $g\geq f$ on the support of g, we have

$$\int_0^s f(t)dt\leq\int_0^s g(t)dt$$

for every $0\leq s\leq B/\|f\|_{\infty}$. The integrals of f and g on $[0,+\infty)$ are both equal to B. So,

$$\int_s^{\infty}g(t)dt\leq\int_s^{\infty}f(t)dt \tag{1.5.14}$$

for every $s \geq 0$. It follows that

$$\begin{aligned}\int_0^\infty t^2 f(t)dt &= \int_0^\infty \int_0^t 2sf(t)dsdt \\ &= \int_0^\infty 2s\left(\int_s^\infty f(t)dt\right)ds \\ &\geq \int_0^\infty 2s\left(\int_s^\infty g(t)dt\right)ds \\ &= \int_0^\infty t^2 g(t)dt \\ &= \int_0^{B/\|f\|_\infty} t^2\|f\|_\infty dt = \frac{B^3}{3\|f\|_\infty^2}.\end{aligned}$$

In the same way, if $A = \int_{-\infty}^0 f(t)dt$, we see that

$$\int_{-\infty}^0 t^2 f(t)dt \geq \frac{A^3}{3\|f\|_\infty^2}. \tag{1.5.15}$$

Summing the estimates we conclude that

$$\int_K \langle x,\theta\rangle^2 dx \geq \frac{B^3+A^3}{3\|f\|_\infty^2}, \tag{1.5.16}$$

and since $A+B = |K| = 1$, it follows that

$$\left(\int_K \langle x,\theta\rangle^2 dx\right)^{1/2} \geq \frac{1}{2\sqrt{3}}\frac{1}{\|f\|_\infty}. \tag{1.5.17}$$

Taking into account Proposition 1.5.4, we get the result. □

Proposition 1.5.6. *Let K be a convex body in $\mathbb{R}^n$ with volume 1 and center of mass at the origin. For every $\theta \in S^{n-1}$ we have*

$$\left(\int_K \langle x,\theta\rangle^2 dx\right)^{1/2} \leq c\frac{1}{|K\cap\theta^\perp|}, \tag{1.5.18}$$

where $c > 0$ is an absolute constant.

Proof. We define f, A and B as in Proposition 1.5.5 and distinguish two cases. Assume first that there exists $s > 0$ such that $f(s) = f(0)/2$. Since f is log-concave on $[0,s]$ and $f(0) \geq f(s)$, we see that $f(t) \geq f(s)$ for all $t \in [0,s]$. Therefore,

$$1 \geq B = \int_0^\infty f(t)dt \geq \int_0^s f(t)dt \geq sf(s) = sf(0)/2. \tag{1.5.19}$$

If $t > s$, using the fact that f is log-concave, we have $f(s) \geq [f(0)]^{1-\frac{s}{t}}[f(t)]^{\frac{s}{t}}$, which implies $f(t)\leq f(0)2^{-t/s}$. We write

$$\begin{aligned}\int_0^\infty t^2f(t)dt &= \int_0^s t^2f(t)dt+\int_s^\infty t^2f(t)dt\\ &\leq \|f\|_\infty\int_0^s t^2dt+\int_s^\infty t^2f(0)2^{-t/s}dt\\ &\leq f(0)\left(e\frac{s^3}{3}+s^3\int_1^\infty u^22^{-u}du\right)\\ &\leq (c/f(0))^2,\end{aligned}$$

where we have used Proposition 1.5.4 and the estimate $s\leq 2/f(0)$.

On the other hand, if for all $s>0$ on the support of f we have $f(s)>f(0)/2$, then the role of s is played by $s_0=\max(\mathrm{supp}f\cap\mathbb{R}^+)$. We have $1\geq B\geq f(0)s_0/2$ and

$$\int_0^\infty t^2f(t)dt=\int_0^{s_0}t^2f(t)dt\leq ef(0)s_0^3/3, \tag{1.5.20}$$

which leads to the same upper bound $(c/f(0))^2$.

The same reasoning applies to $(-\infty,0]$. It follows that

$$\begin{aligned}\int_K\langle x,\theta\rangle^2dx &= \int_0^\infty t^2f(t)dt+\int_{-\infty}^0 t^2f(t)dt\\ &\leq (c_1/f(0))^2,\end{aligned}$$

where $c_1>0$ is an absolute constant. □

Assume that K is isotropic. Then, Propositions 1.5.5 and 1.5.6 show that all $(n-1)$-dimensional sections $K\cap\theta^\perp$ of K have "the same volume".

Theorem 1.5.7. *Let K be an isotropic convex body in $\mathbb{R}^n$. For every $\theta\in S^{n-1}$ we have*

$$\frac{c_1}{L_K}\leq|K\cap\theta^\perp|\leq\frac{c_2}{L_K}, \tag{1.5.21}$$

where $c_1,c_2>0$ are absolute constants. □

This establishes the connection to the slicing problem.

Conjecture 3 (slicing problem): There exists an absolute constant $c>0$ with the following property: if K is a convex body in $\mathbb{R}^n$ with volume 1 and center of mass at the origin, there exists $\theta\in S^{n-1}$ such that

$$|K\cap\theta^\perp|\geq c. \tag{1.5.22}$$

It is now not hard to see that this conjecture is equivalent to the conjecture about the isotropic constant. Assume that the slicing problem has an affirmative answer. If K is isotropic, Theorem 1.5.7 shows that *all* sections $K\cap\theta^\perp$ have volume bounded by c_2/L_K. Since (1.5.22) must be true for at least one $\theta\in S^{n-1}$, we get $L_K\leq c_2/c$.

Conversely, if there exists an absolute bound C^2 for the isotropic constant, then the slicing conjecture follows. An easy way to see this is through the Binet ellipsoid of inertia. Assume that K is a convex body in $\mathbb{R}^n$ with volume 1 and center of mass at the origin. According to Corollary 1.4.2, there exists $\theta \in S^{n-1}$ such that

$$\int_K \langle x, \theta\rangle^2 dx \leq L_K^2 \leq C^2. \tag{1.5.23}$$

Now, Proposition 1.5.5 shows that

$$|K \cap \theta^{\perp}| \geq c := \frac{1}{2\sqrt{3}eC}. \tag{1.5.24}$$

1.6 Connection with the M-position

The following theorem of Milman ([40], see also [41]) establishes the existence of "M-ellipsoids" associated to any convex body.

Theorem 1.6.1. *There exists an absolute constant $c > 0$ with the following property: For every convex body K in $\mathbb{R}^n$ with center of mass at the origin, there exists an origin symmetric ellipsoid M_K such that $|K| = |M_K|$ and for every convex body T in $\mathbb{R}^n$*

$$\frac{1}{c}\big|M_K + T\big|^{1/n} \leq \big|K + T\big|^{1/n} \leq c\big|M_K + T\big|^{1/n} \tag{1.6.1}$$

and

$$\frac{1}{c}|M_K^{\circ} + T|^{1/n} \leq |K^{\circ} + T|^{1/n} \leq c|M_K^{\circ} + T|^{1/n}. \tag{1.6.2}$$

A consequence of Theorem 1.6.1 is that for every convex body K in $\mathbb{R}^n$ there exists a position $\tilde{K} = u_K(K)$ of volume $|\tilde{K}| = |K|$ such that for every pair of convex bodies K_1 and K_2 in $\mathbb{R}^n$ and for all $t_1, t_2 > 0$,

$$|t_1\tilde{K_1} + t_2\tilde{K_2}|^{1/n} \leq c\left(t_1|\tilde{K_1}|^{1/n} + t_2|\tilde{K_2}|^{1/n}\right), \tag{1.6.3}$$

where $c > 0$ is an absolute constant. This statement is the reverse Brunn-Minkowski inequality.

Recall the definition of the covering number $N(A, B)$ of two convex bodies A and B: this is the least integer N for which there exist N translates of B whose union covers A. It is quite easy to check that $|A + B| \leq N(A, B)|2B|$ and if B is symmetric, $|A + B/2| \geq N(A, B)|B/2|$.

Interchanging the roles of K and M_K, let us assume that the assertion of Theorem 1.6.1 is satisfied by $M_K = B_2^n$. This is always possible if we apply a linear transformation to K. Then,

$$N(K, B_2^n) \leq \frac{2^n|K + B_2^n|}{|B_2^n|} \leq \exp(c_1 n), \tag{1.6.4}$$

where $c_1 > 0$ is a constant depending only on c. This condition on $N(K, B_2^n)$ implies (see e.g. [44]) that

$$\max\{N(B_2^n, K), N(K^{\circ}, B_2^n), N(B_2^n, K^{\circ})\} \leq \exp(c_2 n) \tag{1.6.5}$$

for some constant c_2 which again depends only on c_1. In other words, we have the following.

Proposition 1.6.2. *There exists an absolute constant $\beta > 0$ such that every convex body K in $\mathbb{R}^n$ with center of mass at the origin has a linear image $\tilde{K}$ with $|\tilde{K}| = |B_2^n|$ which satisfies*

$$\max\{N(\tilde{K}, B_2^n), N(B_2^n, \tilde{K}), N(\tilde{K}^\circ, B_2^n), N(B_2^n, \tilde{K}^\circ)\} \leq \exp(\beta n). \tag{1.6.6}$$

We say that a convex body K in $\mathbb{R}^n$ which has volume $|K| = |B_2^n|$, center of mass at the origin and satisfies (1.6.6) is in *M-position with constant β*. If K_1 and K_2 are two such convex bodies, then it is easily checked that

$$|K_1 + K_2|^{1/n} \leq C\left(|K_1|^{1/n} + |K_2|^{1/n}\right) \tag{1.6.7}$$

and

$$|K_1^\circ + K_2^\circ|^{1/n} \leq C\left(|K_1^\circ|^{1/n} + |K_2^\circ|^{1/n}\right) \tag{1.6.8}$$

where C is a constant depending only on β. If K is in M-position with constant β, setting $K_1 = K$, $K_2 = B_2^n$ and using the reverse Santaló inequality, we get

$$c^n|K| \cdot |K^\circ| \leq |K \cap B_2^n| \cdot |\mathrm{co}(K^\circ \cup B_2^n)| \leq |K \cap B_2^n| \cdot |K^\circ + B_2^n|, \tag{1.6.9}$$

which, combined with (1.6.8), gives $|K \cap B_2^n| \geq c^n|K|$.

Pisier (see [51], Chapter 7) has given a different approach to these results, which provides a construction of special M-ellipsoids with regularity estimates on the covering numbers. The precise statement is as follows.

Theorem 1.6.3. *For every $\alpha > 1/2$ and every convex body K there exists an affine image $\tilde{K}$ of K which satisfies $|\tilde{K}| = |B_2^n|$ and*
(1.6.10)

$$\max\{N(\tilde{K}, tB_2^n), N(B_2^n, t\tilde{K}), N(\tilde{K}^\circ, tB_2^n), N(B_2^n, t\tilde{K}^\circ)\} \leq \exp\left(\frac{c(\alpha)n}{t^{1/\alpha}}\right)$$

for every $t \geq 1$, where $c(\alpha)$ is a constant depending only on α, with $c(\alpha) = O\big((\alpha - \frac{1}{2})^{-1/2}\big)$ as $\alpha \to \frac{1}{2}$. □

We then say that $\tilde{K}$ is in M-position of order α (or α-regular M-position).

The next Theorem shows that if the isotropic constant is uniformly bounded, then isotropic convex bodies are in M-position of order 1. This would give a description of the M-position in classical terms. Our method is based on an estimate of the covering numbers $N(K, tB_2^n)$ in terms of the quantity

$$M(K, B_2^n) = \int_K |x| dx. \tag{1.6.11}$$

Note that

$$M(K, B_2^n) \leq \left(\int_K |x|^2 dx\right)^{1/2} = \sqrt{n} L_K. \tag{1.6.12}$$

Theorem 1.6.4. *Let K be an isotropic convex body in $\mathbb{R}^n$. For every $t > 0$ we have*

$$N(K, tB_2^n) \leq 2\exp\left(\frac{6n^{3/2}L_K}{t}\right). \tag{1.6.13}$$

Proof. Consider the Minkowski functional $p_K(x) = \inf\{\lambda > 0 : x \in \lambda K\}$. It is clear that p_K is subadditive and positively homogeneous. We define a Borel probability measure on $\mathbb{R}^n$ by

$$\mu(A) = \frac{1}{c_K}\int_A e^{-p_K(x)}dx, \tag{1.6.14}$$

where $c_K = \int_{\mathbb{R}^n}\exp(-p_K(x))dx$. Let $\{x_1,\ldots,x_N\}$ be a subset of K which is maximal with respect to the condition

$$i \neq j \Longrightarrow |x_i - x_j| \geq t. \tag{1.6.15}$$

Then, the balls $x_i + (t/2)B_2^n$ have disjoint interiors, and $K \subseteq \bigcup_{i\leq N}(x_i + tB_2^n)$. Consequently, $N(K, tB_2^n) \leq N$.
We choose $b > 0$ so that $\mu(bB_2^n) \geq 1/2$. If we set $y_i = (2b/t)x_i$, then

$$\begin{aligned}
\mu(y_i + bB_2^n) &= \frac{1}{c_K}\int_{bB_2^n} e^{-p_K(x+y_i)}dx \geq \frac{1}{c_K}\int_{bB_2^n} e^{-p_K(x)}e^{-p_K(y_i)}dx \\
&= e^{-p_K(y_i)}\frac{1}{c_K}\int_{bB_2^n} e^{-p_K(x)}dx = e^{-\frac{2b}{t}p_K(x_i)}\mu(bB_2^n) \\
&\geq e^{-2b/t}\mu(bB_2^n),
\end{aligned}$$

since $p_K(x_i) \leq 1$, $i = 1,\ldots,N$. The balls $y_i + bB_2^n$ have disjoint interiors, therefore

$$Ne^{-2b/t}\mu(bB_2^n) \leq \sum_{i=1}^N \mu(y_i + bB_2^n) = \mu\left(\bigcup_{i=1}^N (y_i + bB_2^n)\right) \leq 1. \tag{1.6.16}$$

It follows that

$$N(K, tB_2^n) \leq e^{2b/t}(\mu(bB_2^n))^{-1} \leq 2e^{2b/t}. \tag{1.6.17}$$

What remains is to estimate b. We first compute the constant

$$\begin{aligned}
c_K &= \int_{\mathbb{R}^n} e^{-p_K(x)}dx = \int_{\mathbb{R}^n}\int_{p_K(x)}^\infty e^{-s}dsdx \\
&= \int_0^\infty e^{-s}|\{x : p_K(x) \leq s\}|ds = \int_0^\infty s^n e^{-s}ds = n!.
\end{aligned}$$

It follows that

$$\begin{aligned}
J &:= \int_{\mathbb{R}^n}|x|\mu(dx) = \frac{1}{c_K}\int_{\mathbb{R}^n}|x|\int_{p_K(x)}^\infty e^{-s}dsdx \\
&= \frac{1}{n!}\int_0^\infty s^{n+1}e^{-s}ds \cdot \int_K |x|dx = (n+1)M(K, B_2^n).
\end{aligned}$$

From Markov's inequality, $\mu(x \in \mathbb{R}^n : |x| > 2J) \leq 1/2$, which shows thar $\mu(2JB_2^n) \geq 1/2$. If we choose $b = 2J$, we get

$$\begin{aligned}
N(K, tB_2^n) &\leq 2\exp(4J/t) \leq 2\exp(4(n+1)M(K, B_2^n)/t) \\
&\leq 2\exp(6n^{3/2}L_K/t),
\end{aligned}$$

which is the assertion of the Theorem. □

Note that $|K| = 1$, and hence $|K| = |r_n B_2^n|$ for some $r_n > 0$ with $r_n \simeq \sqrt{n}$. Therefore,

$$N(K, t(r_n B_2^n)) \leq 2\exp\left(\frac{cL_K n}{t}\right) \tag{1.6.18}$$

for all $t > 0$, where $c > 0$ is an absolute constant. This proves that if L_K is uniformly bounded, then isotropic convex bodies are in M-position of order 1. In particular, every pair of isotropic convex bodies K and T we would satisfy the reverse Brunn-Minkowski inequality. We will show that the converse is also true.

Theorem 1.6.5. *Assume that there exists a constant $A > 0$ such that for every n the following holds true: if K and T are isotropic convex bodies in $\mathbb{R}^n$, then*

$$|K + T|^{1/n} \leq 2A = A\left(|K|^{1/n} + |T|^{1/n}\right). \tag{1.6.19}$$

Then, for every convex body in $\mathbb{R}^n$ we have

$$L_K \leq cA^4, \tag{1.6.20}$$

where $c > 0$ is an absolute constant.

We first prove a simple Lemma.

Lemma 1.6.6. *Let K and T be two isotropic convex bodies in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Then, $W := (L_T/L_K)^{\frac{m}{n+m}} K \times (L_K/L_T)^{\frac{n}{n+m}} T$ is an isotropic convex body in $\mathbb{R}^{n+m}$, and*

$$L_{K\times T} = L_K^{\frac{n}{n+m}} L_T^{\frac{m}{n+m}}. \tag{1.6.21}$$

Proof. Let E be the subspace spanned by the first n standard unit vectors in $\mathbb{R}^{n+m}$. We define $W = aK \times bT$ where $a^n b^m = 1$ so that $|W| = 1$, and write M for the operator $M_W \in L(\mathbb{R}^{n+m})$ defined by

$$M(z) = \int_W \langle w, z\rangle z dw. \tag{1.6.22}$$

It is clear that if $z \in E$ then $M(z) \in E$. Also,
(1.6.23)

$$\langle M(z), z\rangle = \int_W \langle w, z\rangle^2 dw = b^m \int_{aK} \langle x, z\rangle^2 dx = b^m a^{n+2} L_K^2 |z|^2 = a^2 L_K^2 |z|^2.$$

The same argument shows that if $z \in E^\perp$ then $M(z) \in E^\perp$ and

$$\langle Mz, z\rangle = b^2 L_T^2 |z|^2. \tag{1.6.24}$$

Since M acts as a multiple of the identity on both E and $E^\perp$, we see that W will be isotropic provided that

$$aL_K = bL_T. \tag{1.6.25}$$

Since $a^n b^m = 1$ this condition gives

$$a = \left(\frac{L_T}{L_K}\right)^{\frac{m}{n+m}} \quad \text{and} \quad b = \left(\frac{L_K}{L_T}\right)^{\frac{n}{n+m}}. \tag{1.6.26}$$

Also, $L_{K\times T} = L_W = aL_K$ and inserting the value of a we complete the proof. $\square$

Proof of Theorem 1.6.5: Let $D_s = r_s B_2^s$ be the Euclidean ball of volume 1 in $\mathbb{R}^s$. Let K be an isotropic convex body in $\mathbb{R}^n$. According to Lemma 1.6.6, the body $W = (L_{D_n}/L_K)^{1/2} K \times (L_K/L_{D_n}) D_n$ is an isotropic convex body in $\mathbb{R}^{2n}$ (and $L_W = (L_K L_{D_n})^{1/2}$). Applying (1.6.19) to W and D_{2n} we get

(1.6.27) $$|W + D_{2n}|^{1/n} \leq 2A.$$

On the other hand,

(1.6.28) $$W + D_{2n} \supseteq (L_K/L_{D_n})^{1/2} D_n + D_{2n} \supseteq (L_K/L_{D_n})^{1/2} D_n \times (r_{2n}/r_n) D_n.$$

Since $r_{2n}/r_n \geq c_1$ for some absolute constant $c_1 > 0$, we get

(1.6.29) $$\left(\frac{L_K}{L_{D_n}}\right)^{1/4} c_1^{1/2} \leq 2A.$$

Since $L_{D_n} \leq c_2$ for some absolute constant $c_2 > 0$, we have $L_K \leq cA^4$ (where $c = 16c_2/c_1^2$). □

1.7 Connection with the Busemann-Petty problem

Let K be a convex body in $\mathbb{R}^n$. Assume that $0 \in \mathrm{int}(K)$. For every $\theta \in S^{n-1}$ consider the section $K \cap \theta^\perp$ of K, and the normalization of $S_1(\cdot)$

(1.7.1)
$$S_1(K \cap \theta^\perp) = \frac{1}{|K \cap \theta^\perp|^n} \int_{K\cap\theta^\perp} \cdots \int_{K\cap\theta^\perp} |\mathrm{co}(0, x_1, \ldots, x_{n-1})| dx_{n-1} \ldots dx_1.$$

Busemann proved a formula which connects the volume of K with the areas of the $(n-1)$-dimensional sections $K \cap \theta^\perp$, $\theta \in S^{n-1}$:

Theorem 1.7.1. *If K is a convex body in $\mathbb{R}^n$ with $0 \in \mathrm{int}(K)$, then*

(1.7.2) $$|K|^{n-1} = \frac{n!\,\omega_n}{2} \int_{S^{n-1}} |K \cap \theta^\perp|^n S_1(K \cap \theta^\perp) \sigma(d\theta).$$

Proof. Consider $\mathbb{R}^{n(n-1)} = \mathbb{R}_1^n \times \cdots \times \mathbb{R}_{n-1}^n$, and set $\tilde{K} = K \times \cdots \times K \subset \mathbb{R}^{n(n-1)}$. We will write the coordinates of any $x_i \in \mathbb{R}_i^n$ in the form $x_i = (x_i^1, \ldots, x_i^n)$. Then,

(1.7.3) $$|K|^{n-1} = \int_K \cdots \int_K dx_{n-1} \ldots dx_1 = \int_{\tilde{K}} dx_{n-1}^1 \ldots dx_{n-1}^n \ldots dx_1^1 \ldots dx_1^n.$$

Given $x_1, \ldots, x_{n-1}$, let $a_1, \ldots, a_{n-1}$ be the solution of the linear system

(1.7.4) $$x_i^n = \sum_{j=1}^{n-1} a_j x_i^j\ , \quad i = 1, \ldots, n-1,$$

and consider the new coordinates $x_1^1, \ldots, x_1^{n-1}, a_1, \ldots, x_{n-1}^1, \ldots, x_{n-1}^{n-1}, a_{n-1}$ on $\mathbb{R}^{n(n-1)}$. The Jacobian of this change of variables is

(1.7.5) $$J = \det[(x_i^j)_{i,j=1,\ldots,n-1}].$$

The set where $J=0$ has measure 0 and outside this set the transformation is well-defined.

If $b=(1+\sum a_j^2)^{-1/2}$, then the unit vector θ normal to the hyperplane $x^n = a_1x^1+\cdots+a_{n-1}x^{n-1}$ is described either by

$$\theta^j = ba_j \ (j=1,\dots,n-1) \ \text{ and } \ \theta^n=-b, \tag{1.7.6}$$

or by

$$\theta^j = -ba_j \ (j=1,\dots,n-1) \ \text{ and } \ \theta^n=b. \tag{1.7.7}$$

If ϕ is the angle formed by θ and the x^n-axis, then $b=|\cos\phi|$, and hence, $d\theta=\frac{1}{b}d\theta^1\dots d\theta^{n-1}$. Again, we may ignore those hyperplanes that are parallel to the x^n-axis. We can then check that

$$\det[(\partial\theta^j/\partial a_i)_{i,j=1,\dots,n-1}] = b^{n+1}, \tag{1.7.8}$$

which gives

$$d\theta = \frac{1}{b}d\theta^1\dots d\theta^{n-1} = b^n da_1\dots da_{n-1} = |\cos^n\phi| da_1\dots da_{n-1}. \tag{1.7.9}$$

Note also that for the hyperplane $x_j^n = a_1x_j^1+\cdots+a_{n-1}x_j^{n-1}$ we have

$$dx_j = |\sec\phi| dx_j^1\dots dx_j^{n-1}. \tag{1.7.10}$$

Taking into account all the above, we write

$$\begin{aligned} |K|^{n-1} &= \int_{J^{-1}\tilde{K}} |J| dx_1^1\dots dx_1^{n-1}\dots dx_{n-1}^1,\dots, dx_{n-1}^{n-1} da_1\dots da_{n-1} \\ &= \frac{n\omega_n}{2}\int_{S^{n-1}}\int_{K\cap\theta^\perp}\cdots\int_{K\cap\theta^\perp} |J\sec\phi| dx_1\dots dx_{n-1}\sigma(d\theta), \end{aligned}$$

where $x_1,\dots,x_{n-1}$ are considered as points of $\theta^\perp$. Observe that the projection of the simplex $\mathrm{co}(0,x_1,\dots,x_{n-1})$ onto $x^n=0$ has volume $|J|/(n-1)!$. Since all these points belong to $\theta^\perp$, we get

$$|\mathrm{co}(0,x_1,\dots,x_{n-1})||\cos\phi| = \frac{|J|}{(n-1)!}. \tag{1.7.11}$$

It follows that

$$\begin{aligned} |K|^{n-1} &= \frac{n!\,\omega_n}{2}\int_{S^{n-1}}\int_{K\cap\theta^\perp}\cdots\int_{K\cap\theta^\perp} |\mathrm{co}(0,x_1,\dots,x_{n-1})| dx_{n-1}\dots dx_1\sigma(d\theta) \\ &= \frac{n!\,\omega_n}{2}\int_{S^{n-1}} |K\cap\theta^\perp|^n S_1(K\cap\theta^\perp)\sigma(d\theta). \end{aligned}$$

$\square$

If we assume that K is symmetric, then the results of §1.3 show that

$$S_1(K\cap\theta^\perp) \simeq S_2(K\cap\theta^\perp) = L^{n-1}_{K\cap\theta^\perp}/\sqrt{(n-1)!}. \tag{1.7.12}$$

Therefore, Theorem 1.7.1 has the following immediate consequences.

Corollary 1.7.2. *Let K be a symmetric convex body in $\mathbb{R}^n$. Then,*

$$|K|^{n-1} \leq c^n \int_{S^{n-1}} L^{n-1}_{K\cap\theta^\perp} |K\cap\theta^\perp|^n \sigma(d\theta), \tag{1.7.13}$$

where $c > 0$ is an absolute constant. □

Corollary 1.7.3. *Let K be a symmetric convex body in $\mathbb{R}^n$. Then,*

$$|K|^{\frac{n-1}{n}} \geq c_1 \left(\int_{S^{n-1}} |K\cap\theta^\perp|^n \sigma(d\theta)\right)^{1/n}, \tag{1.7.14}$$

where $c_1 > 0$ is an absolute constant.

These last results bring us to the Busemann-Petty problem which was originally formulated as follows:

Assume that K_1 and K_2 are symmetric convex bodies in $\mathbb{R}^n$ and satisfy

$$|K_1\cap\theta^\perp| \leq |K_2\cap\theta^\perp| \tag{1.7.15}$$

for all $\theta \in S^{n-1}$. Does it follow that $|K_1| \leq |K_2|$?

The answer is positive if $n \leq 4$ and negative for all higher dimensions. What remains open is the asymptotic version of the problem.

Conjecture 4 (Busemann-Petty problem): There exists an absolute constant $c > 0$ such that if K_1 and K_2 are symmetric convex bodies in $\mathbb{R}^n$ and $|K_1\cap\theta^\perp| \leq |K_2\cap\theta^\perp|$ for all $\theta \in S^{n-1}$, then $|K_1| \leq c|K_2|$.

The results of this Section show that this conjecture is equivalent to the boundedness of the isotropic constant, at least if we restrict ourselves to the class of symmetric convex bodies. Let us first assume that there is a constant $C > 0$ such that $L_W \leq C$ for every symmetric convex body W. If K_1 and K_2 satisfy

$$|K_1\cap\theta^\perp| \leq |K_2\cap\theta^\perp| \tag{1.7.16}$$

for all $\theta \in S^{n-1}$, then Corollaries 1.7.1 and 1.7.2 show that

$$\begin{aligned} |K_1|^{n-1} &\leq c^n C^{n-1} \int_{S^{n-1}} |K_1\cap\theta^\perp|^n \sigma(d\theta) \\ &\leq c^n C^{n-1} \int_{S^{n-1}} |K_2\cap\theta^\perp|^n \sigma(d\theta) \\ &\leq (c/c_1)^n C^{n-1} |K_2|^{n-1}, \end{aligned}$$

which gives

$$|K_1| \leq c_3 |K_2| \tag{1.7.17}$$

for an absolute constant $c_3 > 0$. Conversely, let us assume that Conjecture 4 is correct and let K be an isotropic symmetric convex body in $\mathbb{R}^n$. Let $\theta_0 \in S^{n-1}$ be such that

$$|K\cap\theta_0^\perp| = \max_{\theta\in S^{n-1}} |K\cap\theta^\perp|. \tag{1.7.18}$$

We choose $r>0$ so that $\omega_{n-1}r^{n-1}=|K\cap\theta_0^{\perp}|$. Then,

$$|K\cap\theta^{\perp}|\leq r^{n-1}\omega_{n-1}=|(rB_2^n)\cap\theta^{\perp}| \tag{1.7.19}$$

for all $\theta\in S^{n-1}$, therefore

$$|K|^{n-1}\leq c^{n-1}|rB_2^n|^{n-1}=\frac{c^{n-1}\omega_n^{n-1}}{\omega_{n-1}^n}|K\cap\theta_0^{\perp}|^n\leq c_1^n|K\cap\theta_0^{\perp}|^n, \tag{1.7.20}$$

for some absolute constant $c_1>0$. Since $|K|=1$, we see that

$$|K\cap\theta_0^{\perp}|\geq 1/c_1. \tag{1.7.21}$$

Since K is isotropic, we have $|K\cap\theta^{\perp}|\simeq 1/L_K$ for every $\theta\in S^{n-1}$. It follows that $L_K\leq C$ for some absolute constant $C>0$.

Notes and References

A basic reference on the isotropic position is the paper of Milman and Pajor [43]. It contains more information on the history of this topic (see also [3], [27] and [9]), on basic properties of isotropic convex bodies and on the connection of the isotropic constant conjecture with other questions (slicing problem, Sylvester's problem, Busemann-Petty problem) The fact that if L_K is bounded then K is in M-position was observed in [42], [43]. The argument in these notes is from [28] (see also [36]) and works for not necessarily symmetric convex bodies. Theorem 1.6.5 is recent (see [14]).

Chapter 2

An upper bound for the isotropic constant

2.1 Khintchine type inequalities for polynomials

Let K be a convex body in $\mathbb{R}^n$ with volume $|K| = 1$. For every $p > 0$ and $\theta \in S^{n-1}$ we define

$$I_p(K,\theta) = \left(\int_K |\langle x,\theta\rangle|^p dx\right)^{1/p}. \tag{2.1.1}$$

Proposition 2.1.1. *There exists an absolute constant $c > 0$ with the following property: If K is a convex body in $\mathbb{R}^n$ with volume 1, then for all $\theta \in \mathbb{R}^n$ and all $p > 1$ we have*

$$I_p(K,\theta) \leq cpI_1(K,\theta). \tag{2.1.2}$$

The proof is based on Borell's lemma.

Lemma 2.1.2. *Let K be a convex body of volume 1 in $\mathbb{R}^n$. Assume that A is a symmetric convex subset of $\mathbb{R}^n$ such that $|K \cap A| = \theta > 1/2$. Then, for every $t > 1$ we have*

$$|(\mathbb{R}^n \setminus tA) \cap K| \leq \theta\left(\frac{1-\theta}{\theta}\right)^{\frac{t+1}{2}}. \tag{2.1.3}$$

Proof. Observe that

$$\mathbb{R}^n \setminus A \supseteq \frac{2}{t+1}(\mathbb{R}^n \setminus tA) + \frac{t-1}{t+1}A,$$

therefore

$$(\mathbb{R}^n \setminus A) \cap K \supseteq \frac{2}{t+1}\big((\mathbb{R}^n \setminus tA) \cap K\big) + \frac{t-1}{t+1}\big(A \cap K\big).$$

Then, apply the Brunn-Minkowski inequality. $\square$

Proof of Proposition 2.1.1: We set $I := I_1(K,\theta)$ and

$$A = \{x \in \mathbb{R}^n : |\langle x,\theta\rangle| \leq 3I\}. \tag{2.1.4}$$

Then, A is a symmetric convex subset of $\mathbb{R}^n$, and Markov's inequality shows that

$$|A \cap K| \geq 2/3. \tag{2.1.5}$$

We observe that $\{x \in K : |\langle x,\theta\rangle| \geq t\} = K \cap \big(\mathbb{R}^n \setminus (t/3I)A\big)$, and write

$$\begin{aligned}\int_K |\langle x,\theta\rangle|^p dx &= \int_0^{3I} pt^{p-1}|K \cap (\mathbb{R}^n \setminus (t/3I)A)|dt \\ &\quad + \int_{3I}^{\infty} pt^{p-1}|K \cap (\mathbb{R}^n \setminus (t/3I)A)|dt.\end{aligned}$$

The first integral is bounded by

$$\int_0^{3I} pt^{p-1}dt = (3I)^p, \tag{2.1.6}$$

while for the second one, making the change of variables $t = 3Is$ and using Lemma 2.1.2 we get

$$\begin{aligned}\int_{3I}^{\infty} pt^{p-1}|K \cap (\mathbb{R}^n \setminus (t/3I)A)|dt &= (3I)^p \int_1^{\infty} ps^{p-1}|K \cap (\mathbb{R}^n \setminus sA)|ds \\ &\leq (3I)^p \int_1^{\infty} ps^{p-1}2^{-s/2}ds.\end{aligned}$$

Combining the above and estimating the last integral, we see that

$$\int_K |\langle x,\theta\rangle|^p dx \leq (3I)^p\big[1 + (c_1p)^p\big] \tag{2.1.7}$$

for some absolute constant $c_1 > 0$. □

Remark 1: The same argument works if we replace the function $x \mapsto |\langle x,\theta\rangle|$ by any seminorm.

Proposition 2.1.1 can be equivalently stated as follows.

Proposition 2.1.3. *Let K be a convex body of volume 1 in $\mathbb{R}^n$. For every $\theta \in S^{n-1}$ we have*

$$\int_K \exp(|\langle x,\theta\rangle|/CI_1(K,\theta))dx \leq 2, \tag{2.1.8}$$

where $C > 0$ is an absolute constant. □

Corollary 2.1.4. *Let K be a convex body of volume 1 in $\mathbb{R}^n$. For every $\theta \in S^{n-1}$ and every $s > 0$,*

$$\mathrm{Prob}\,(x \in K : |\langle x,\theta\rangle| \geq CI_1(K,\theta)s) \leq 2e^{-s}, \tag{2.1.9}$$

where $C > 0$ is the constant in Proposition 2.1.3. □

More generally, write $\mathcal{P}_{d,n}$ for the space of polynomials $f : \mathbb{R}^n \to \mathbb{R}$ of degree less than or equal to d. Bourgain [10] (see also Bobkov [5]) proved that for every $1 \leq q, r \leq \infty$ there exists a constant $c_{q,r,d} > 0$ depending only on q, r and d, such that $\|f\|_{L^q(K)} \leq c_{q,r,d}\|f\|_{L^r(K)}$ for every $f \in \mathcal{P}_{d,n}$ and every convex body K of volume 1 in $\mathbb{R}^n$. Carbery and Wright (see [19]) have recently established the best possible dependence of the constant $c_{q,r,d}$ on q, r and d. In order to formulate their result, for every $f \in \mathcal{P}_{d,n}$ we define $f^{\#}(x) = |f(x)|^{1/d}$.

Theorem 2.1.5. *There exists an absolute constant $C > 0$ such that for every convex body K of volume 1 in $\mathbb{R}^n$ and for every $f \in \mathcal{P}_{d,n}$ the following hold true:*
(i) *If $n \leq r \leq q \leq \infty$, then*

$$\|f^{\#}\|_q \leq C\|f^{\#}\|_r. \tag{2.1.10}$$

(ii) *If $1 \leq r \leq n \leq q \leq \infty$, then*

$$\|f^{\#}\|_q \leq C\frac{n}{r}\|f^{\#}\|_r. \tag{2.1.11}$$

(iii) *If $1 \leq r \leq q \leq n$, then*

$$\|f^{\#}\|_q \leq C\frac{q}{r}\|f^{\#}\|_r. \tag{2.1.12}$$

In particular,

$$\|f^{\#}\|_q \leq C\frac{q}{r}\|f^{\#}\|_r \tag{2.1.13}$$

whenever $1 \leq r \leq q < \infty$. □

These inequalities allow us to obtain tail estimates for polynomials $f \in \mathcal{P}_{d,n}$.

Lemma 2.1.6. *Let K be a convex body of volume 1 in $\mathbb{R}^n$ and let $f \in \mathcal{P}_{d,n}$. Then,*

$$\mathrm{Prob}\left(x \in K : f^{\#}(x) \geq 3C\|f^{\#}\|_q \cdot s\right) \leq e^{-qs} \tag{2.1.14}$$

for all $q \geq 1$ and $s \geq 1$, where C is the constant in Theorem 2.1.5.

Proof. From Theorem 2.1.5 we have

$$\int_K f^{\#}(x)^{qp}dx \leq (Cp)^{qp}\|f^{\#}\|_q^{qp}. \tag{2.1.15}$$

for every $p \geq 1$. Markov's inequality gives

$$(3C\|f^{\#}\|_q s)^{qp}\mathrm{Prob}\left(x \in K : f^{\#}(x) \geq 3C\|f^{\#}\|_q \cdot s\right) \leq (Cp)^{qp}\|f^{\#}\|_q^{qp}, \tag{2.1.16}$$

which shows that

$$\mathrm{Prob}\left(x \in K : f^{\#}(x) \geq 3C\|f^{\#}\|_q \cdot s\right) \leq \left(\frac{p}{3s}\right)^{qp}. \tag{2.1.17}$$

If we choose $p = 3s/e \geq 1$, we get

$$\mathrm{Prob}\left(x \in K : f^{\#}(x) \geq 3C\|f^{\#}\|_q \cdot s\right) \leq e^{-3qs/e} \leq e^{-qs} \tag{2.1.18}$$

which is the assertion of the Lemma. □

For every $p > 0$ we define

$$I_p(K) = \left(\int_K |x|^p dx\right)^{1/p}. \tag{2.1.19}$$

Applying Lemma 2.1.6 to linear functionals $f(x) = \langle x, \theta\rangle$ and to the polynomial $f(x) = |x|^2$, we have the following immediate consequence.

Proposition 2.1.7. *Let K be a convex body of volume 1 in $\mathbb{R}^n$. If $q \geq 1$, then*

$$\mathrm{Prob}\,(x \in K : |\langle x, \theta\rangle| \geq 3CI_q(K,\theta)s) \leq e^{-qs} \tag{2.1.20}$$

for all $\theta \in S^{n-1}$ and $s \geq 1$, and

$$\mathrm{Prob}\,(x \in K : |x| \geq 3CI_q(K)s) \leq e^{-qs} \tag{2.1.21}$$

for all $s \geq 1$, where C is the constant in Theorem 2.1.5. □

2.2 Ψ_α-estimates

Let K be a convex body of volume 1 in $\mathbb{R}^n$. If $f : K \to \mathbb{R}$ is a bounded measurable function and if $\alpha \geq 1$, the Orlicz norm $\|f\|_{\psi_\alpha}$ of f is defined by

$$\|f\|_{\psi_\alpha} = \inf\left\{t > 0 : \int_K \exp\left((|f(x)|/t)^\alpha\right) dx \leq 2\right\}. \tag{2.2.1}$$

An equivalent description is the following.

Lemma 2.2.1. *Let K be a convex body of volume 1 in $\mathbb{R}^n$, $f : K \to \mathbb{R}$ be a bounded measurable function and $\alpha \geq 1$. Then,*

$$\|f\|_{\psi_\alpha} \simeq \sup\left\{\frac{\|f\|_p}{p^{1/\alpha}} : p \geq \alpha\right\}. \tag{2.2.2}$$

Let $\alpha \geq 1$ and $y \neq 0$ in $\mathbb{R}^n$. We say that K satisfies a ψ_α-estimate with constant b_α in the direction of y if

$$\|\langle \cdot, y\rangle\|_{\psi_\alpha} \leq b_\alpha \|\langle \cdot, y\rangle\|_1. \tag{2.2.3}$$

Proposition 2.1.3 is then taking the following form.

Proposition 2.2.2. *There exists an absolute constant $C > 0$ such that for every convex body K of volume 1 in $\mathbb{R}^n$ and every $y \neq 0$,*

$$\|\langle \cdot, y\rangle\|_{\psi_1} \leq C\|\langle \cdot, y\rangle\|_1. \tag{2.2.4}$$

Next, we concentrate on isotropic convex bodies and give some first results on the ψ_α-behavior of linear functionals and of the Euclidean norm. First, note that in the case of an isotropic convex body Proposition 2.2.2 states that

$$\|\langle \cdot, \theta\rangle\|_{\psi_1} \leq CL_K \tag{2.2.5}$$

for all $\theta \in S^{n-1}$. We say that "every isotropic convex body is a ψ_1-body with constant C".

Proposition 2.2.3. *Let $\alpha \geq 1$. Every isotropic convex body K is a ψ_α-body with constant*

$$b_\alpha \leq c\big(R(K)/L_K\big)^{1-\frac{1}{\alpha}}. \tag{2.2.6}$$

Proof. For every $\theta \in S^{n-1}$ and every $p \geq \alpha$ we have

$$\begin{aligned} I_p(K,\theta) &= \left(\int_K |\langle x,\theta\rangle|^{\frac{p}{\alpha}} |\langle x,\theta\rangle|^{p\left(1-\frac{1}{\alpha}\right)} dx\right)^{1/p} \\ &\leq R(K)^{1-\frac{1}{\alpha}} I_{\frac{p}{\alpha}}(K,\theta)^{\frac{1}{\alpha}} \\ &\leq R(K)^{1-\frac{1}{\alpha}} \left(c\frac{p}{\alpha} I_1(K,\theta)\right)^{\frac{1}{\alpha}} \\ &\leq R(K)^{1-\frac{1}{\alpha}} \left(c\frac{p}{\alpha} L_K\right)^{\frac{1}{\alpha}} \\ &\leq c_1 R(K)^{1-\frac{1}{\alpha}} L_K^{\frac{1}{\alpha}} p^{\frac{1}{\alpha}}. \end{aligned}$$

The result now follows from Lemma 2.2.1. □

The next Theorem (see [1]) asserts that the Euclidean norm satisfies a ψ_2-estimate on isotropic convex bodies.

Theorem 2.2.4. *Let K be an isotropic convex body in $\mathbb{R}^n$. If $f(x) = |x|$, then*

$$\|f\|_{\psi_2} \leq c\sqrt{n} L_K, \tag{2.2.7}$$

where $c > 0$ is an absolute constant.

We will use the following Lemma.

Lemma 2.2.5. *Let K be a convex body in $\mathbb{R}^n$ with volume 1 and center of mass at the origin. For every $q \geq 1$,*

$$\left(\int_{S^{n-1}} \int_K |\langle x,\theta\rangle|^q dx\sigma(d\theta)\right)^{1/q} \simeq \sqrt{\frac{q}{q+n}} I_q(K). \tag{2.2.8}$$

Proof. For every $q \geq 1$ and $x \in \mathbb{R}^n$, we check that

$$\left(\int_{S^{n-1}} |\langle x,\theta\rangle|^q \sigma(d\theta)\right)^{1/q} \simeq \frac{\sqrt{q}}{\sqrt{q+n}} |x|. \tag{2.2.9}$$

A simple application of Fubini's theorem gives the result. □

Proof of Theorem 2.2.4: It suffices to prove that for every $q > 1$

$$\left(\int_K |x|^q dx\right)^{1/q} \leq c_1 \sqrt{q}\sqrt{n} L_K \tag{2.2.10}$$

for some absolute constant $c_1 > 0$. From Proposition 2.1.1 we know that for every $\theta \in S^{n-1}$

$$\int_K |\langle x,\theta\rangle|^q dx \leq c_2^q q^q L_K^q. \tag{2.2.11}$$

Integrating on the sphere we get

$$\int_{S^{n-1}}\int_K |\langle x,\theta\rangle|^q dx\sigma(d\theta) \leq c_2^q q^q L_K^q. \tag{2.2.12}$$

Taking into account Lemma 2.2.5, we see that

$$\left(\int_K |x|^q dx\right)^{1/q} \leq c_3 q\sqrt{\frac{n+q}{q}}L_K \leq c_4\sqrt{q}\sqrt{n}L_K, \tag{2.2.13}$$

provided that $q \leq n$. On the other hand, if $q > n$, using the fact that $R(K) \leq (n+1)L_K$, we get

$$\left(\int_K |x|^q dx\right)^{1/q} \leq cnL_K \leq c\sqrt{q}\sqrt{n}L_K. \tag{2.2.14}$$

Combining the above we see that there is an absolute constant $c_1 > 0$ such that

$$\left(\int_K |x|^q dx\right)^{1/q} \leq c_1\sqrt{q}\sqrt{n}L_K \tag{2.2.15}$$

for all $q > 1$. □

Corollary 2.2.6. *There exists an absolute constant $c > 0$ such that: if K is an isotropic convex body in $\mathbb{R}^n$ then*

$$\mathrm{Prob}(x \in K : |x| \geq c\sqrt{n}L_K s) \leq 2\exp(-s^2) \tag{2.2.16}$$

for every $s > 0$. □

2.3 Reduction to small diameter

Proposition 2.3.1. *Let K be an isotropic convex body in $\mathbb{R}^n$. Then, there exists an isotropic convex body Q in $\mathbb{R}^n$ with $L_Q \simeq L_K$ and $R(Q) \leq c\sqrt{n}L_Q$, where $c > 0$ is an absolute constant.*

For the proof we need a simple observation about the stability of the isotropic position

Lemma 2.3.2. *Let K be a convex body of volume 1 in $\mathbb{R}^n$. Assume that for some constant $L > 0$ and some $a, b > 0$ we have*

$$a^{-2}L^2 \leq \int_K \langle x,\theta\rangle^2 dx \leq b^2L^2 \tag{2.3.1}$$

for all $\theta \in S^{n-1}$. Then,

$$a^{-1}L \leq L_K \leq bL. \tag{2.3.2}$$

Proof. Consider the Binet ellipsoid $E_B(K)$ which is defined by

$$\|\theta\|_{E_B(K)}^2 = \int_K \langle x, \theta\rangle^2 dx. \tag{2.3.3}$$

Our assumption implies that

$$\frac{1}{bL}\omega_n^{1/n} \leq |E_B(K)|^{1/n} \leq \frac{a}{L}\omega_n^{1/n}. \tag{2.3.4}$$

On the other hand,

$$|E_B(K)|^{1/n} = L_K^{-1}\omega_n^{1/n} \tag{2.3.5}$$

from Lemma 1.4.1. Combining the above we conclude the proof. □

Proof of Proposition 2.3.1: Since K is isotropic, we have

$$\int_K |x|^2 dx = nL_K^2. \tag{2.3.6}$$

For every $r > 0$ we define

$$K_r = \{x \in K : |x| \leq r\sqrt{n}L_K\}. \tag{2.3.7}$$

Markov's inequality shows that $|K_r| \geq 1 - r^{-2}$. Then, for every $\theta \in S^{n-1}$,

$$\begin{aligned} \int_{K_r} \langle x, \theta\rangle^2 dx &= \int_K \langle x, \theta\rangle^2 dx - \int_{K\setminus K_r} \langle x, \theta\rangle^2 dx \\ &\geq L_K^2 - |K \setminus K_r|^{1/2} \left(\int_K \langle x, \theta\rangle^4 dx\right)^{1/2} \\ &\geq L_K^2 - r^{-1}(4c)^2 L_K^2, \end{aligned}$$

where $c > 0$ is the constant in Proposition 2.1.1. If we choose $r = 32c^2$, we have

$$\frac{L_K^2}{2} \leq \int_{K_r} \langle x, \theta\rangle^2 dx \leq L_K^2 \tag{2.3.8}$$

for all $\theta \in S^{n-1}$. We find $a \geq 1$ with $a^n \leq c_1 := 1/(1-r^{-2})$, such that $W = aK_r$ has volume $|W| = 1$. Then,

$$\frac{L_K^2}{2}|y|^2 \leq \int_W \langle x, y\rangle^2 dx \leq c_1^2 L_K^2 |y|^2 \tag{2.3.9}$$

for all $y \in \mathbb{R}^n$, and Lemma 2.3.2 shows that $L_W \simeq L_K$.

Let $T \in SL(n)$ be such that $Q = T(W)$ is isotropic. From (2.3.9), for every $\theta \in S^{n-1}$ we have

$$L_W^2 = \int_Q \langle x, \theta\rangle^2 dx = \int_W \langle x, T^*\theta\rangle^2 dx \simeq |T^*\theta|^2 L_K^2. \tag{2.3.10}$$

Since $L_Q = L_W \simeq L_K$, we obtain

$$c_2 \leq |T^*\theta| \leq c_3 \tag{2.3.11}$$

for every $\theta \in S^{n-1}$. Therefore,

$$R(Q) \leq \|T : \ell_2^n \to \ell_2^n\| R(W) \leq c_3 ar\sqrt{n}L_K \leq c_5\sqrt{n}L_K \leq c\sqrt{n}L_Q \tag{2.3.12}$$

where $c > 0$ is an absolute constant. □

2.4 Upper bound for the isotropic constant

Bourgain's argument (see [10]) may now be described for isotropic convex bodies with small diameter.

Theorem 2.4.1. *Let K be an isotropic convex body in $\mathbb{R}^n$. Then,*

$$L_K \leq c\sqrt[4]{n}\log n, \tag{2.4.1}$$

where $c > 0$ is an absolute constant.

Proof. From Proposition 2.3.1 we can find an isotropic convex body Q in $\mathbb{R}^n$ with $L_Q \simeq L_K$ and $R(Q) \leq c\sqrt{n}L_Q$. Recall that

$$\int_Q \langle x, Tx\rangle dx = (\mathrm{tr}T)\cdot L_Q^2 \tag{2.4.2}$$

for every $T \in L(\mathbb{R}^n)$. The extremal property (Theorem 1.1.2) of the isotropic position, combined with the arithmetic-geometric means inequality, shows that for every symmetric positive definite $T \in SL(n)$ we have

$$nL_Q^2 \leq (\mathrm{tr}T)\cdot L_Q^2 = \int_Q \langle x, Tx\rangle dx \leq \int_Q \max_{y\in TQ}\langle y, x\rangle dx. \tag{2.4.3}$$

Since $R(Q) \leq c\sqrt{n}L_Q$, Proposition 2.2.3 shows that

$$\left\| \frac{\langle \cdot, y\rangle}{c_1\sqrt[4]{n}L_Q|y|} \right\|_{\psi_2} \leq 1, \tag{2.4.4}$$

for every $y \neq 0$, where $c_1 > 0$ is an absolute constant. It follows that

$$\mathrm{Prob}(\{x \in Q : \langle x, y\rangle \geq c_1\sqrt[4]{n}L_Q t\}) \leq 2\exp(-t^2/|y|^2) \tag{2.4.5}$$

for every $y \neq 0$ and every $t > 0$.
We define a process $\mathcal{X} = (X_y)_{y\in TQ}$, where $X_y : Q \to \mathbb{R}$ is given by

$$X_y(x) = \frac{\langle x, y\rangle}{c_1\sqrt[4]{n}L_Q}. \tag{2.4.6}$$

Then, for every $y \neq z \in TQ$ and every $t > 0$, we have

$$\begin{aligned} \mathrm{Prob}(|X_y - X_z| \geq t) &= \mathrm{Prob}(\{x \in Q : |\langle y - z, x\rangle| \geq c_1\sqrt[4]{n}L_Q t\}) \\ &\leq 2\exp\left(-t^2/|y-z|^2\right), \end{aligned}$$

that is, $\mathcal{X}$ is subgaussian with respect to the Euclidean metric on TQ.

Let $g_1, \ldots, g_n$ be independent standard Gaussian random variables on some probability space Ω, and consider the Gaussian process $\mathcal{Z} = (Z_y)_{y\in TQ}$ with

$$Z_y(\omega) = \langle G(\omega), y\rangle, \tag{2.4.7}$$

where $G = (g_1, \ldots, g_n)$. Then, $\mathcal{X}$ and $\mathcal{Z}$ satisfy the assumptions of Talagrand's comparison theorem (see Appendix A) . Therefore,

$$\mathbb{E}\sup_{y\in TQ} X_y \leq C\cdot\mathbb{E}\sup_{y\in TQ} Z_y, \tag{2.4.8}$$

where $C > 0$ is an absolute constant. A simple computation (see Appendix A) shows that

$$\mathbb{E} \sup_{y \in TQ} Z_y \simeq \sqrt{n} w(TQ). \tag{2.4.9}$$

Therefore,

$$nL_Q^2 \leq c_1 \sqrt[4]{n} L_Q \cdot \mathbb{E} \sup_{y \in TQ} X_y \leq c_1 C \sqrt[4]{n} L_Q \cdot \sqrt{n} w(TQ). \tag{2.4.10}$$

In other words,

$$L_Q \leq c_2 w(TQ) / \sqrt[4]{n}, \tag{2.4.11}$$

where $c_2 > 0$ is an absolute constant. Well-known results of Lewis [34] , Figiel and Tomczak-Jaegermann [23], combined with Pisier's inequality [50] on the norm of the Rademacher projection, show that there exists a symmetric positive definite $T \in SL(n)$ for which $w(TQ) = O(\sqrt{n} \log n)$. This completes the proof. $\square$

The argument above makes use of the simple ψ_2-estimate from Proposition 2.2.3. It is natural to ask if linear functionals on an isotropic convex body with small diameter exhibit better ψ_2-behavior. One may ask for the worst direction $\theta \in S^{n-1}$ or, at least, for probability estimates on $\|\langle \cdot, \theta \rangle\|_{\psi_2}$ with respect to σ. We study these two questions in the next Chapter.

2.5 Alternative proof

We sketch one more proof of the bound $L_K = O(\sqrt[4]{n} \log n)$, which is based on the ψ_1-behavior of linear functionals on convex bodies. This argument is more elementary, since it involves simpler entropy considerations.

1. Expectation of the maximum of linear functionals on convex bodies

Let K be an isotropic convex body in $\mathbb{R}^n$, which satisfies the ψ_α-estimate

$$\|\langle \cdot, \theta \rangle\|_{\psi_\alpha} \leq B \|\langle \cdot, \theta \rangle\|_1 \leq B L_K \tag{2.5.1}$$

for all $\theta \in S^{n-1}$, where $\alpha \in [1, 2]$ and $B > 0$.

Proposition 2.5.1. *If $\theta_1, \ldots, \theta_N \in S^{n-1}$, then*

$$\int_K \max_{1 \leq i \leq N} |\langle x, \theta_i \rangle| \, dx \leq C B L_K (\log N)^{1/\alpha}, \tag{2.5.2}$$

where $C > 0$ is an absolute constant.

Proof. For every $t > 0$,

$$\begin{aligned} \mathrm{Prob}\left(x \in K : \max_{1 \leq i \leq N} |\langle x, \theta_i \rangle| \geq t \right) &\leq \sum_{i=1}^N \mathrm{Prob}\left(x \in K : |\langle x, \theta_i \rangle| \geq t \right) \\ &\leq 2N \exp\left(-(t/BL_K)^\alpha \right). \end{aligned}$$

Then, given $A > 0$ we may write

$$\begin{aligned}\int_K \max_{1\le i\le N} |\langle x,\theta_i\rangle| dx &= \int_0^\infty \mathrm{Prob}\left(x\in K : \max_{1\le i\le N} |\langle x,\theta_i\rangle| \ge t\right) dt \\ &\le A + \int_A^\infty \mathrm{Prob}\left(x\in K : \max_{1\le i\le N} |\langle x,\theta_i\rangle| \ge t\right) dt \\ &\le A + 2N \int_A^\infty \exp\left(-(t/BL_K)^\alpha\right) dt.\end{aligned}$$

Choosing $A = 4BL_K(\log N)^{1/\alpha}$ we get

$$\begin{aligned}\int_A^\infty \exp\left(-(t/BL_K)^\alpha\right) dt &= 4BL_K(\log N)^{1/\alpha} \int_1^\infty \exp(-4^\alpha s\log N) ds \\ &\le 4BL_K(\log N)^{1/\alpha} \exp(-2\log N) \int_1^\infty e^{-s} ds \\ &\le 4BL_K(\log N)^{1/\alpha} N^{-2},\end{aligned}$$

where we have used the fact that

$$\exp(-4^\alpha s\log N) \le \exp(-2\log N)\cdot e^{-s} \tag{2.5.3}$$

is valid for all $s \ge 1$. It follows that

$$\int_K \max_{1\le i\le N} |\langle x,\theta_i\rangle| dx \le CBL_K(\log N)^{1/\alpha}, \tag{2.5.4}$$

with $C = 12$. □

2. Dudley-Fernique decomposition

Let K be a convex body in $\mathbb{R}^n$. We assume that $0 \in K$ and write R for the circumradius of K. For every $j \in \mathbb{N}$ we may find a subset N_j of K such that

$$|N_j| = N(K, (R/2^j)B_2^n), \tag{2.5.5}$$

and

$$K \subseteq \bigcup_{y\in N_j} (y + (R/2^j)B_2^n). \tag{2.5.6}$$

Sudakov's inequality (see Appendix A) shows that

$$\log |N_j| \le cn \left(\frac{2^j w(K)}{R}\right)^2. \tag{2.5.7}$$

We define $N_0 = \{0\}$ and

$$W_j = N_j - N_{j-1} = \{y - y' \mid y\in N_j, y'\in N_{j-1}\} \tag{2.5.8}$$

for every $j \ge 1$.

Lemma 2.5.2. *For every $x \in K$ and any $m \in \mathbb{N}$ we can find $z_j \in W_j \cap (3R/2^j)B_2^n$, $j = 1, \dots, m$ and $w_m \in (R/2^m)B_2^n$ such that*

$$x = z_1 + \cdots + z_m + w_m. \tag{2.5.9}$$

Proof. Let $x \in K$. By the definition of N_j, we can find $y_j \in N_j$, $j = 1, \dots, m$, such that

$$|x - y_j| \leq \frac{R}{2^j}. \tag{2.5.10}$$

We write

$$x = 0 + (y_1 - 0) + (y_2 - y_1) + \cdots + (y_m - y_{m-1}) + (x - y_m). \tag{2.5.11}$$

We set $y_0 = 0$ and $w_m = x - y_m$, $z_j = y_j - y_{j-1}$ for $j = 1, \dots, m$. Then, $|w_m| = |x - y_m| \leq R/2^m$, and $z_j \in N_j - N_{j-1} = W_j$. Also,

$$|z_j| \leq |x - y_j| + |x - y_{j-1}| \leq \frac{R}{2^j} + \frac{R}{2^{j-1}} = \frac{3R}{2^j}. \tag{2.5.12}$$

Finally, $x = z_1 + \cdots + z_m + w_m$. $\square$

We set $Z_j = W_j \cap (3R/2^j)B_2^n$. Then, taking into account (2.5.7), we may state the following.

Theorem 2.5.3. *Let K be a convex body in $\mathbb{R}^n$, with $0 \in K$ and circumradius equal to R. There exist $Z_j \subseteq (3R/2^j)B_2^n$, $j \in \mathbb{N}$, such that*

$$\log |Z_j| \leq cn \left(\frac{2^j w(K)}{R} \right)^2, \tag{2.5.13}$$

with the following property: for every $x \in K$ and any $m \in \mathbb{N}$ we can find $z_j \in Z_j$, $j = 1, \dots, m$ and $w_m \in (R/2^m)B_2^n$ such that $x = z_1 + \cdots + z_m + w_m$. $\square$

Theorem 2.5.4. *Let K be an isotropic convex body in $\mathbb{R}^n$ and let $B > 0$. Assume that, for some $1 \leq \alpha < 2$, K satisfies the ψ_α-estimate*

$$\|\langle \cdot, \theta \rangle\|_{\psi_\alpha} \leq B \|\langle \cdot, \theta \rangle\|_1 \leq BL_K \tag{2.5.14}$$

for every $\theta \in S^{n-1}$. Then,

$$L_K \leq CB^{\frac{\alpha}{2}} (2 - \alpha)^{-\frac{\alpha}{2}} n^{\frac{1}{2} - \frac{\alpha}{4}} \log n \tag{2.5.15}$$

where $C > 0$ is an absolute constant.

Proof. There exists a symmetric and positive $T \in SL(n)$ such that

$$w(TK) \leq c\sqrt{n} \log n. \tag{2.5.16}$$

We write

$$nL_K^2 = \int_K |x|^2 dx \leq \frac{\mathrm{tr} T}{n} \int_K |x|^2 = \int_K \langle x, Tx \rangle dx. \tag{2.5.17}$$

Therefore,

$$nL_K^2 \leq \int_K \max_{y\in TK} |\langle y,x\rangle| dx. \tag{2.5.18}$$

We now use Theorem 2.5.3 for TK. If R is the circumradius of TK, for every $j=1,\dots,m$, $m\in\mathbb{N}$ we can find $Z_j\subset (3R/2^j)B_2^n$ such that

$$\log |Z_j| \leq cn\left(\frac{w(TK)2^j}{R}\right)^2, \tag{2.5.19}$$

and every $y\in TK$ is written in the form $y=z_1+\cdots+z_m+w_m$, where $z_j\in Z_j$ and $w_m\in (R/2^m)B_2^n$. This implies that

$$\begin{aligned}\max_{y\in TK}|\langle y,x\rangle| &\leq \sum_{j=1}^m \max_{z\in Z_j}|\langle z,x\rangle| + \max_{w\in (R/2^m)B_2^n}|\langle w,x\rangle|\\ &\leq \sum_{j=1}^m \frac{3R}{2^j}\max_{z\in Z_j}|\langle \overline{z},x\rangle| + \frac{R}{2^m}|x|,\end{aligned}$$

where $\overline{z}$ denotes the unit vector in the direction of z. Noting that $\int_K |x|dx \leq \sqrt{n}L_K$ and using the above, we see that

$$\begin{aligned}nL_K^2 &\leq \sum_{j=1}^m \frac{3R}{2^j}\int_K \max_{z\in Z_j}|\langle \overline{z},x\rangle| dx + \frac{R}{2^m}\int_K |x|dx\\ &\leq \sum_{j=1}^m \frac{3R}{2^j}\int_K \max_{z\in Z_j}|\langle \overline{z},x\rangle| dx + \frac{R}{2^m}\sqrt{n}L_K.\end{aligned}$$

From Proposition 2.5.1 we get

$$nL_K^2 \leq \sum_{j=1}^m \frac{3R}{2^j} c'' n^{\frac{1}{\alpha}} L_K B\left(\frac{w(TK)2^j}{R}\right)^{\frac{2}{\alpha}} + \frac{R}{2^m}\sqrt{n}L_K. \tag{2.5.20}$$

The sum on the right is bounded by

$$\frac{c_1}{(2-\alpha)} L_K B n^{\frac{1}{\alpha}} w(TK)^{\frac{2}{\alpha}} 2^{m(\frac{2}{\alpha}-1)} R^{1-\frac{2}{\alpha}}. \tag{2.5.21}$$

Solving the equation

$$B\frac{n^{\frac{1}{\alpha}} w(TK)^{\frac{2}{\alpha}} 2^{m\left(\frac{2}{\alpha}-1\right)}}{R^{\frac{2}{\alpha}-1}(2-\alpha)} = \frac{R\sqrt{n}}{2^m} \tag{2.5.22}$$

we see that the optimal value of m satisfies the equation

$$\frac{1}{2^m} = \frac{n^{\frac{1}{2}-\frac{\alpha}{4}} w(TK)}{(2-\alpha)^{\frac{\alpha}{2}} R} B^{\frac{\alpha}{2}}. \tag{2.5.23}$$

Going back to (2.5.20), we obtain

$$nL_K^2 \leq c_2 (2-\alpha)^{-\frac{\alpha}{2}} n^{1-\frac{\alpha}{4}} w(TK) L_K B^{\frac{\alpha}{2}}. \tag{2.5.24}$$

Since $w(TK)\leq c_3\sqrt{n}\log n$, we get the result. $\square$

Notes and References

Bourgain's theorem 2.4.1 (see [10]) is the best known general upper bound on the isotropic constant. Reduction to small diameter is implicit in his argument (and explicitly described in [48]). Originally, the Theorem was proved for symmetric convex bodies. An alternative proof, based on the same ideas, was given by Dar (see [20]). A modification of Dar's argument (see [46]) leads to the upper bound $O(\sqrt[4]{n}\log n)$ for every convex body in $\mathbb{R}^n$ (the previously known general estimate was $O(\sqrt{n})$; see e.g. [21]). Bourgain [12] has also proved that $L_K \leq cb\log b$ for ψ_2-bodies with constant b.

Chapter 3

Isotropic convex bodies with small diameter

3.1 L_q-centroid bodies

Let K be a convex body of volume 1 in $\mathbb{R}^n$. For every $q \geq 1$ we define the q-centroid body $Z_q(K)$ of K by

$$(3.1.1) \qquad h_{Z_q(K)}(y) = I_q(K,y) := \left(\int_K |\langle x,y\rangle|^q dx \right)^{1/q}.$$

For simplicity we will sometimes write $H_q(y) := h_{Z_q(K)}(y)$. Since $|K| = 1$, it is easy to check that

$$(3.1.2) \qquad Z_1(K) \subseteq Z_p(K) \subseteq Z_q(K) \subseteq Z_\infty(K)$$

for every $1 \leq p \leq q \leq \infty$, where $Z_\infty(K) = \hat{K} := \mathrm{co}\{K, -K\}$. Another simple observation is that

$$(3.1.3) \qquad Z_q(TK) = T(Z_q(K))$$

for every $T \in SL(n)$ and $q \in [1,\infty]$.

If K has its center of mass at the origin, then $Z_q(K) \simeq \hat{K}$ for all $q \geq n$. This follows from the next Lemma.

Lemma 3.1.1. *Let K be a convex body in $\mathbb{R}^n$ with volume 1 and center of mass at the origin. Then,*

$$(3.1.4) \qquad \int_K |\langle x,\theta\rangle|^q dx \geq \frac{\Gamma(q+1)\Gamma(n)}{2e\Gamma(q+n+1)} \max\left\{h_K^q(\theta), h_K^q(-\theta)\right\}$$

for every $\theta \in S^{n-1}$. In particular,

$$(3.1.5) \qquad Z_q(K) \supseteq c\hat{K}$$

for all $q \geq n$, where $c > 0$ is an absolute constant.

Proof. Consider the function $f(t) := f_\theta(t) = |K \cap (\theta^\perp + t\theta)|$. Since $f^{\frac{1}{n-1}}$ is concave, we have $f(t) \geq \left(1 - \frac{t}{h_K(\theta)}\right)^{n-1} f(0)$ for all $t \in [0, h_K(\theta)]$. It follows that

$$\begin{aligned}
\int_K |\langle x, \theta\rangle|^q dx &= \int_0^{h_K(\theta)} t^q f(t)dt + \int_0^{h_K(-\theta)} t^q f_{-\theta}(t)dt \\
&\geq \int_0^{h_K(\theta)} t^q \left(1 - \frac{t}{h_K(\theta)}\right)^{n-1} f(0)dt \\
&\quad + \int_0^{h_K(-\theta)} t^q \left(1 - \frac{t}{h_K(-\theta)}\right)^{n-1} f(0)dt \\
&= f(0)\big(h_K^{q+1}(\theta) + h_K^{q+1}(-\theta)\big) \int_0^1 s^q (1-s)^{n-1} ds \\
&= \frac{\Gamma(q+1)\Gamma(n)}{\Gamma(q+n+1)} f(0) \left(h_K^{q+1}(\theta) + h_K^{q+1}(-\theta)\right) \\
&\geq \frac{\Gamma(q+1)\Gamma(n)}{2\Gamma(q+n+1)} f(0)(h_K(\theta) + h_K(-\theta)) \max\{h_K^q(\theta), h_K^q(-\theta)\}.
\end{aligned}$$

Since K has its center of mass at the origin, we have $\|f\|_\infty \leq e f(0)$, therefore

$$1 = |K| = \int_{-h_K(-\theta)}^{h_K(\theta)} f(t)dt \leq e\left(h_K(\theta) + h_K(-\theta)\right) f(0).$$

This completes the proof of (3.1.4). Now, if $q \geq n$ we have

$$\begin{aligned}
h_{Z_q(K)}(\theta) &\geq h_{Z_n(K)}(\theta) \\
&\geq \left[\frac{\Gamma(n+1)\Gamma(n)}{2e\Gamma(2n+1)}\right]^{1/n} \max\left\{h_K(\theta), h_K(-\theta)\right\} \\
&\geq c \max\left\{h_K(\theta), h_K(-\theta)\right\},
\end{aligned}$$

where $c > 0$ is an absolute constant. Since $h_{\hat{K}}(\theta) = \max\left\{h_K(\theta), h_K(-\theta)\right\}$, this proves that $h_{Z_q(K)} \geq c h_{\hat{K}}$ for all $q \geq n$. $\square$

Let V be a symmetric convex body in $\mathbb{R}^n$, and let $\|\cdot\|$ be the norm induced by V to $\mathbb{R}^n$. For every $p \geq 1$ we define

$$M_p := M_p(V) = \left(\int_{S^{n-1}} \|\theta\|^p \sigma(d\theta)\right)^{1/p}. \tag{3.1.6}$$

The parameters M_p were studied by Litvak, Milman and Schechtman in [37].

Theorem 3.1.2. *Let V be a symmetric convex body in $\mathbb{R}^n$ and let $\|\cdot\|$ be the corresponding norm on $\mathbb{R}^n$. We denote by b the smallest constant for which $\|x\| \leq b|x|$ holds true for every $x \in \mathbb{R}^n$. Then,*

$$\max\left\{M_1, c_1 \frac{b\sqrt{p}}{\sqrt{n}}\right\} \leq M_p \leq \max\left\{2M_1, c_2 \frac{b\sqrt{p}}{\sqrt{n}}\right\} \tag{3.1.7}$$

for all $p \in [1, n]$, where $c_1, c_2 > 0$ are absolute constants.

Proof. The function $\|x\| : S^{n-1} \to \mathbb{R}$ is Lipschitz continuous with constant b. By the spherical isoperimetric inequality (see [22] and [45]) it follows that

$$\sigma\big(x \in S^{n-1} : \big|\, \|x\| - M_1 \,\big| > t\big) \leq 2\exp(-ct^2n/b^2) \tag{3.1.8}$$

for all $t > 0$. Then,

$$\begin{aligned} \int_{S^{n-1}} \big|\, \|x\| - M_1 \,\big|^p \sigma(dx) &\leq 2p\int_0^\infty t^{p-1}\exp(-ct^2n/b^2)dt \\ &= \left(\frac{b}{\sqrt{cn}}\right)^p 2p\int_0^\infty s^{p-1}\exp(-s^2)ds \\ &\leq \left(C\frac{b\sqrt{p}}{\sqrt{n}}\right)^p, \end{aligned}$$

for some absolute constant $C > 0$. The triangle inequality on $L^p(S^{n-1})$ implies that

$$M_p - M_1 \leq \|\, \|x\| - M_1\|_p \leq C\frac{b\sqrt{p}}{\sqrt{n}}. \tag{3.1.9}$$

In other words,

$$M_p \leq 2\max\left\{M_1, C\frac{b\sqrt{p}}{\sqrt{n}}\right\}. \tag{3.1.10}$$

For the left hand side inequality we observe that $1/b$ is the minimal width of A, so there exists $z \in S^{n-1}$ such that $A \subset \{y : |\langle y, z\rangle| \leq 1/b\}$. It follows that

$$\{x \in S^{n-1} : \|x\| \geq t\} \supset C_t := \{x \in S^{n-1} : |\langle x, z\rangle| \geq t/b\} \tag{3.1.11}$$

for every $t > 0$. A simple computation shows that

$$\sigma(C_t) \geq c\sqrt{n}\frac{t}{b}\exp(-cnt^2/b^2) \tag{3.1.12}$$

if $t \leq b/3$, where $c > 0$ is an absolute constant. Then,

$$\begin{aligned} M_p &= \left(p\int_0^\infty t^{p-1}\sigma(C_t)dt\right)^{1/p} \geq s[\sigma(C_s)]^{1/p} \\ &\geq cs\frac{\sqrt{n}}{b}\exp(-cns^2/pb^2) \end{aligned}$$

for all $s \leq b/3$, and the choice $s = b\sqrt{p}/3\sqrt{n}$ completes the proof. □

The change of behavior of M_p happens when $p \simeq n(M_1/b)^2$. This value is equivalent to the largest integer $k = k(V)$ for which the majority of k-dimensional sections of V are 4-Euclidean (the *Dvoretzky number* of V; see [22]). It is clear that $M_p \leq M_r$ if $p \leq r$. Therefore, $M_r \simeq b$ if $r \geq n$. In other words, we have a second change of behavior of M_p at the point $p = n$.

We shall apply Theorem 3.1.2 to the symmetric convex body $V = [Z_q(K)]^\circ$. Then, $b = R(Z_q(K))$ and $M_1 = w(Z_q(K))$. For every $p, q \geq 1$ we define

$$w_p(Z_q(K)) = \left(\int_{S^{n-1}} h^p_{Z_q(K)}(\theta)\sigma(d\theta)\right)^{1/p}. \tag{3.1.13}$$

With this notation, we may rephrase Theorem 3.1.2 as follows.

Theorem 3.1.3. *Let K be a convex body in $\mathbb{R}^n$ with volume 1 and center of mass at the origin. For every $p \in [1,n]$ and $q \geq 1$ we have*

$$\begin{aligned}\max\left\{w(Z_q(K)), c_1\frac{R(Z_q(K))\sqrt{p}}{\sqrt{n}}\right\} &\leq w_p(Z_q(K)) \\ &\leq \max\left\{2w(Z_q(K)), c_2\frac{R(Z_q(K))\sqrt{p}}{\sqrt{n}}\right\}.\end{aligned}$$

If $p \geq n$, then $w_p(Z_q(K)) \simeq R(Z_q(K))$. □

The quantities $I_q(K)$ are related to the q-centroid bodies of K through the following Proposition.

Proposition 3.1.4. *Let K be a convex body in $\mathbb{R}^n$ with volume 1 and center of mass at the origin. For every $q \geq 1$,*

$$w_q(Z_q(K)) \simeq \sqrt{\frac{q}{q+n}} I_q(K). \tag{3.1.14}$$

Proof. Since

$$w_q(Z_q(K)) = \left(\int_{S^{n-1}}\int_K |\langle x,\theta\rangle|^q dx\sigma(d\theta)\right)^{1/q}, \tag{3.1.15}$$

this follows by Lemma 2.2.5. □

3.2 Isotropic convex bodies with small diameter

In this Section we assume that K is an isotropic convex body in $\mathbb{R}^n$ with radius $R(K) = \alpha\sqrt{n}L_K$, where $\alpha \geq 1$ is a positive constant. Our main result will be that most directions $\theta \in S^{n-1}$ are "good ψ_2-directions" if α is uniformly bounded. The precise statement is as follows.

Theorem 3.2.1. *Let K be an isotropic convex body in $\mathbb{R}^n$ with $R(K) = \alpha\sqrt{n}L_K$ for some $\alpha \geq 1$. Then,*

$$\sigma\big(\theta \in S^{n-1} : \|\langle\cdot,\theta\rangle\|_{\psi_2} \geq c_1\alpha L_K t\big) \leq \exp\left(-c_2\sqrt{n}t^2/\alpha^2\right) \tag{3.2.1}$$

for all $t \geq 1$, where $c_1, c_2 > 0$ are absolute constants.

The proof starts with the following observation. By the definition of $Z_q(K)$, for every $\theta \in S^{n-1}$ we have

$$\|\langle\cdot,\theta\rangle\|_{\psi_2} \simeq \sup_{q\geq 1}\frac{H_q(\theta)}{\sqrt{q}}. \tag{3.2.2}$$

So, our first task will be to fix $q \geq 1$ and give an upper bound for probabilities of the form

$$\sigma\left(\theta \in S^{n-1} : H_q(\theta) \geq c_1\alpha\sqrt{q}L_K t\right),$$

where the constant $c_1 > 0$ will be specified. Here, we are using for the first time the fact that K has small diameter.

Lemma 3.2.2. *For every $q \geq 1$ we have*

$$w(Z_q(K)) \leq w_q(Z_q(K)) \leq c_1 \alpha \sqrt{q} L_K, \tag{3.2.3}$$

where $c_1 > 0$ is an absolute constant.

Proof. The first inequality is a simple consequence of Hölder's inequality, while for the second one we may assume that $q \leq n$. From Proposition 3.1.4 we have

$$w_q(Z_q(K)) \leq c_1 \sqrt{\frac{q}{n}} I_q(K) \leq c_1 \sqrt{\frac{q}{n}} \cdot \alpha \sqrt{n} L_K = c_1 \alpha \sqrt{q} L_K, \tag{3.2.4}$$

since $I_q(K) \leq R(K) = \alpha \sqrt{n} L_K$ for every $q \geq 1$. □

Lemma 3.2.2 shows that

$$\sigma\left(\theta \in S^{n-1} : H_q(\theta) \geq c_1 \alpha \sqrt{q} L_K t\right) \leq \sigma\left(\theta \in S^{n-1} : H_q(\theta) \geq w(Z_q(K)) t\right), \tag{3.2.5}$$

for every $q \geq 1$ and $t \geq 1$. At this point, the spherical isoperimetric inequality may be used through the following Lemma.

Lemma 3.2.3. *Let M be a convex body in $\mathbb{R}^n$ and assume that for some $R > 0$ we have $h_M(\theta) \leq R$ for all $\theta \in S^{n-1}$. Then,*

$$\sigma\big(\theta \in S^{n-1} : h_M(\theta) \geq 3tw(M)\big) \leq \exp\left(-c \frac{w^2(M) t^2 n}{R^2}\right) \tag{3.2.6}$$

for all $t \geq 1$, where $c > 0$ is an absolute constant.

Proof. Let m be the Lévy mean of h_M on S^{n-1}. Since h_M is Lipschitz with constant R, the spherical isoperimetric inequality gives

$$\sigma\big(\theta \in S^{n-1} : h_M(\theta) \geq m + tw(M)\big) \leq \exp\left(-c \frac{w^2(M) t^2 n}{R^2}\right) \tag{3.2.7}$$

for all $t > 0$. On observing that $m \leq 2w(M)$, we conclude the proof. □

Applying this for $M = Z_q(K)$ and taking into account (3.2.5), we have proved the following fact.

Proposition 3.2.4. *Let K be an isotropic convex body in $\mathbb{R}^n$ with $R(K) = \alpha \sqrt{n} L_K$ for some $\alpha \geq 1$. Then,*

$$\sigma\big(\theta \in S^{n-1} : H_q(\theta) \geq c_1 \alpha \sqrt{q} L_K t\big) \leq \exp\left(-ct^2 n \frac{w(Z_q(K))^2}{R(Z_q(K))^2}\right), \tag{3.2.8}$$

for every $q \geq 1$ and $t \geq 1$, where $c, c_1 > 0$ are absolute constants. □

It is now natural to study the quantity $nw(Z_q(K))^2 / R(Z_q(K))^2$ as a function of q. To this end, we specify an absolute constant $\beta > 0$ such that $R^2(V) < \beta n w^2(V)$ for every symmetric convex body V in $\mathbb{R}^n$ (note that $R(M) \leq c\sqrt{n} w(M)$ in general), and for every convex body K in $\mathbb{R}^n$ with volume 1 and center of mass at the origin, we define a function $m_K : [1, \infty) \to (1, \infty)$ by

$$m_K(q) = \beta n \frac{w(Z_q(K))^2}{R(Z_q(K))^2}. \tag{3.2.9}$$

It is clear that m_K takes values in the interval $(1, \beta n]$. We consider the sets

$$\begin{aligned} P_K &= \{q \in [1, 2\beta n] : q \geq m_K(q)\} \\ N_K &= \{q \in [1, 2\beta n] : q \leq m_K(q)\} \end{aligned}$$

Observe that $[\beta n, 2\beta n] \subseteq P_K$ and $1 \in N_K$. Since m_K is continuous and the sets P_K, N_K are non-empty, $q_0 := \min P_K$ is well-defined, and $m_K(q_0) = q_0$. We shall repeatedly use the following observation.

Lemma 3.2.5. *If $p, q \in [1, 2\beta n]$ and $p \leq m_K(q)$, then*

$$w(Z_q(K)) \geq c w_p(Z_q(K)). \tag{3.2.10}$$

Proof. Since $p \leq m_K(q)$, we have

$$w(Z_q(K)) \geq R(Z_q(K))\sqrt{p}/\sqrt{\beta n}. \tag{3.2.11}$$

The result follows from Theorem 3.1.3. □

Some first observations on the function m_K follow.

Proposition 3.2.6. *There exist absolute constants $c_1, c_2, c_3 > 0$ such that: for every convex body K in $\mathbb{R}^n$ with volume 1 and center of mass at the origin,*
(i) *If $q \in N_K$, then*

$$m_K(q) \geq c_1 \frac{m_K(2)}{q}. \tag{3.2.12}$$

(ii) *If $m_K(2) > 1/c_3$, then*

$$[1, c_3\sqrt{m_K(2)}] \subseteq N_K. \tag{3.2.13}$$

(iii) *If $q \in P_K$, then*

$$R(Z_q(K)) \geq c_2\sqrt{n}\, w(Z_2(K)). \tag{3.2.14}$$

Proof. (i) Let $q \in N_K$. Since $q \leq m_K(q)$, Lemma 3.2.5 and Proposition 3.1.4 give

$$\begin{aligned} w(Z_q(K)) &\geq c w_q(Z_q(K)) \geq c'\sqrt{\frac{q}{n}} I_q(K) \geq c'\sqrt{\frac{q}{n}} I_2(K) \\ &\geq c''\sqrt{q}\, w_2(Z_2(K)) \geq c''\sqrt{q}\, w(Z_2(K)). \end{aligned}$$

By Proposition 2.1.1 we have $H_q(\theta) \leq cqH_2(\theta)$ for all $\theta \in S^{n-1}$, therefore

$$R(Z_q(K)) \leq cqR(Z_2(K)). \tag{3.2.15}$$

It follows that

$$m_K(q) = \beta n \frac{w(Z_q(K))^2}{R(Z_q(K))^2} \geq c_1 \beta n \frac{q w(Z_2(K))^2}{q^2 R(Z_2(K))^2} = c_1 \frac{m_K(2)}{q}. \tag{3.2.16}$$

(ii) If $q_0 = \min P_K$ we have $m_K(q_0) = q_0$ and the previous claim shows that

$$q_0^2 \geq c_1 m_K(2). \tag{3.2.17}$$

Therefore, $[1, c_3\sqrt{m_K(2)}] \subseteq N_K$, where $c_3 = \sqrt{c_1}$.
(iii) For the value q_0, using Lemma 3.2.5 and Proposition 3.1.4 we have
(3.2.18)
$$R(Z_{q_0}(K)) = \sqrt{\beta n}\frac{w(Z_{q_0}(K))}{\sqrt{q_0}} \geq c\sqrt{n}\frac{w_{q_0}(Z_{q_0}(K))}{\sqrt{q_0}} \simeq I_{q_0}(K) \geq cI_2(K).$$

By Proposition 3.1.4 we have $I_2(K) \simeq \sqrt{n}w_2(Z_2(K))$, and hence,

$$R(Z_{q_0}(K)) \geq c\sqrt{n}w(Z_2(K)). \tag{3.2.19}$$

If $q \in P_K$ then $R(Z_q(K)) \geq R(Z_{q_0}(K))$, which completes the proof. □

If we assume that K is isotropic, we have $R(Z_2(K)) = w(Z_2(K)) = L_K$. Then, Proposition 3.2.6 takes the following form.

Proposition 3.2.7. *Let K be an isotropic convex body in $\mathbb{R}^n$. Then,*
(i) *For every $q \in N_K$ we have $m_K(q) \geq c_1 n/q$, where $c_1 > 0$ is an absolute constant.*
(ii) $[1, c_2\sqrt{n}] \subset N_K$*, where $c_2 > 0$ is an absolute constant.*
(iii) *For every $q \in P_K$ we have $R(Z_q(K)) \geq c_3\sqrt{n}L_K$, where $c_3 > 0$ is an absolute constant.* □

We now use again the additional assumption that K has small diameter.

Proposition 3.2.8. *Let K be an isotropic convex body in $\mathbb{R}^n$ with $R(K) = \alpha\sqrt{n}L_K$ for some $\alpha \geq 1$. For every $q \geq 1$ we have*

$$m_K(q) \geq c_3\sqrt{n}/\alpha^2, \tag{3.2.20}$$

where $c_3 > 0$ is an absolute constant.

Proof. From Proposition 3.2.7 there exists $q_1 \simeq \sqrt{n}$ with the property $[1, q_1] \subset N_K$. From Lemma 3.2.5 we have
(3.2.21)
$$w(Z_{q_1}(K)) \geq cw_{q_1}(Z_{q_1}(K)) \simeq \sqrt{q_1}I_{q_1}(K)/\sqrt{n} \geq \sqrt{q_1}I_2(K)/\sqrt{n} = \sqrt{q_1}L_K.$$

Let $q \geq q_1$. Then,

$$w(Z_q(K)) \geq w(Z_{q_1}(K)) \geq c\sqrt{q_1}L_K \geq c'\sqrt[4]{n}L_K. \tag{3.2.22}$$

On the other hand,

$$R(Z_q(K)) \leq R(K) \leq \alpha\sqrt{n}L_K, \tag{3.2.23}$$

therefore

$$m_K(q) = \beta n\frac{w(Z_q(K))^2}{R(Z_q(K))^2} \geq c\sqrt{n}/\alpha^2. \tag{3.2.24}$$

If $q \leq q_1$ then $q \in N_K$, and Proposition 3.2.7 shows that

$$m_K(q) \geq c_1\frac{n}{q} \geq c_1\frac{n}{q_1} \geq c_1'\sqrt{n}. \tag{3.2.25}$$

The result follows from (3.2.24) and (3.2.25). □

Proof of Theorem 3.2.1: Proposition 3.2.4 shows that for every $q, t \geq 1$ we have

$$\sigma\big(\theta \in S^{n-1} : H_q(\theta) \geq c_1\alpha\sqrt{q}L_K t\big) \leq \exp(-c_2 m_K(q)t^2). \tag{3.2.26}$$

From Proposition 3.2.8, $m_K(q) \geq c_3\sqrt{n}/\alpha^2$ for all $q \geq 1$. Therefore,

$$\sigma\big(\theta \in S^{n-1} : H_q(\theta) \geq c_1\alpha\sqrt{q}L_K t\big) \leq \exp(-c_4\sqrt{n}t^2/\alpha^2). \tag{3.2.27}$$

Lemma 3.1.1 and (3.2.2) show that

$$\|\langle\cdot,\theta\rangle\|_{\psi_2} \simeq \sup_{1\leq q\leq n} \frac{H_q(\theta)}{\sqrt{q}}. \tag{3.2.28}$$

It easily follows that there exists an absolute constant $c_5 > 0$ such that

$$\{\theta \in S^{n-1} : \|\langle\cdot,\theta\rangle\|_{\psi_2} \geq c_5\alpha t L_K\} \subseteq \bigcup_{q=2}^{n} J_q, \tag{3.2.29}$$

where, for the integer $q \in \{2, \ldots, n\}$ we define

$$J_q := \{\theta \in S^{n-1} : H_q(\theta) \geq 2c_1\alpha t\sqrt{q}L_K\}. \tag{3.2.30}$$

Taking into account (3.2.27) we see that

$$\sigma(\theta \in S^{n-1} : \|\langle\cdot,\theta\rangle\|_{\psi_2} \geq c_5\alpha t L_K) \leq n\exp(-c_6\sqrt{n}t^2/\alpha^2) \tag{3.2.31}$$

for some absolute constant $c_6 > 0$. This completes the proof for $n \geq n_0(\alpha)$. $\square$

3.3 An example

We consider symmetric convex bodies in $\mathbb{R}^n$ of the form

$$K = \{(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R} : |t| \leq R, |x| \leq f(|t|)\}, \tag{3.3.1}$$

where $f : [0,R] \to \mathbb{R}^+$ is a decreasing linear function. We may write $f(t) = a - bt$, where $a > 0$ and $0 \leq b \leq a/R$. We assume that $|K| = 1$, which gives the condition

$$2\omega_{n-1}\int_0^R (a-bt)^{n-1}dt = 1. \tag{3.3.2}$$

We also write I for the quantity

$$I := I_1(K, e_n) = \int_K |t|d(x,t) = 2\omega_{n-1}\int_0^R t(a-bt)^{n-1}dt. \tag{3.3.3}$$

We shall show that the right choice of the parameters R, a and b leads to very simple examples of bad ψ_2-behavior. It is convenient to assume from the beginning that

$$I \ll R \tag{3.3.4}$$

and

$$c_2\frac{a}{nI} \leq b \leq c_3\frac{a}{nI}. \tag{3.3.5}$$

where $c_i > 0$ are absolute constants. One can actually show that (3.3.4) implies (3.3.5). In the sequel, K denotes the convex body defined by (3.3.1), where $f(t) = a - bt$.

Lemma 3.3.1. *Assume that K satisfies* (3.3.2), (3.3.4) *and* (3.3.5). *If $s \leq c' \min\{R, a/b\}$, then*

$$\mathrm{Prob}(|t| \geq s) \geq c_4 \exp(-c_5 s/I), \tag{3.3.6}$$

where $c', c_4, c_5 > 0$ are absolute constants.

Proof. The probability is equal to

$$\begin{aligned}\mathrm{Prob}(|t| \geq s) &= 2\omega_{n-1} \int_s^R (a - bt)^{n-1} dt \\ &= 2\omega_{n-1} \frac{a^n}{nb} \left(\left(1 - \frac{bs}{a}\right)^n - \left(1 - \frac{bR}{a}\right)^n \right).\end{aligned}$$

Since $|K| = 1$, Propositions 1.5.5 and 2.1.1 show that

$$\omega_{n-1} a^{n-1} = |K \cap e_n^{\perp}| \geq c/I. \tag{3.3.7}$$

Taking into account the assumption $b \leq c_3 a/(nI)$, we see that

$$2\omega_{n-1} \frac{a^n}{nb} \geq c \tag{3.3.8}$$

for some absolute constant $c > 0$. If $s \leq a/2b$, using the numerical inequality $1 - x \geq e^{-2x}$ for $x \in [0, 1/2]$, we obtain

$$\left(1 - \frac{bs}{a}\right)^n \geq \exp(-2bns/a). \tag{3.3.9}$$

On the other hand,

$$\left(1 - \frac{bR}{a}\right)^n \leq \exp(-bRn/a). \tag{3.3.10}$$

If $s \ll R$, then

$$\exp(-bRn/a) \leq \frac{1}{2} \exp(-2bns/a). \tag{3.3.11}$$

Here, we also use the fact that $I \ll R$, which gives $\exp(bRn/a) \geq \exp(cR/I) \gg 1$. It follows that if $s \leq c' \min\{R, a/b\}$, then

$$\mathrm{Prob}(|t| \geq s) \geq (c/2) \exp(-2bns/a) \geq (c/2) \exp(-2c_2 s/I). \tag{3.3.12}$$

This gives the result, with $c_4 = c/2$ and $c_5 = 2c_2$. □

Lemma 3.3.2. *For every $j = 1, \ldots, n-1$ we have*

$$c_6 \frac{\sqrt{n}}{a} \leq |K \cap e_j^{\perp}| \leq c_7 \sqrt[n-1]{R}, \tag{3.3.13}$$

where $c_6, c_7 > 0$ are absolute constants.

Proof. For the upper bound we use Hölder's inequality and (3.3.2):

$$\begin{aligned} |K \cap e_j^{\perp}| &= 2\omega_{n-2}\int_0^R (a-bt)^{n-2}dt \leq 2\omega_{n-2}\left(\int_0^R (a-bt)^{n-1}dt\right)^{\frac{n-2}{n-1}} \sqrt[n-1]{R} \\ &= 2\omega_{n-2}(2\omega_{n-1})^{-\frac{n-2}{n-1}} \sqrt[n-1]{R} \leq c_7 \sqrt[n-1]{R}, \end{aligned}$$

where $c_7 > 0$ is an absolute constant. For the lower bound we observe that

$$\begin{aligned} |K \cap e_j^{\perp}| &= 2\omega_{n-2}\int_0^R (a-bt)^{n-2}dt \geq \frac{\omega_{n-2}}{a\omega_{n-1}} 2\omega_{n-1}\int_0^R (a-bt)^{n-1}dt \\ &= \frac{\omega_{n-2}}{a\omega_{n-1}} \geq c_6\frac{\sqrt{n}}{a}, \end{aligned}$$

where $c_6 > 0$ is an absolute constant. $\square$

Lemma 3.3.3. *There exist $a \simeq \sqrt{n}$ and $b \simeq 1/\sqrt{n}$ such that the symmetric convex body*

$$W = \{y = (x,t) : |t| \leq a, |x| \leq a - b|t|\} \tag{3.3.14}$$

has volume 1 and satisfies

$$c_8 \leq \int_W \langle y, \theta\rangle^2 dy \leq c_9 \tag{3.3.15}$$

for every $\theta \in S^{n-1}$, where $c_8, c_9 > 0$ are absolute constants.

Proof. Let r be the solution of the equation $\omega_{n-1}r^{n-1} = 1$, and consider the body

$$K = \{(x,t) : |t| \leq r, |x| \leq r - |t|/\sqrt{n}\}. \tag{3.3.16}$$

Then,

$$|K| = 2\omega_{n-1}r^{n-1} \cdot \sqrt{n}r \cdot \frac{1 - \left(1 - \frac{1}{\sqrt{n}}\right)^n}{n} \simeq 1, \tag{3.3.17}$$

since $r \simeq \sqrt{n}$. Let $s > 0$ be such that the body $W := sK$ has volume 1. Then, $s^n \simeq 1$, and W is of the form (3.3.14), where $a \simeq \sqrt{n}$ and $b = 1/\sqrt{n}$. Note that

$$I^{-1} \simeq |W \cap e_n^{\perp}| = \omega_{n-1}r^{n-1}s^{n-1} \simeq 1 \tag{3.3.18}$$

where $I = I_1(W, e_n)$, and hence

$$\int_W \langle y, e_n\rangle^2 dy \simeq I^2 \simeq 1. \tag{3.3.19}$$

Since W is symmetric with respect to the coordinate subspaces, (3.3.15) will hold for every $\theta \in S^{n-1}$ provided that

$$c_8 \leq \int_W \langle y, e_j\rangle^2 dy \leq c_9 \tag{3.3.20}$$

for every $j = 1, \dots, n$. Because of (3.3.19), we only need to check the case $j \le n-1$. From Lemma 3.3.2, for every $j = 1, \dots, n-1$ we have

$$c_6' \le c_6 \frac{\sqrt{n}}{a} \le |W \cap e_j^\perp| \le c_7 \sqrt[n-1]{a} \le c_7', \tag{3.3.21}$$

which implies

$$\int_W \langle y, e_j \rangle^2 dy \simeq |W \cap e_j^\perp|^{-2} \simeq 1. \tag{3.3.22}$$

This proves the Lemma. □

Starting with W, we may easily pass to a "similar" isotropic body.

Theorem 3.3.4. *There exist $a_1, R_1 \simeq \sqrt{n}$ and $b_1 \simeq 1/\sqrt{n}$ such that the symmetric convex body*

$$Q = \{y = (x,t) : |t| \le R_1, |x| \le a_1 - b_1|t|\} \tag{3.3.23}$$

is isotropic.

Proof. Consider the body W of the previous Lemma. There exists a diagonal operator $T = \mathrm{diag}(u, \dots, u, v)$ such that $Q = T(W)$ is isotropic. Lemma 2.3.2 and the proof of Proposition 2.3.1 show that $u, v \simeq 1$. Then, Q can be written in the form (3.3.23) with $R_1 = av$, $a_1 = au$ and $b_1 = bu/v$. □

The next two Lemmas describe two "contradictory" properties of Q.

Lemma 3.3.5. *There exist absolute constants $c, C > 0$ such that*

$$c\sqrt{n}B_2^n \subseteq Q \subseteq C\sqrt{n}B_2^n. \tag{3.3.24}$$

Proof. The problem is two-dimensional. For every $y = (x,t) \in Q$ we have

$$\|y\|_2^2 = |x|^2 + t^2 \le a_1^2 + R_1^2 \le C^2 n, \tag{3.3.25}$$

where $C > 0$ is an absolute constant, because $a_1, R_1 \simeq \sqrt{n}$. This shows that $Q \subseteq C\sqrt{n}B_2^n$. For the other inclusion, we observe that the inradius of Q is equal to $\min\{R_1, d\}$, where d is the distance from $(0,0)$ to the line $y = a_1 - b_1 t$ in $\mathbb{R}^2$. We have

$$d = \frac{a_1}{\sqrt{1 + b_1^2}} \simeq \sqrt{n}, \tag{3.3.26}$$

and hence $Q \supseteq c\sqrt{n}B_2^n$ for some absolute constant $c > 0$. □

Lemma 3.3.6. *There exists an absolute constant $c > 0$ such that*

$$\|\langle \cdot, e_n \rangle\|_{\psi_2} \ge c\sqrt[4]{n}. \tag{3.3.27}$$

Proof. For every $q \geq 1$ we have

$$I_q := I_q(Q, e_n) \leq c_1 q I_1(Q, e_n) \leq c_2 q, \tag{3.3.28}$$

where $c_2 > 0$ is an absolute constant. Proposition 2.1.7 shows that

$$\mathrm{Prob}(y \in Q : |\langle y, e_n\rangle| \geq 3CI_q) \leq e^{-q} \tag{3.3.29}$$

for every $q \geq 1$. If $3C \cdot c_2 q \leq c' \min\{R_1, a_1/b_1\}$ where c' is the constant in Lemma 3.3.1, we have

$$\mathrm{Prob}(y \in Q : |\langle y, e_n\rangle| \geq 3CI_q) \geq \exp(-3c_5 CI_q/I). \tag{3.3.30}$$

For these values of q it follows that $qI \leq 3c_5 CI_q$, and since $I \simeq 1$, we conclude that

$$\frac{I_q}{\sqrt{q}} \geq c'' \sqrt{q}, \tag{3.3.31}$$

where $c'' > 0$ is an absolute constant. Since $\min\{R_1, a_1/b_1\} = R_1 \simeq \sqrt{n}$, the largest value of q for which (3.3.31) holds is of the order of $\sqrt{n}$ (note that this is the largest possible, since $R(Q) = O(\sqrt{n})$). It follows that

$$\|\langle \cdot, e_n\rangle\|_{\psi_2} = \sup\left\{ \frac{\|\langle \cdot, e_n\rangle\|_q}{\sqrt{q}} : q \geq 1\right\} \geq c\sqrt[4]{n} \tag{3.3.32}$$

for some absolute constant $c > 0$. □

We summarize in the following Theorem.

Theorem 3.3.7. *There exists an isotropic convex body of revolution Q in $\mathbb{R}^n$ with the following properties:*

$$c_1\sqrt{n}B_2^n \subseteq Q \subseteq c_2\sqrt{n}B_2^n \tag{3.3.33}$$

and

$$\|\langle \cdot, e_n\rangle\|_{\psi_2} \geq c_3\sqrt[4]{n} \tag{3.3.34}$$

where $c_1, c_2, c_3 > 0$ are absolute constants. □

Remark 1: Theorem 3.3.7 shows that the simple ψ_2-estimate of Proposition 2.2.3 cannot be improved, even for bodies which have uniformly bounded geometric distance to a Euclidean ball. We can also check that Proposition 3.2.8 is sharp: if we apply it to Q we have

$$m_Q(q) \geq c_4\sqrt{n} \tag{3.3.35}$$

for every $q \geq 1$. If q_0 is the largest $q \geq 1$ for which (3.3.31) holds, then

$$R(Z_{q_0}(Q)) \geq cq_0 \tag{3.3.36}$$

and Lemma 3.2.2 implies

$$w(Z_{q_0}(Q)) \leq C\sqrt{q_0}. \tag{3.3.37}$$

Therefore,

$$m_Q(q_0) = \beta n \frac{w(Z_{q_0}(Q))^2}{R(Z_{q_0}(Q))^2} \leq \beta n \frac{C^2 q_0}{c^2 q_0^2} \leq C_1 n/q_0. \tag{3.3.38}$$

Since $q_0 \simeq \sqrt{n}$, we conclude that

$$\inf_{q\geq 1} m_Q(q) \simeq \sqrt{n}. \tag{3.3.39}$$

In other words, all the basic estimates of §3.2 are exact.

We can actually give a complete description of the L_q-centroid bodies of Q, using the following consequence of the "L_q-affine isoperimetric inequality" of Lutwak, Yang and Zhang (see [39], also [18]).

Proposition 3.3.8. *Let K be a convex body of volume 1 in $\mathbb{R}^n$. Then,*

$$|Z_q(K)|^{1/n} \geq c\sqrt{q/n} \tag{3.3.40}$$

for every $1 \leq q \leq n$, where $c > 0$ is an absolute constant. □

Theorem 3.3.9. *Let Q be the isotropic convex body in Theorem 3.3.7. There exists $q_0 \simeq \sqrt{n}$ such that:*

(i) *For every $1 \leq q \leq n$,*

$$w(Z_q(Q)) \simeq \sqrt{q}.$$

(ii) *If $1 \leq q \leq q_0$ then $R(Z_q(Q)) \simeq q$, and if $q \geq q_0$ then $R(Z_q(Q)) \simeq \sqrt{n}$.*

(iii) *If $1 \leq q \leq q_0$ then $m_Q(q) \simeq n/q$, and if $q_0 \leq q \leq n$ then $m_Q(q) \simeq q$.*

Proof. Let q_0 be the largest $q \geq 1$ for which (3.3.31) holds.

1. From Proposition 3.3.8 and Urysohn's inequality, for every $1 \leq q \leq n$ we have

$$w(Z_q(Q)) \geq c\sqrt{n}|Z_q(Q)|^{1/n} \geq c'\sqrt{q}. \tag{3.3.41}$$

On the other hand, since $R(Q) = O(\sqrt{n})$, we see that

$$w(Z_q(Q)) \leq w_q(Z_q(Q)) \simeq \sqrt{q/n}I_q(Q) \leq c''\sqrt{q}. \tag{3.3.42}$$

2. In the case $q \leq q_0$, (3.3.31) shows that

$$R(Z_q(Q)) \simeq q. \tag{3.3.43}$$

In particular, $R(Z_{q_0}(Q)) \simeq q_0$. It follows that if $q \geq q_0$, then

$$\sqrt{n} \simeq q_0 \simeq R(Z_{q_0}(Q)) \leq R(Z_q(Q)) \leq R(Q) \simeq \sqrt{n}. \tag{3.3.44}$$

3. Since we have determined $R(Z_q(Q))$ and $w(Z_q(Q))$ for every value of $q \in [1, n]$, we can compute the parameter $m_Q(q)$. □

Notes and References

L_q-centroid bodies were introduced by Lutwak and G. Zhang in [38]. The idea to use q-centroid bodies (and the result of [37]) for ψ_2-estimates comes from the thesis of Paouris. All the results of §3.2 and §3.3 can be found in [48].

Chapter 4

Reduction to bounded volume ratio

4.1 Symmetrization of isotropic convex bodies

Let K be a convex body in $\mathbb{R}^n$. Suppose that E is a k-dimensional subspace of $\mathbb{R}^n$ and T is a convex body in E with volume 1 and center of mass at the origin. The (T,E)-*symmetrization* of K is the unique convex body $K(T,E)$ which has the following two properties:

1. If $y \in E^{\perp}$ then $|K \cap (y+E)| = |K(T,E) \cap (y+E)|$.

2. If $y \in P_{E^{\perp}}(K)$, then $(K(T,E)-y) \cap E$ is homothetic to T and has its center of mass at 0.

Lemma 4.1.1. *$K(T,E)$ is a convex body.*

Proof. We set $K_1 := K(T,E)$. It suffices to prove that for every $y_1, y_2 \in P_{E^{\perp}}(K_1) = P_{E^{\perp}}(K)$ and every $\lambda \in (0,1)$,

$$\lambda[K_1 \cap (y_1+E)] + (1-\lambda)[K_1 \cap (y_2+E)] \subseteq K_1 \cap (\lambda y_1 + (1-\lambda)y_2 + E). \tag{4.1.1}$$

Since $(K_1 - y) \cap E$ is homothetic to T for every $y \in P_{E^{\perp}}(K)$, it suffices to prove that

$$\lambda|K_1 \cap (y_1+E)|^{\frac{1}{k}} + (1-\lambda)|K_1 \cap (y_2+E)|^{\frac{1}{k}} \leq |K_1 \cap (\lambda y_1 + (1-\lambda)y_2 + E)|^{\frac{1}{k}}. \tag{4.1.2}$$

By the definition of K_1,

$$|K_1 \cap (y+E)|^{\frac{1}{k}} = |K \cap (y+E)|^{\frac{1}{k}} \tag{4.1.3}$$

for every $y \in P_{E^{\perp}}(K)$, so the Lemma follows from the Brunn-Minkowski inequality applied to K. $\square$

Definition: Let A be a convex body in $\mathbb{R}^n$ with volume 1 and center of mass at the origin. We say that $e \in S^{n-1}$ is an axis of inertia of A if it is an eigenvector of the operator $M_A(y) = \int_A \langle x, y \rangle x dx$. Since M_A is symmetric and positive, there exists an orthonormal basis consisting of eigenvectors of M_A. Since $\det M_A = L_A^{2n}$, it is clear that if $\{e_1, \ldots, e_n\}$ is such a basis, then

$$L_K^{2n} = \prod_{i=1}^{n} \int_A \langle x, e_i \rangle^2 dx. \tag{4.1.4}$$

We will say that such a basis is a *basis of inertia* for A.

Remark 1: Fubini's theorem shows that if $\theta_1 \in \mathbb{R}^n$ then

$$\int_{K_1} \langle x, \theta \rangle^2 = \int_{P_{E^\perp}(K)} \int_{(K_1 - y)\cap E} \langle y + z, \theta \rangle^2 dz dy. \tag{4.1.5}$$

If $\theta \in E^\perp$, then the inner integral is independent of z, and since $|(K_1 - y) \cap E| = |(K - y) \cap E|$ for every $y \in P_{E^\perp}(K)$, we get

$$\int_{K_1} \langle x, \theta \rangle^2 = \int_K \langle x, \theta \rangle^2 dx. \tag{4.1.6}$$

If $\theta \in E$, then (4.1.5) gives

$$\begin{aligned} \int_{K_1} \langle x, \theta \rangle^2 &= \int_{P_{E^\perp}(K)} \int_{(K_1 - y)\cap E} \langle z, \theta \rangle^2 dz dy \\ &= \left(\int_{P_{E^\perp}(K)} |(K - y) \cap E|^{1+\frac{2}{k}} dy \right) \left(\int_T \langle z, \theta \rangle^2 dz \right). \end{aligned}$$

It follows that: if $\theta_1, \theta_2 \in E^\perp$ then

$$\langle M_{K_1}(\theta_1), \theta_2 \rangle = \langle M_K(\theta_1), \theta_2 \rangle, \tag{4.1.7}$$

and if $\theta_1, \theta_2 \in E$ then

$$\langle M_{K_1}(\theta_1), \theta_2 \rangle = c(K, E) \langle M_T(\theta_1), \theta_2 \rangle, \tag{4.1.8}$$

where

$$c(K, E) = \int_{P_{E^\perp}(K)} |K \cap (y + E)|^{1+\frac{2}{k}} dy. \tag{4.1.9}$$

Lemma 4.1.2. *Let K be an isotropic convex body in $\mathbb{R}^n$. Let E be a k-dimensional subspace of $\mathbb{R}^n$ and let T be a convex body in E with volume 1 and center of mass at the origin. If $e_1, \ldots, e_k$ are axes of inertia of T and $\{e_{k+1}, \ldots, e_n\}$ is any orthonormal basis of $E^\perp$, then $\{e_1, \ldots, e_n\}$ is a basis of inertia for $K(T, E)$.*

Proof. Set $K_1 := K(T, E)$. We first observe that the restriction of $M_{K(T,E)}$ onto $E^\perp$ is a multiple of the identity: from (4.1.7) we have

$$\langle M_{K_1}(\theta_1), \theta_2 \rangle = L_K^2 \langle \theta_1, \theta_2 \rangle \tag{4.1.10}$$

for all $\theta_1, \theta_2 \in E^\perp$, which shows that $M_{K_1}|_{E^\perp} = L_K^2 I_{E^\perp}$. Then, any orthonormal basis $\{e_{k+1}, \ldots, e_n\}$ of $E^\perp$ consists of axes of inertia of K_1. Also, since M_{K_1} is symmetric, we have that E is invariant under M_{K_1}.

Now, from (4.1.8) we have

$$\langle M_{K_1}(e_i),\theta\rangle = c(K,E)\langle M_T(e_i),\theta\rangle$$

for every $\theta\in E$ and every $i=1,\dots,k$. Since $e_1,\dots,e_k$ are eigenvectors of M_T, this shows that $e_1,\dots,e_k$ are axes of inertia of K_1. □

Proposition 4.1.3. *Let $f:\mathbb{R}^n\to\mathbb{R}^+$ be a function with compact support, such that $f^{1/k}$ is concave on* supp(f)*. Assume also that $\int f(x)dx=1$. Then,*

$$\frac{(k+1)(k+2)}{(n+k+1)(n+k+2)}\|f\|_\infty^{2/k} \leq \int f(x)^{1+2/k}dx \leq \|f\|_\infty^{2/k}. \tag{4.1.11}$$

Remark 2: The right hand side inequality is trivial:

$$\int f(x)^{1+2/k}dx \leq \|f\|_\infty^{2/k}\int f(x)dx = \|f\|_\infty^{2/k}. \tag{4.1.12}$$

For the left hand side inequality, we will use the following Lemma.

Lemma 4.1.4. *Let $f:\mathbb{R}^+\to\mathbb{R}^+$ be a function with compact support, such that $f^{1/k}$ is concave on* supp(f) *and $a=f(0)>0$. Let $n\in\mathbb{N}$ and choose $b>0$ such that*

$$\int_0^\infty f(x)x^n dx = \int_0^\infty (a^{1/k}-bx)_+^k x^n dx, \tag{4.1.13}$$

where $x_+=\max\{x,0\}$. Then,

$$\int_0^\infty f(x)^p x^n dx \geq \int_0^\infty (a^{1/k}-bx)_+^{pk} x^n dx \tag{4.1.14}$$

for every $p>1$.

Proof. The existence of b is clear because $\int_0^\infty f(x)x^n dx<+\infty$ (f has compact support). Define $h(x)=a^{1/k}-f(x)^{1/k}$. Then, h is a convex function with $h(0)=0$. Therefore, $h_1(x)=h(x)/x$ is increasing. Writing (4.1.13) in the form

$$\int_0^\infty (a^{1/k}-xh_1(x))_+^k x^n dx = \int_0^\infty (a^{1/k}-bx)_+^k x^n dx, \tag{4.1.15}$$

we see that $h_1(x)$ cannot be everywhere less or everywhere greater than b. Since h_1 is increasing, there exists $x_0\in[0,\infty)$ such that $h_1\leq b$ on $[0,x_0]$ and $h_1\geq b$ on $[x_0,\infty)$. It follows that

$$(g(x)-f(x))(x-x_0)\geq 0 \tag{4.1.16}$$

for all $x\geq 0$, where $g(x)=(a^{1/k}-bx)_+^k$. Since g^{p-1} is a decreasing function, we have

$$\int_0^{x_0} x^n\int_{g(x)}^{f(x)} y^{p-1}dydx \geq \int_0^{x_0} x^n\int_{g(x)}^{f(x)} g(x_0)^{p-1}dydx \tag{4.1.17}$$

and

$$\int_{x_0}^\infty x^n\int_{f(x)}^{g(x)} y^{p-1}dydx \leq \int_{x_0}^\infty x^n\int_{f(x)}^{g(x)} g(x_0)^{p-1}dydx. \tag{4.1.18}$$

Subtracting these two inequalities, we get

$$\int_0^\infty x^n \int_{g(x)}^{f(x)} y^{p-1} dy dx \geq g(x_0)^{p-1} \int_0^\infty (f(x) - g(x)) x^n dx = 0. \tag{4.1.19}$$

This shows that

$$\int_0^\infty x^n \int_0^{f(x)} p y^{p-1} dy dx \geq \int_0^\infty x^n \int_0^{g(x)} p y^{p-1} dy dx, \tag{4.1.20}$$

which is equivalent to (4.1.14). □

Proof of Proposition 4.1.3: Since the inequality is translation invariant, we may assume that $\|f\|_\infty = f(0)$. We write

$$\int f(x)^{1+2/k} dx = n\omega_n \int_{S^{n-1}} \int_0^\infty f(r\theta)^{1+2/k} r^{n-1} dr \sigma(d\theta), \tag{4.1.21}$$

we fix $\theta \in S^{n-1}$ and define $g(r) = f(r\theta)$ on $[0, \infty)$ (note that $g^{1/k}$ is concave). Lemma 4.1.4 shows that

$$\int_0^\infty g(x)^{1+2/k} x^{n-1} dx \geq \int_0^\infty (a^{1/k} - bx)_+^{k+2} x^{n-1} dx \tag{4.1.22}$$

where $b > 0$ is chosen so that

$$\int_0^\infty g(x) x^{n-1} dx = \int_0^\infty (a^{1/k} - bx)_+^k x^{n-1} dx \tag{4.1.23}$$

and $a = g(0)$. Direct computation shows that

$$\int_0^\infty (a^{1/k} - bx)_+^{k+2} x^{n-1} dx = c_{n,k} a^{2/k} \int_0^\infty (a^{1/k} - bx)_+^k x^{n-1} dx, \tag{4.1.24}$$

where

$$c_{n,k} = \frac{(k+1)(k+2)}{(n+k+1)(n+k+2)}. \tag{4.1.25}$$

Combining with (4.1.22) and (4.1.23) we get

$$\int_0^\infty g(x)^{1+2/k} x^{n-1} dx \geq c_{n,k} g(0)^{2/k} \int_0^\infty g(x) x^{n-1} dx. \tag{4.1.26}$$

In other words,

$$\int_0^\infty f(r\theta)^{1+2/k} r^{n-1} dr \geq c_{n,k} \|f\|_\infty^{2/k} \int_0^\infty f(r\theta) r^{n-1} dr \tag{4.1.27}$$

for every $\theta \in S^{n-1}$. Integrating this last inequality with respect to θ we get

$$\int f(x)^{1+2/k} dx \geq c_{n,k} \|f\|_\infty^{2/k} \int f(x) dx = c_{n,k} \|f\|_\infty^{2/k}. \tag{4.1.28}$$

We will also need the following generalization of Proposition 1.5.4 (see [24]).

Proposition 4.1.5. *Let K be a convex body in $\mathbb{R}^n$ with center of mass at the origin. Let $k < n$ and let E be a k-dimensional subspace of $\mathbb{R}^n$. Then,*

$$\max_{y\in E^\perp} |K\cap (E+y)| \leq \left(\frac{n+1}{k+1}\right)^k |K\cap E|. \tag{4.1.29}$$

□

Proposition 4.1.6. *Let K be a convex body in $\mathbb{R}^n$ with volume 1 and center of mass at the origin. Let E be a k-dimensional subspace of $\mathbb{R}^n$ and let T be a convex body in E with volume 1 and center of mass at the origin. Then, for every $\theta\in E$,*

$$\left(\frac{k+1}{n+1}\right)^2 |K\cap E|^{\frac{2}{k}} \int_T \langle z,\theta\rangle^2 dz \leq \int_{K(T,E)} \langle x,\theta\rangle^2 dx \tag{4.1.30}$$

and

$$\int_{K(T,E)} \langle x,\theta\rangle^2 dx \leq \left(\frac{n+1}{k+1}\right)^2 |K\cap E|^{\frac{2}{k}} \int_T \langle z,\theta\rangle^2 dz. \tag{4.1.31}$$

Proof. From Remark 1 we have

$$\int_{K(T,E)} \langle x,\theta\rangle^2 dx = \int_{P_{E^\perp}(K)} |K\cap (E+w)|^{1+\frac{2}{k}} dw \cdot \int_T \langle z,\theta\rangle^2 dz. \tag{4.1.32}$$

Let $g(y) = |K\cap (E+y)|$ on $P_{E^\perp(K)}$. By the Brunn-Minkowski inequality, $g^{1/k}$ is concave and $\int g(y)dy = |K| = 1$. Proposition 4.1.3 shows that

$$\frac{(k+1)(k+2)}{(n+1)(n+2)}\|g\|_\infty^{\frac{2}{k}} \int_T \langle z,\theta\rangle^2 dy \leq \int_{K(T,E)} \langle x,\theta\rangle^2 dx \leq \|g\|_\infty^{\frac{2}{k}} \int_T \langle z,\theta\rangle^2 dy. \tag{4.1.33}$$

Since K has its center of mass at the origin, Proposition 4.1.5 shows that

$$g(0) \leq \|g\|_\infty \leq \left(\frac{n+1}{k+1}\right)^k g(0). \tag{4.1.34}$$

On observing that $g(0) = |K\cap E|$ and

$$\frac{(k+1)(k+2)}{(n+1)(n+2)} \geq \left(\frac{k+1}{n+1}\right)^2,$$

we conclude the proof. □

Theorem 4.1.7. *Let K be an isotropic convex body in $\mathbb{R}^n$. Let E be a k-dimensional subspace of $\mathbb{R}^n$ and let T be a convex body in E with volume 1 and center of mass at the origin. Then,*

$$\left(\frac{k+1}{n+1}\right)^{\frac{k}{n}} L_{K(T,E)} \leq L_K^{1-\frac{k}{n}} L_T^{\frac{k}{n}} |K\cap E|^{\frac{1}{n}} \leq \left(\frac{n+1}{k+1}\right)^{\frac{k}{n}} L_{K(T,E)}. \tag{4.1.35}$$

Proof. We choose an orthonormal basis $\{e_1, \dots, e_n\}$ as in Lemma 4.1.2. Then,

$$L_{K(T,E)}^{2n} = \prod_{i=1}^{n} \int_{K(T,E)} \langle x, e_i\rangle^2 dx = L_K^{2(n-k)} \prod_{i=1}^{k} \int_{K(T,E)} \langle x, e_i\rangle^2 dx. \tag{4.1.36}$$

By the left hand side inequality of Proposition 4.1.6, this gives

$$L_{K(T,E)}^{2n} \geq \left(\frac{k+1}{n+1}\right)^{2k} L_K^{2(n-k)} |K \cap E|^2 \prod_{i=1}^{k} \int_T \langle x, e_i\rangle^2 dx, \tag{4.1.37}$$

and since $\{e_1, \dots, e_k\}$ is a basis of axes of inertia of T,

$$L_{K(T,E)}^{2n} \geq \left(\frac{k+1}{n+1}\right)^{2k} L_K^{2(n-k)} L_T^{2k} |K \cap E|^2. \tag{4.1.38}$$

It follows that

$$L_{K(T,E)} \geq \left(\frac{k+1}{n+1}\right)^{k/n} L_K^{1-\frac{k}{n}} L_T^{\frac{k}{n}} |K \cap E|^{\frac{1}{n}}. \tag{4.1.39}$$

Using the right hand side inequality of Proposition 4.1.6 and following the same argument, we get

$$L_{K(T,E)} \leq \left(\frac{n+1}{k+1}\right)^{k/n} L_K^{1-\frac{k}{n}} L_T^{\frac{k}{n}} |K \cap E|^{\frac{1}{n}}. \tag{4.1.40}$$

Note that $[(k+1)/(n+1)]^{k/n} \geq c$ for all $k < n$, where $c > 0$ is an absolute constant. □

4.2 Monotonicity in the dimension

For every $n \in \mathbb{N}$ we define

$$L_n = \sup\{L_K : K \text{ is an isotropic convex body in } \mathbb{R}^n\}. \tag{4.2.1}$$

In §1.6 we saw that if K and T are isotropic convex bodies in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, then

$$L_{K\times T} = L_K^{\frac{n}{n+m}} L_T^{\frac{m}{n+m}}. \tag{4.2.2}$$

Choosing K and T so that $L_K = L_n$ and $L_T = L_m$, we readily see that

$$L_{n+m}^{n+m} \geq L_n^n L_m^m \tag{4.2.3}$$

for all $n, m \in \mathbb{N}$. In particular, we have the following.

Lemma 4.2.1. *Let $k, n \in \mathbb{N}$. If k divides n, then $L_k \leq L_n$.* □

In this Section we will prove that the sequence L_n is "monotone" in the following precise sense.

Theorem 4.2.2. *There exists an absolute constant $C > 0$ such that: if $m, n \in \mathbb{N}$ and $m < n$ then $L_m \leq CL_n$.*

The proof will be based on Theorem 4.1.7 and on the following fact.

Proposition 4.2.3. *Let K be a convex body in $\mathbb{R}^n$ with volume 1 and center of mass at the origin. For every $k < n$ there exists a k-dimensional subspace E of $\mathbb{R}^n$ such that*

$$|K \cap E|^{1/n} \geq c, \tag{4.2.4}$$

where $c > 0$ is an absolute constant.

Proof. There exists an ellipsoid $\mathcal{E}$ such that $|K| = |\mathcal{E}|$ and

$$N(K, \mathcal{E}) \leq \exp(\beta n), \tag{4.2.5}$$

where $\beta > 0$ is an absolute constant (in the terminology of §1.6 this is an M-ellipsoid with constant β).

Averaging over all $(n-k)$-dimensional "coordinate subspaces" with respect to the axes of $\mathcal{E}$ we see that there exists a $(n-k)$-dimensional subspace F such that

$$|P_F(\mathcal{E})| \leq \frac{\omega_{n-k}}{\omega_n^{(n-k)/n}} \leq \left(\frac{n}{n-k}\right)^{\frac{n-k}{2}}. \tag{4.2.6}$$

From (4.2.5) we see that

$$|P_F(K)|^{1/n} \leq e^{\beta} |P_F(\mathcal{E})|^{1/n} \leq e^{\beta} \left(\frac{n}{n-k}\right)^{\frac{n-k}{2n}} \leq C = C(\beta). \tag{4.2.7}$$

Define $E = F^{\perp}$. Using Proposition 4.1.5 and Fubini's theorem we get

$$1 = |K| \leq \max_{y \in E^{\perp}} |K \cap (y + E)| |P_F(K)| \leq \left(\frac{n+1}{k+1}\right)^k |K \cap E|. \tag{4.2.8}$$

Then, (4.2.7) shows that

$$|K \cap E|^{1/n} \geq \frac{1}{C} \left(\frac{k+1}{n+1}\right)^{k/n} \geq c, \tag{4.2.8}$$

where $c > 0$ is an absolute constant. $\square$

Proof of Theorem 4.2.2: Let $m, n \in \mathbb{N}$ with $m < n$. Consider the largest integer s for which $2^s m \leq n$. Lemma 4.2.1 shows that

$$L_{2^s m} \geq L_m. \tag{4.2.9}$$

Set $k = 2^s m$. Let K be an isotropic convex body in $\mathbb{R}^n$ such that $L_K = L_n$. From Proposition 4.2.3 there exists a k-dimensional subspace E satisfying

$$|K \cap E|^{1/n} \geq c. \tag{4.2.10}$$

Let T be an isotropic convex body in E such that $L_T = L_k$. Since K has extremal isotropic constant, using Theorem 4.1.7 and (4.2.10) we get

$$L_K \geq L_{K(T,E)} \geq L_K^{1-\frac{k}{n}} L_T^{\frac{k}{n}} |K \cap E|^{\frac{1}{n}} \left(\frac{n+1}{k+1}\right)^{-\frac{k}{n}}, \tag{4.2.11}$$

which shows that

$$L_T \leq L_K \cdot \frac{n+1}{k+1} \left(\frac{1}{c}\right)^{\frac{n}{k}} \leq CL_K \tag{4.2.12}$$

because $n < 2k$. By the definition of K and T we have

$$L_m \leq L_k = L_T \leq CL_K = CL_n. \tag{4.2.13}$$

□

4.3 Generalizations of Busemann's inequality

Let K be a symmetric convex body in $\mathbb{R}^n$. Busemann's inequality states that the function

$$x \mapsto \frac{|x|}{|K \cap x^{\perp}|} \tag{4.3.1}$$

is a norm. This fact is a special case (take $k = 2$) of the following Theorem.

Theorem 4.3.1. *Let K be a symmetric convex body in $\mathbb{R}^n$. Let $2 \leq k \leq n-1$ and let E be a k-codimensional subspace of $\mathbb{R}^n$. Let $F = E^{\perp}$ and, for every $z \in F$ define $E(z) = \{x + tz : x \in E, t > 0\}$. Then, the function*

$$z \mapsto \frac{|z|}{|K \cap E(z)|} \tag{4.3.2}$$

is a norm on F.

Proof. Let z_1 and z_2 be linearly independent vectors in F. Set $z_3 = z_1 + z_2$ and define

$$f_i(t) = \big|K \cap \big(t(z_i/|z_i|) + E\big)\big| \tag{4.3.3}$$

for all $t > 0$, $i = 1, 2, 3$. Then,

$$F_i = |K \cap E(z_i)| = \int_0^{\infty} f_i(t)dt. \tag{4.3.4}$$

We will prove that

$$\frac{|z_3|}{F_3} \leq \frac{|z_1|}{F_1} + \frac{|z_2|}{F_2}. \tag{4.3.5}$$

Let $t_1, t_2 > 0$ and let $y_i = t_i z_i/|z_i|$, $i = 1, 2$. The segment $[y_1, y_2]$ intersects the ray in the direction of z_3 at the point $y_3 = t_3 z_3/|z_3|$. Writing $y_3 = \alpha y_1 + (1-\alpha)y_2$, we see that

$$\alpha = \frac{t_2/|z_2|}{t_1/|z_1| + t_2/|z_2|}. \tag{4.3.6}$$

Then, from the equation

$$\frac{\alpha t_1}{|z_1|}z_1 + \frac{(1-\alpha)t_2}{|z_2|}z_2 = \frac{t_3}{|z_3|}z_3 \tag{4.3.7}$$

we get

$$\frac{|z_3|}{t_3} = \frac{|z_1|}{t_1} + \frac{|z_2|}{t_2}. \tag{4.3.8}$$

For every $s \in [0,1]$ we define $t_1(s)$ and $t_2(s)$ by the equations

$$s = \frac{1}{F_1}\int_0^{t_1(s)} f_1(u)du = \frac{1}{F_2}\int_0^{t_2(s)} f_2(u)du. \tag{4.3.9}$$

We have

$$\frac{dt_i}{ds} = \frac{F_i}{f_i(t_i(s))}, \tag{4.3.10}$$

and differentiating (4.3.8) we see that

$$\frac{|z_3|}{t_3^2(s)}\frac{dt_3}{ds} = \frac{|z_1|}{t_1^2(s)}\frac{F_1}{f_1(t_1(s))} + \frac{|z_2|}{t_2^2(s)}\frac{F_2}{f_2(t_2(s))}. \tag{4.3.11}$$

Applying the Brunn-Minkowski inequality (in log-concave form) we see that

$$f_3(t_3(s)) \geq f_1(t_1(s))^{\alpha} f_2(t_2(s))^{1-\alpha}. \tag{4.3.12}$$

We write

$$\frac{F_3}{|z_3|} \geq \int_0^1 \frac{f_3(t_3(s))}{|z_3|}\frac{dt_3}{ds}ds. \tag{4.3.13}$$

Now, the integrand is greater than or equal to

$$\frac{t_3^2(s)}{|z_3|^2}\left(\frac{|z_1|}{t_1^2(s)}\frac{F_1}{f_1(t_1(s))} + \frac{|z_2|}{t_2^2(s)}\frac{F_2}{f_2(t_2(s))}\right) f_1(t_1(s))^{\alpha} f_2(t_2(s))^{1-\alpha}. \tag{4.3.14}$$

If we set $a = |z_1|/t_1$ and $b = |z_2|/t_2$, from (4.3.6) and (4.3.8) we may write the last expression in the form

$$\frac{1}{(a+b)^2}\left(a^2\frac{F_1}{|z_1|f_1(t_1(s))} + b^2\frac{F_2}{|z_2|f_2(t_2(s))}\right) f_1(t_1(s))^{\frac{a}{a+b}} f_2(t_2(s))^{\frac{b}{a+b}}. \tag{4.3.15}$$

By the arithmetic-geometric means inequality,

$$a\left(\frac{aF_1}{|z_1|f_1(t_1)}\right) + b\left(\frac{bF_2}{|z_2|f_2(t_2)}\right) \geq (a+b)\left(\frac{aF_1}{|z_1|f_1(t_1)}\right)^{\frac{a}{a+b}}\left(\frac{bF_2}{|z_2|f_2(t_2)}\right)^{\frac{b}{a+b}},$$

so the integrand in (4.3.13) is greater than

$$\frac{1}{(a+b)}\left(\frac{aF_1}{|z_1|}\right)^{\frac{a}{a+b}}\left(\frac{bF_2}{|z_2|}\right)^{\frac{b}{a+b}}. \tag{4.3.16}$$

Applying once again the arithmetic-geometric means inequality, we see that

$$\begin{aligned}\left(\frac{aF_1}{|z_1|}\right)^{\frac{a}{a+b}}\left(\frac{bF_2}{|z_2|}\right)^{\frac{b}{a+b}} &\geq \left(\frac{a}{a+b}\frac{|z_1|}{aF_1}+\frac{b}{a+b}\frac{|z_2|}{bF_2}\right)^{-1}\\ &= (a+b)\left(\frac{|z_1|}{F_1}+\frac{|z_2|}{F_2}\right)^{-1}.\end{aligned}$$

This shows that the integrand in (4.3.13) is greater $\left(\frac{|z_1|}{F_1}+\frac{|z_2|}{F_2}\right)^{-1}$, which proves (4.3.5). □

A generalization of Theorem 4.3.1 (whose proof follows the same lines) is given in the next Theorem.

Theorem 4.3.2. *Let K be a symmetric convex body in $\mathbb{R}^n$. Let $2 \leq k \leq n-1$ and let E be a k-codimensional subspace of $\mathbb{R}^n$. Let $F = E^{\perp}$ and, for every $z \in F$ define $E(z) = \{x + tz : x \in E, t > 0\}$. Then, for every $p \geq 0$ the function*

$$\|z\|_p = \frac{|z|^{1+\frac{p}{p+1}}}{\left(\int_{K\cap E(z)} |\langle x, z\rangle|^p dx\right)^{\frac{1}{p+1}}} \tag{4.3.17}$$

is a norm on F. □

Theorem 4.3.3. *Let K be an isotropic symmetric convex body in $\mathbb{R}^n$. Let $2 \leq k \leq n-1$ and let E be a k-codimensional subspace of $\mathbb{R}^n$. Let $F = E^{\perp}$ and write W for the unit ball of $(F, \|\cdot\|_{k+1})$, where $\|\cdot\|_p$ is the norm defined in Theorem 4.3.2. Then,*

$$c_1\frac{L_W}{L_K} \leq |K\cap E|^{1/k} \leq c_2\frac{L_W}{L_K} \tag{4.3.18}$$

where $c_1, c_2 > 0$ are absolute constants.

Proof. Let $y \in F$. For every $x \in \mathbb{R}^n$ we write $x = x_1 + \rho\theta$ where $x_1 \in E$, $\rho \geq 0$ and $\theta \in S(F)$. Then,

$$\int_K \langle x, y\rangle^2 dx = k\omega_k \int_{S(F)} \langle \theta, y\rangle^2 \left(\int_{K\cap E(\theta)} |\langle w, \theta\rangle|^{k+1} dw\right)\sigma_F(d\theta). \tag{4.3.19}$$

In other words,

$$L_K^2 = k\omega_k \int_{S(F)} \langle\theta, y\rangle^2 \frac{1}{\|\theta\|_W^{k+2}}\sigma_F(d\theta) = (k+2)\int_W \langle x, y\rangle^2 dx. \tag{4.3.20}$$

This means that $W/|W|^{1/k}$ is isotropic, and

$$L_K^2 = (k+2)|W|^{1+\frac{2}{k}} L_W^2. \tag{4.3.21}$$

Observe that

$$|W|^{1+\frac{2}{k}} = \left(\omega_k \int_{S(F)} \left(\int_{K\cap E(\theta)} |\langle x, \theta\rangle|^{k+1} dx\right)^{\frac{k}{k+2}} \sigma_F(d\theta)\right)^{\frac{k+2}{k}}, \tag{4.3.22}$$

while

$$1 = |K| = \omega_k \int_{S(F)} \int_{K\cap E(\theta)} |\langle x,\theta\rangle|^{k-1} dx \sigma_F(d\theta). \tag{4.3.23}$$

Using the results of §2.1 we check that

$$(k+2)|W|^{1+\frac{2}{k}} \simeq |K\cap E|^{-\frac{2}{k}}. \tag{4.3.24}$$

The Theorem is a consequence of (4.3.21) and (4.3.24). □

Remark 1: All the results of this Section can be stated for not necessarily symmetric convex bodies. For example, in Theorem 4.3.2 the conclusion would be that

$$\rho_p(\theta) = \left(\int_{K\cap E(\theta)} \langle x,\theta\rangle^p dx \right)^{\frac{1}{p+1}} \tag{4.3.25}$$

defined on $F \setminus \{0\}$ is the radial function of a convex body. In fact, the only assumption needed is that $0 \in \mathrm{int}(K)$. Then, Theorem 4.3.3 is also true (with the obvious modification).

4.4 Reduction to bounded volume ratio

Recall the definition of volume ratio: If K is a convex body in $\mathbb{R}^n$ with center of mass at the origin, then

$$\mathrm{vr}(K) = \inf \left\{ \left(\frac{|K|}{|\mathcal{E}|} \right)^{1/n} : \mathcal{E} \text{ is an 0-symmetric ellipsoid in } K \right\}. \tag{4.4.1}$$

For every $\alpha > 1$ define

$$L_n(\alpha) = \sup\{L_K : K \text{ is isotropic in } \mathbb{R}^n \text{ and } \mathrm{vr}(K) \le \alpha\}. \tag{4.4.2}$$

In this Section we will prove the following reduction of the slicing problem.

Theorem 4.4.1. *There exist two constants $c > 0$ and $\alpha > 1$ such that*

$$L_n \le c[L_n(\alpha)]^4 \tag{4.4.3}$$

for all n.

In the rest of this Section we assume that K is an isotropic convex body in $\mathbb{R}^n$ such that

$$L_K = L_n = \sup\{L_K : K \text{ is an isotropic convex body in } \mathbb{R}^n\}. \tag{4.4.4}$$

As a consequence of Theorem 4.3.3 and of the monotonicity of the sequence $\{L_s\}$, we have the following property of K.

Proposition 4.4.2. *Let K be an isotropic convex body in $\mathbb{R}^n$ with $L_K = L_n$. For every k-codimensional subspace E of $\mathbb{R}^n$,*

$$|K\cap E|^{1/k} \le C_1, \tag{4.4.5}$$

where $C_1 > 0$ is an absolute constant.

Proof. Theorem 4.3.3 shows that there exists a k-dimensional symmetric convex body W such that

$$|K \cap E|^{1/k} \leq c_2 \frac{L_W}{L_K} \tag{4.4.6}$$

where $c_2 > 0$ is an absolute constant. On the other hand, by Theorem 4.2.2 we have

$$L_W \leq L_k \leq C L_n = C L_K \tag{4.4.7}$$

for some absolute constant $C > 0$, and the result follows. $\square$

Proposition 4.4.2 imposes strong conditions on the axes of any M-ellipsoid of K.

Proposition 4.4.3. *Let K be an isotropic convex body in $\mathbb{R}^n$ with $L_K = L_n$. Let $\mathcal{E}$ be an ellipsoid with $|K| = |\mathcal{E}|$ and $N(K, \mathcal{E}) \leq \exp(\beta n)$. If*

$$\mathcal{E} = \Big\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} \frac{\langle x, e_i \rangle^2}{n\lambda_i^2} \leq 1 \Big\}, \tag{4.4.8}$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of $\mathbb{R}^n$ and $\lambda_1 \leq \ldots \leq \lambda_n$, then

$$\lambda_{[n/2]+1} \leq C_2(\beta) \tag{4.4.9}$$

for some constant $C_2(\beta) \leq c \exp(2\beta)$.

Proof. Let $k = [n/2]$ and let $E = \mathrm{span}\{e_1, \ldots, e_k\}$. In the proof of Proposition 4.2.3 we saw that

$$1 = |K| \leq \left(\frac{n+1}{k+1} \right)^k |K \cap E^{\perp}| \cdot |P_E(K)|. \tag{4.4.10}$$

Proposition 4.4.2 shows that

$$|K \cap E^{\perp}| \leq C_1^k. \tag{4.4.11}$$

Also, $P_E(K)$ is covered by at most $\exp(\beta n)$ translates of $P_E(\mathcal{E})$, so

$$|P_E(K)| \leq \exp(\beta n) \omega_k n^{k/2} \prod_{i=1}^{k} \lambda_i. \tag{4.4.12}$$

It follows that

$$1 \leq \left(\frac{n+1}{k+1} \right)^k \exp(\beta n) \omega_k n^{k/2} C_1^k \prod_{i=1}^{k} \lambda_i. \tag{4.4.13}$$

On the other hand, since $|\mathcal{E}| = 1$, we have

$$\prod_{i=1}^{n} \lambda_i = \frac{1}{n^{n/2} \omega_n}. \tag{4.4.14}$$

Therefore,

$$\lambda_{k+1}^{n-k} \leq \prod_{i=k+1}^{n} \lambda_i \leq \left(\frac{n+1}{k+1}\right)^k C_1^k \exp(\beta n) \frac{\omega_k n^{k/2}}{\omega_n n^{n/2}}, \tag{4.4.15}$$

which implies

$$\lambda_{k+1} \leq C(\beta) \tag{4.4.16}$$

for some constant depending only on β. □

The last ingredient of the proof is the fact that every convex body which is in M-position has orthogonal projections of proportional dimension with bounded volume ratio. For the proof we need the "volume ratio theorem" (see [30], [55] and [56]).

Theorem 4.4.4. *Let K be a convex body in $\mathbb{R}^n$ such that $B_2^n \subseteq K$ and $|K| = \alpha^n |B_2^n|$ for some $\alpha > 1$. For every $1 \leq k \leq n$, a random subspace $E \in G_{n,k}$ satisfies with probability greater than $1 - e^{-n}$:*

$$B_E \subseteq K \cap E \subseteq (c\alpha)^{\frac{n}{n-k}} B_E, \tag{4.4.17}$$

where $c > 0$ is an absolute constant.

For completeness we give a proof of the volume ratio theorem which was recently given by Klartag (see [31]).

Lemma 4.4.5. *Let K be a convex body in $\mathbb{R}^n$. For every $1 \leq k \leq n$,*

$$\int_{G_{n,k}} \int_{K \cap E} |x|^{n-k} dx \nu_{n,k}(dE) = \frac{k\omega_k}{n\omega_n} |K|. \tag{4.4.18}$$

Proof. Let χ_K be the indicator function of K. Integrating in polar coordinates we get

$$\begin{aligned} \int_{G_{n,k}} \int_{K \cap E} |x|^{n-k} dx d\nu_{n,k} &= \int_{G_{n,k}} k\omega_k \int_{S(E)} \int_0^\infty \chi_K(r\theta) r^{n-1} dr d\sigma_E d\nu_{n,k} \\ &= k\omega_k \int_{S^{n-1}} \int_0^\infty \chi_K(r\theta) r^{n-1} \sigma(d\theta) \\ &= \frac{k\omega_k}{n\omega_n} \int_{\mathbb{R}^n} \chi_K(x) dx \\ &= \frac{k\omega_k}{n\omega_n} |K|. \end{aligned}$$

□

Lemma 4.4.6. *Let K be a convex body in $\mathbb{R}^n$ with $0 \in \mathrm{int}(K)$ and let $R(K) = \max\{|x| : x \in K\}$. Then,*

$$|\{x \in K : |x| > R(K)/2\}| \geq c^n |K|, \tag{4.4.19}$$

where $c > 0$ is an absolute constant.

Proof. We may assume that $|K| = 1$. There exists $\theta \in S^{n-1}$ such that $h_K(\theta) = r = R(K)$. Since

$$|\{x \in K : |x| > R(K)/2\}| \geq |\{x \in K : \langle x, \theta\rangle > r/2\}|, \tag{4.4.20}$$

it is enough to prove that

$$\int_{r/2}^{3r/4} f(t)dt \geq c^n, \tag{4.4.21}$$

where $f = f_{K,\theta}$.
Let $y_0 \in [-r, r]$ with $f(y_0) = \|f\|_\infty$. Suppose that $r/2 \leq x \leq 3r/4$.
Case 1: $y_0 \leq x \leq r$. Then, we write

$$x = \frac{r-x}{r-y_0}y_0 + \frac{x-y_0}{r-y_0}r, \tag{4.4.22}$$

and applying the Brunn-Minkowski inequality we get

$$f(x)^{\frac{1}{n-1}} \geq \frac{r-x}{r-y_0}f(y_0)^{\frac{1}{n-1}} \geq \frac{1}{8}\|f\|_\infty^{\frac{1}{n-1}}, \tag{4.4.23}$$

because $r - x \geq r/4$ and $r - y_0 \leq 2r$.
Case 2: $0 \leq x \leq y_0$. We write

$$f(x)^{\frac{1}{n-1}} \geq \frac{x}{y_0}f(y_0)^{\frac{1}{n-1}} + \frac{y_0 - x}{y_0}f(0)^{\frac{1}{n-1}} \geq \frac{1}{2}\|f\|_\infty^{\frac{1}{n-1}} \tag{4.4.24}$$

because $x \geq y_0/2$.
Now, observe that $1 = \int_{-r}^{r} f(t)dt \leq 2r\|f\|_\infty$, therefore

$$\int_{r/2}^{3r/4} f(t)dt \geq \frac{1}{8^{n-1}}\frac{r}{4}\|f\|_\infty \geq \frac{1}{8^n}. \tag{4.4.25}$$

This proves the Lemma, with $c = 1/8$. □

Proposition 4.4.7. *Let K be a convex body in $\mathbb{R}^n$ with $0 \in \mathrm{int}(K)$. For every $1 \leq k \leq n$, a random subspace $E \in G_{n,k}$ satisfies with probability greater than $1 - e^{-n}$:*

$$\frac{|K \cap E|}{\omega_k}(R(K \cap E))^{n-k} \leq c^n \frac{|K|}{\omega_n}, \tag{4.4.26}$$

where $c > 0$ is an absolute constant.

Proof. By Lemma 4.4.6, for every $E \in G_{n,k}$ we have

$$\frac{1}{8^k}\frac{R(K \cap E)^{n-k}}{2^{n-k}}|K \cap E| \leq \int_{K \cap E} |x|^{n-k}dx \leq R(K \cap E)^{n-k}|K \cap E|. \tag{4.4.27}$$

Integrating over $G_{n,k}$ and using Lemma 4.4.5, we get

$$\int_{G_{n,k}} \frac{|K \cap E|}{\omega_k} R(K \cap E)^{n-k}\nu_{n,k}(dE) \leq 16^n \frac{|K|}{\omega_n}. \tag{4.4.28}$$

The result now follows from Markov's inequality. □

Proof of Theorem 4.4.4: Let $E \in G_{n,k}$ be a subspace satisfying (4.4.26). Since $B_2^n \subseteq K$, we have $|K \cap E| \geq \omega_k$. Therefore,

$$(R(K \cap E))^{n-k} \leq (c\alpha)^n, \tag{4.4.29}$$

which means: $K \cap E \subseteq (c\alpha)^{\frac{n}{n-k}} B_E$. □

Proposition 4.4.8. *Let A be a convex body in $\mathbb{R}^k$. Assume that $|A| = |B_2^k|$ and $N(A, B_2^k) \leq \exp(\beta k)$ for some constant $\beta > 0$. Then, for any $1 \leq s \leq k$, a random orthogonal projection $P_F(A)$ of A onto an s-dimensional subspace F of $\mathbb{R}^k$ has volume ratio bounded by a constant $C(\beta, s/k)$.*

Proof. Recall that $|\mathrm{co}(A^\circ \cup B_2^k)|^{1/k} \leq C|B_2^k|^{1/k}$ where C depends on β. In other words, $W = \mathrm{co}(A^\circ \cup B_2^k)$ has bounded volume ratio, and Theorem 4.4.4 shows that for a random $E \in G_{k,s}$,

$$A^\circ \cap F \subseteq W \cap F \subseteq C(\beta)^{\frac{k}{k-s}} B_F. \tag{4.4.30}$$

By duality, this means that

$$P_F(A) \supseteq rB_F, \tag{4.4.31}$$

where $r = C(\beta)^{-\frac{k}{k-s}}$. Since

$$|P_F(A)| \leq N(P_F(A), B_F)|B_F| \leq N(A, B_2^k)|B_F| \leq \exp(\beta s)|B_F|, \tag{4.4.32}$$

this shows that

$$\big(|P_F(A)|/|rB_F|\big)^{1/s} \leq C(\beta, s/k) = \exp(\beta k/s) C(\beta)^{\frac{k}{k-s}}. \tag{4.4.33}$$

□

Proof of Theorem 4.4.1: There exists an absolute constant $\beta > 0$ and an ellipsoid $\mathcal{E}$ described as in Proposition 4.4.3, such that $|K| = |\mathcal{E}| = 1$ and $N(K, \mathcal{E}) \leq \exp(\beta n)$. Let $k = [n/2] + 1$. If $E = \mathrm{span}\{e_1, \ldots, e_k\}$, from Proposition 4.4.2 and (4.4.10) we have

$$|P_E(K)| \geq c_1^k. \tag{4.4.34}$$

Now,

$$N(P_E(K), P_E(\mathcal{E})) \leq N(K, \mathcal{E}) \leq \exp(2\beta k) \tag{4.4.35}$$

and Proposition 4.4.3 shows that

$$P_E(\mathcal{E}) \subseteq c\sqrt{k} B_E. \tag{4.4.36}$$

Let $\rho > 0$ be defined so that $|P_E(K)| = |\rho B_E|$. From (4.4.34) we see that $\rho \simeq \sqrt{k}$. This means that

$$\rho B_E \supseteq c_1 P_E(\mathcal{E}), \tag{4.4.37}$$

and (4.4.35) gives

$$N(P_E(K), \rho B_E) \leq \exp(c_2 \beta k). \tag{4.4.38}$$

So, we may apply Proposition 4.4.8 to find a subspace F of E with dimension $\dim F = s = [k/2]+1$, such that

$$\text{vr}(P_F(K)) = \text{vr}(P_F(P_E(K))) \leq C = C(\beta). \tag{4.4.39}$$

Actually, Proposition 4.4.8 shows that $P_F(K)$ contains a ball rB_F such that

$$|P_F(K)| \leq C(\beta)^s |rB_F|. \tag{4.4.40}$$

Consider the $(D_{F^\perp}, F^\perp)$-symmetrization of K, where $D_{F^\perp}$ is the Euclidean ball of volume 1 in $F^\perp$. We denote this body by K_1.

Claim: $\text{vr}(K_1) \leq \alpha$, *where* $\alpha > 1$ *is an absolute constant.*
Proof of the claim: We define the ellipsoid

$$\mathcal{F} = \{ax + by : a^2 + b^2 \leq 1, x \in rB_F, y \in D_{F^\perp}\}. \tag{4.4.41}$$

Note that $D_{F^\perp} = K_1 \cap F^\perp$. Then, it is not hard to check that

$$\frac{1}{\sqrt{2}}\mathcal{F} \subseteq \text{co}\{rB_F, K_1 \cap F^\perp\} \subseteq K_1. \tag{4.4.42}$$

On the other hand, $|\sqrt{2}\mathcal{F}| \geq |rB_F| \cdot |K_1 \cap F^\perp|$ and (4.4.40) shows that

$$|\mathcal{F}|^{1/n} \geq \frac{1}{\sqrt{2}C(\beta)^{s/n}} |P_F(K_1)|^{1/n} |K_1 \cap F^\perp|^{1/n} \geq \frac{1}{\sqrt{2}C'(\beta)}. \tag{4.4.43}$$

This shows that $\text{vr}(K_1) \leq \alpha$, where $\alpha = 2C'(\beta)$. □

We now use Theorem 4.1.7: Since $s \geq n/4$, we have

$$L_K^{\frac{1}{4}} L_{D_{F^\perp}}^{\frac{3}{4}} |K \cap F^\perp|^{\frac{1}{n}} \leq c_1 L_{K_1}. \tag{4.4.43}$$

Also, from (4.4.38) we have $|P_F(K)| \leq \exp(c\beta n)|\rho B_F|$, which gives

$$|P_F(K)|^{1/n} \leq C''(\beta). \tag{4.4.44}$$

This means that

$$|K \cap F^\perp|^{\frac{1}{n}} \geq c|P_F(K)|^{-\frac{1}{n}} \geq c(\beta). \tag{4.4.45}$$

Going back to (4.4.43) and taking into account the fact that $L_{D_{F^\perp}} \geq c$, we get

$$L_K \leq c_1(\beta) L_{K_1}^4. \tag{4.4.46}$$

Since $L_K = L_n$ and $L_{K_1} \leq L_n(\alpha)$, the proof is complete. □

Notes and References

All the results of this Chapter come from [14] (see also [13]). See [3] and [43] for complete proofs of the generalizations of Busemann's inequality which are outlined in §4.3.

Chapter 5

The 1-unconditional case

5.1 Bound for the isotropic constant

In this Chapter we study the case of symmetric convex bodies which generate a norm with 1-unconditional basis. After a linear transformation, we may assume that the canonical orthonormal basis $\{e_1, \dots, e_n\}$ is 1-unconditional basis for $\|\cdot\|_K$. That is, for every choice of real numbers $t_1, \dots, t_n$ and every choice of signs $\varepsilon_i = \pm 1$,

$$\big\|\varepsilon_1 t_1 e_1 + \cdots + \varepsilon_n t_n e_n\big\|_K = \big\|t_1 e_1 + \cdots + t_n e_n\big\|_K. \tag{5.1.1}$$

Geometrically, this means that if $x = (x_1, \dots, x_n) \in K$ then the whole rectangle $[-|x_1|, |x_1|] \times \cdots \times [-|x_n|, |x_n|]$ is contained in K.

Note that the matrix of inertia of such a body is diagonal, therefore one can bring it to the isotropic position by a diagonal operator. This explains that for every 1-unconditional convex body K in $\mathbb{R}^n$ there exists a linear image $\tilde{K}$ of K which has the following properties:

1. The volume of $\tilde{K}$ is equal to 1.
2. If $x = (x_1, \dots, x_n) \in \tilde{K}$ then $[-|x_1|, |x_1|] \times \cdots \times [-|x_n|, |x_n|] \subseteq \tilde{K}$.
3. For every $j = 1, \dots, n$,

$$\int_{\tilde{K}} x_j^2 dx = L_K^2. \tag{5.1.2}$$

This last condition implies that $\tilde{K}$ is in isotropic position, because

$$\int_{\tilde{K}} x_i x_j dx = 0 \ \text{ for all } i \neq j \tag{5.1.3}$$

by Property 2.

In the rest of this Chapter we assume that K has these three properties. It is not hard to prove that $L_K \leq C$ for some absolute constant $C > 0$. One way to see this is using the Loomis-Whitney inequality (which even holds without the convexity assumption) .

Lemma 5.1.1. *For every convex body K in $\mathbb{R}^n$,*

$$|K|^{n-1} \leq \prod_{i=1}^{n} |P_i(K)|, \tag{5.1.4}$$

where $P_i(K)$ is the orthogonal projection of K onto $e_i^{\perp}$. □

Now, since K has Property 2, it is clear that

$$P_i(K) = K \cap e_i^{\perp} \tag{5.1.5}$$

for all $i = 1, \ldots, n$. The Loomis-Whitney inequality implies that

$$|K \cap e_i^{\perp}| \geq 1 \tag{5.1.6}$$

for some $i \leq n$. From Proposition 1.5.6,

$$L_K^2 = \int_K x_i^2 dx \leq \frac{c}{|K \cap e_i^{\perp}|^2} \tag{5.1.7}$$

where $c > 0$ is an absolute constant. This proves that $L_K \leq C$, with $C = \sqrt{c}$.

Theorem 5.1.2. *There exists an absolute constant $C > 0$ such that $L_K \leq C$ for every n and every 1-unconditional convex body K in $\mathbb{R}^n$.* □

We shall give an independent proof of Theorem 5.1.2. It will be convenient to consider the normalized part

$$K^+ = 2K \cap \mathbb{R}_+^n \tag{5.1.8}$$

of K in $\mathbb{R}_+^n = [0, +\infty)^n$. In other words, if $x = (x_1, \ldots, x_n)$ is uniformly distributed in K, then $(2|x_1|, \ldots, 2|x_n|)$ is uniformly distributed in K^+. It is easy to check that K^+ has the following three properties:

4. The volume of K^+ is equal to 1.
5. If $x = (x_1, \ldots, x_n) \in K^+$ and $0 \leq y_j \leq x_j$ for all $1 \leq j \leq n$, then $y = (y_1, \ldots, y_n) \in K^+$.
6. For every $j = 1, \ldots, n$,

$$\int_{K^+} x_j^2 dx = 4L_K^2. \tag{5.1.9}$$

Theorem 5.1.3. *Let K be an isotropic 1-unconditional convex body in $\mathbb{R}^n$. Then, $L_K^2 \leq 1/2$.*

Proof. Property 5 shows that if $x = (x_1, \ldots, x_n) \in K^+$, then $[0, x_1] \times \cdots \times [0, x_n] \subseteq K^+$. It follows that

$$\prod_{j=1}^{n} x_j \leq 1 \tag{5.1.10}$$

for every $x \in K^+$. Define

$$V = \left\{ x \in \mathbb{R}_+^n : \prod_{j=1}^{n} x_j \geq 1 \right\}. \tag{5.1.11}$$

Then, V is convex and the sets K^+ and V have disjoint interiors. So, there exists a hyperplane H of the form

$$\lambda_1 x_1 + \cdots + \lambda_n x_n = \alpha \tag{5.1.12}$$

with $\lambda_j > 0$, which touches V and separates it from K^+. Since H touches V we can choose the λ_i's so that $\alpha = n$ and $\prod_{j=1}^n \lambda_j = 1$.

Since H separates V from K^+, we have

$$K^+ \subseteq \big\{x \in \mathbb{R}^N_+ : \lambda_1 x_1 + \cdots + \lambda_n x_n \leq n\big\}. \tag{5.1.13}$$

The arithmetic-geometric means inequality gives

$$\prod_{j=1}^n \left(\int_{K^+} x_j dx\right)^{\lambda_j/n} \leq \int_{K^+} \frac{\lambda_1 x_1 + \cdots + \lambda_n x_n}{n} dx \leq 1. \tag{5.1.14}$$

Now, Khintchine type inequalities for linear functionals (see §2.1) show that

$$4L_K^2 = \int_{K^+} x_j^2 dx \leq 2 \left(\int_{K^+} x_j dx\right)^2 \tag{5.1.15}$$

for all $j = 1, \ldots, n$ (for the exact constant 2 one needs to use a different version of the proof of Proposition 2.1.1, in the spirit of Lemma 4.1.4). If we set $d = \frac{\lambda_1 + \cdots + \lambda_n}{n}$, then

$$(2L_K^2)^d = \prod_{j=1}^n (2L_K^2)^{\lambda_j/n} \leq \prod_{j=1}^n \left(\int_{K^+} x_j dx\right)^{\lambda_j/n} \leq 1, \tag{5.1.16}$$

which shows that $L_K^2 \leq 1/2$. □

Combining with Proposition 1.5.5 we have:

Corollary 5.1.4. *Let K be an isotropic 1-unconditional convex body in $\mathbb{R}^n$. Then,*

$$|K \cap \theta^\perp| \geq \frac{1}{\sqrt{6}} \tag{5.1.17}$$

for every $\theta \in S^{n-1}$.

Proof. Let $\theta \in S^{n-1}$. Proposition 1.5.5 shows that

$$L_K = \left(\int_K \langle x, \theta\rangle^2 dx\right)^{1/2} \geq \frac{1}{2\sqrt{3}} \frac{1}{|K \cap \theta^\perp|}. \tag{5.1.18}$$

Note that if K is symmetric, then the factor e is not needed in the statement of Proposition 1.5.5. □

The basic source of all the results in this Chapter is the distributional inequality in the next Theorem.

Theorem 5.1.5. *Let K be an isotropic 1-unconditional convex body in $\mathbb{R}^n$. Then,*

$$|\{x \in K^+ : x_1 \geq \alpha_1, \ldots, x_n \geq \alpha_n\}| \leq \left(1 - \frac{\alpha_1 + \cdots + \alpha_n}{\sqrt{6}n}\right)^n, \tag{5.1.19}$$

for all $(\alpha_1, \ldots, \alpha_n) \in K^+$.

Proof. We define a function $u : K^+ \to \mathbb{R}^+$ by

$$u(\alpha_1, \ldots, \alpha_n) = |\{x \in K^+ : x_1 \geq \alpha_1, \ldots, x_n \geq \alpha_n\}|. \tag{5.1.20}$$

The Brunn-Minkowski inequality shows that the function $h = u^{\frac{1}{n}}$ is concave on K^+. Observe that $u(0) = 1$ and

$$\frac{\partial u}{\partial \alpha_j}(0) = -|K \cap e_j^\perp| \leq -\frac{1}{\sqrt{6}}, \tag{5.1.21}$$

where the last inequality comes from Corollary 5.1.4. Let $\alpha \in K^+$ and consider the function $h_\alpha : [0,1] \to \mathbb{R}$ defined by $h_\alpha(t) = h(\alpha t)$. Note that

$$h'_\alpha(0) = \sum_{j=1}^n \alpha_j \frac{\partial h}{\partial \alpha_j}(0) = \sum_{i=1}^n \alpha_i \cdot \frac{1}{n} \frac{\partial u}{\partial \alpha_j}(0) \leq -\frac{\alpha_1 + \cdots + \alpha_n}{\sqrt{6}n} \tag{5.1.22}$$

by (5.1.21). Since h is concave, h_α is concave on $[0,1]$. This implies that h'_α is decreasing on $[0,1]$, and hence,

$$h(\alpha) - 1 = h_\alpha(1) - h_\alpha(0) \leq h'_\alpha(0) \leq -\frac{\alpha_1 + \cdots + \alpha_n}{\sqrt{6}n} \tag{5.1.23}$$

for all $\alpha \in K^+$. This proves the Theorem. $\square$

As a direct consequence we get the following statement, which is valid for all $\alpha_j \geq 0$.

Corollary 5.1.6. *Let K be an isotropic 1-unconditional convex body in $\mathbb{R}^n$. Then,*

$$|\{x \in K^+ : x_1 \geq \alpha_1, \ldots, x_n \geq \alpha_n\}| \leq \exp(-c(\alpha_1 + \cdots + \alpha_n)), \tag{5.1.24}$$

for all $\alpha_1, \ldots, \alpha_n \geq 0$, where $c = 1/\sqrt{6}$.

Proof. If $(\alpha_1, \ldots, \alpha_n) \in K^+$ we just note that $1 - x \leq e^{-x}$ for all $x \geq 0$. If not, then the quantity on the left is equal to zero. $\square$

Another consequence of Theorem 5.1.5 is that an interior point $(\alpha_1, \ldots, \alpha_n)$ of K^+ necessarily satisfies

$$\alpha_1 + \cdots + \alpha_n < \sqrt{6}n. \tag{5.1.25}$$

This observation can be equivalently stated as follows.

Proposition 5.1.7. *Let K be an isotropic 1-unconditional convex body in $\mathbb{R}^n$. Then $K \subseteq \sqrt{3/2}nB_1^n$, where $B_1^n = \{x \in \mathbb{R}^n : |x_1| + \cdots + |x_n| \leq 1\}$ is the unit ball of ℓ_1^n.* $\square$

One last observation is that, on the other hand, K contains a large cube:

Proposition 5.1.8. *Let K be an isotropic 1-unconditional convex body in $\mathbb{R}^n$. Then,*

$$K \supseteq [-L_K/\sqrt{2}, L_K/\sqrt{2}]^n. \tag{5.1.26}$$

Proof. The center of mass $v = (v_1, \ldots, v_n)$ of K^+ is in K^+, therefore the rectangle $[0, v_1] \times \cdots \times [0, v_n]$ is contained in K^+, where

$$v_j = \int_{K^+} x_j dx \geq \sqrt{2} L_K. \tag{5.1.26}$$

Then,

$$[-v_1/2, v_1/2] \times \cdots \times [-v_n/2, v_n/2] \subseteq K, \tag{5.1.27}$$

and this proves the Proposition. Note that $L_K/\sqrt{2} \geq L_{B_2^n}/\sqrt{2} \geq 1/(2\sqrt{\pi e})$. $\square$

5.2 Tail estimates for the Euclidean norm

Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n_+$. We write $X_1, \ldots, X_n$ for its coordinates in decreasing order. That is,

$$\max x_j = X_1 \geq X_2 \geq \cdots \geq X_n = \min x_j. \tag{5.2.1}$$

Let μ_K^+ denote the uniform distribution on K^+. Corollary 5.1.6 has the following consequence.

Proposition 5.2.1. *Let K be an isotropic 1-unconditional convex body in $\mathbb{R}^n$. Then,*

$$\mu_K^+(\{x \in \mathbb{R}^n_+ : X_k \geq \alpha\}) \leq \binom{n}{k} e^{-ck\alpha} \tag{5.2.2}$$

for all $\alpha \geq 0$ and $1 \leq k \leq n$, where $c = 1/\sqrt{6}$.

Proof. Let $1 \leq j_1 < \cdots < j_k \leq n$. From Corollary 5.1.6 we have

$$\mu_K^+(\{x \in \mathbb{R}^n_+ : x_{j_1} \geq \alpha, \ldots, x_{j_k} \geq \alpha\}) \leq \exp(-ck\alpha). \tag{5.2.3}$$

Since

$$\{x \in \mathbb{R}^n_+ : X_k \geq \alpha\} = \bigcup_{1 \leq j_1 < \cdots < j_k \leq n} \{x \in \mathbb{R}^n_+ : x_{j_1} \geq \alpha, \ldots, x_{j_k} \geq \alpha\}, \tag{5.2.4}$$

we get

$$\begin{aligned} \mu_K^+(\{x : X_k \geq \alpha\}) &\leq \sum_{1 \leq j_1 < \cdots < j_k \leq n} \mu_K^+(\{x : x_{j_1} \geq \alpha, \ldots, x_{j_k} \geq \alpha\}) \\ &\leq \binom{n}{k} e^{-ck\alpha}. \end{aligned}$$

$\square$

Theorem 5.2.2. *Let K be an isotropic 1-unconditional convex body in $\mathbb{R}^n$. Then, for every $t > 0$*

$$|\{x \in K : \|x\|_1 \geq c_1(1+t)n\} \leq n \exp\left(-2t\frac{n}{\log n + 1}\right), \tag{5.2.5}$$

where $c_1 = \sqrt{6}$.

Proof. Let $\alpha_1, \dots, \alpha_n \geq 0$. From Proposition 5.2.1 we have

$$\begin{aligned}
\left|\left\{x \in K : \|x\|_1 \geq \sum_{k=1}^n \alpha_k\right\}\right| &= \mu_K^+\left(\left\{x \in \mathbb{R}_+^n : \sum_{k=1}^n x_k \geq 2\sum_{k=1}^n \alpha_k\right\}\right) \\
&= \mu_K^+\left(\left\{x \in \mathbb{R}_+^n : \sum_{k=1}^n X_k \geq 2\sum_{k=1}^n \alpha_k\right\}\right) \\
&\leq \sum_{k=1}^n \mu_K^+(\{x \in \mathbb{R}_+^n : X_k \geq 2\alpha_k\}) \\
&\leq \sum_{k=1}^n \binom{n}{k} \exp(-2ck\alpha_k),
\end{aligned}$$

where $c = 1/\sqrt{6}$. Since

$$\binom{n}{k} \leq \left(\frac{en}{k}\right)^k, \tag{5.2.6}$$

we get

$$\left|\left\{x \in K : \|x\|_1 \geq \sum_{k=1}^n \frac{\alpha_k}{c}\right\}\right| \leq \sum_{k=1}^n \exp\left(-k\left(2\alpha_k - \log\frac{en}{k}\right)\right). \tag{5.2.7}$$

We choose

$$\alpha_k = \frac{1}{2}\log\frac{en}{k} + \frac{t}{k}\frac{n}{\log n + 1}. \tag{5.2.8}$$

Since $\sum_{k=1}^n \frac{1}{k} \leq \log n + 1$ and $n! \geq (n/e)^n$, we get

$$\sum_{k=1}^n \alpha_k = \frac{1}{2}\log\left(\frac{n^n e^n}{n!}\right) + t\frac{n}{\log n + 1}\sum_{k=1}^n \frac{1}{k} \leq n + nt. \tag{5.2.9}$$

Going back to (5.2.7) we get

$$\left|\left\{x \in K : \|x\|_1 \geq \sqrt{6}(1+t)n\right\}\right| \leq n\exp\left(-2t\frac{n}{\log n + 1}\right). \tag{5.2.10}$$

□

Theorem 5.2.3. *Let K be an isotropic 1-unconditional convex body in $\mathbb{R}^n$. Then, for every $t \geq 4$*

$$|\{x \in K : |x| \geq c_2 t\sqrt{n}\}| \leq \exp\left(-\frac{t\sqrt{n}}{2}\right), \tag{5.2.11}$$

where $c_2 = \sqrt{6}$.

Proof. Let $\alpha_1, \ldots, \alpha_n \geq 0$. From Proposition 5.2.1 we have

$$\begin{aligned}
\left|\left\{x \in K : |x|^2 \geq \sum_{k=1}^n \alpha_k^2\right\}\right| &= \mu_K^+\left(\left\{x \in \mathbb{R}_+^n : \sum_{k=1}^n x_k^2 \geq 4\sum_{k=1}^n \alpha_k^2\right\}\right) \\
&= \mu_K^+\left(\left\{x \in \mathbb{R}_+^n : \sum_{k=1}^n X_k^2 \geq 4\sum_{k=1}^n \alpha_k^2\right\}\right) \\
&\leq \sum_{k=1}^n \mu_K^+(\{x \in \mathbb{R}_+^n : X_k \geq 2\alpha_k\}) \\
&\leq \sum_{k=1}^n \binom{n}{k} \exp(-2c\alpha_k).
\end{aligned}$$

This shows that

$$\left|\left\{x \in K : |x|^2 \geq \sum_{k=1}^n \frac{\alpha_k^2}{c^2}\right\}\right| \leq \sum_{k=1}^n \exp\left(-k\left(2\alpha_k - \log\frac{en}{k}\right)\right), \tag{5.2.12}$$

where $c = 1/\sqrt{6}$. We choose

$$\alpha_k = \frac{1}{2}\log\frac{en}{k} + t\frac{\sqrt{n}}{k}. \tag{5.2.13}$$

We check that if $t \geq 2$ then $\sum_{k=1}^n \alpha_k^2 \leq 4nt^2$, and going back to (5.2.13) we have

$$\left|\left\{x \in K : |x| \geq 2\sqrt{6}t\sqrt{n}\right\}\right| \leq n\exp(-2t\sqrt{n}) \leq \exp(-t\sqrt{n}) \tag{5.2.14}$$

for every $t \geq 2$. This proves the Theorem. □

5.3 Linear functionals on the ℓ_1^n-ball

Let B_1^n, the unit ball of ℓ_1^n, be equipped with the uniform distribution μ_n. The density of μ_n is given by

$$\frac{d\mu_n(x)}{dx} = \frac{n!}{2^n}\chi_{B_1^n}(x). \tag{5.3.1}$$

We also define $\Delta_n = \{x \in \mathbb{R}_+^n : x_1 + \cdots + x_n \leq 1\}$.

Let $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$ and consider the linear functional $f_\theta(x) = \theta_1 x_1 + \cdots + \theta_n x_n$.

Lemma 5.3.1. *For every $q \in \mathbb{N}$,*

$$\int |f_\theta(x)|^{2q} d\mu_n(x) = \frac{n!(2q)!}{(n+2q)!}\sum \theta_1^{2q_1} \ldots \theta_n^{2q_n}, \tag{5.3.2}$$

where the summation is over all non-negative integers $q_1, \ldots, q_n$ with $q_1 + \cdots + q_n = q$.

Proof. Using the fact that $\|\cdot\|_1$ is 1-unconditional, we write

$$\int |f_\theta|^{2q} d\mu_n = \sum \frac{(2q)!}{(2q_1)!\dots(2q_n)!}\theta_1^{2q_1}\dots\theta_n^{2q_n}\int x_1^{2q_1}\dots x_n^{2q_n}d\mu_n(x), \tag{5.3.3}$$

where the summation is over all non-negative integers $q_1,\dots,q_n$ with $q_1+\dots+q_n=q$. Now, a simple computation shows that

$$\int x_1^{2q_1}\dots x_n^{2q_n}d\mu_n(x) = n!\int_{\Delta_n} x_1^{2q_1}\dots x_n^{2q_n}dx = \frac{n!(2q_1)!\dots(2q_n)!}{(n+2q)!}. \tag{5.3.4}$$

This proves the Lemma. □

Theorem 5.3.2. *For every $\theta\in\mathbb{R}^n$,*

$$c_1\|\theta\|_\infty \le \sqrt{n}\|f_\theta\|_{L^{\psi_2}(\mu_n)} \le c_2\|\theta\|_\infty, \tag{5.3.5}$$

where $c_1 = 1/\sqrt{6}$ and $c_2 = 2\sqrt{2}$.

Proof. If $\alpha = \sqrt{n}\|\theta\|_\infty$, we have $\theta_1^{2q_1}\dots\theta_n^{2q_n} \le \alpha^{2q}/n^q$ for all $q_1,\dots,q_n\ge 1$ with $q_1+\dots+q_n=q$. Since the sum in (5.3.2) consists of $\binom{n+q-1}{n-1}$ terms, we have

$$\begin{aligned}\int |f_\theta(x)|^{2q}d\mu_n(x) &\le \frac{n!(2q)!}{(n+2q)!}\frac{(n+q-1)!}{(n-1)!q!}\frac{\alpha^{2q}}{n^q}\\ &= \frac{n}{(n+q)\cdots(n+2q)}\binom{2q}{q}\frac{q!\alpha^{2q}}{n^q}\\ &\le \frac{4^q q!\alpha^{2q}}{n^{2q}}.\end{aligned}$$

Therefore, if $|t|<1/(2\alpha)$, we have

$$\begin{aligned}\int \exp\left((tnf_\theta(x))^2\right)d\mu_n(x) &= 1+\sum_{q=1}^\infty \frac{t^{2q}n^{2q}}{q!}\int |f_\theta(x)|^{2q}d\mu_n(x)\\ &\le 1+\sum_{q=1}^\infty (2t\alpha)^{2q}\\ &= \frac{1}{1-4t^2\alpha^2}.\end{aligned}$$

If we choose $t = 1/(2\sqrt{2}\alpha)$, the last quantity becomes equal to 2. This shows that

$$n\|f_\theta\|_{L^{\psi_2}(\mu_n)} \le 2\sqrt{2}\sqrt{n}\|\theta\|_\infty. \tag{5.3.6}$$

We now turn to the lower bound. We may clearly assume that $\theta_j\ge 0$ for all $1\le j\le n$, and Lemma 5.3.1 shows that for every $q\ge 1$ the function $F_{2q}(\theta) := \|f_\theta\|_{2q}$ is increasing with respect to each coordinate on $\mathbb{R}^n_+$. It follows that $F_{\psi_2}(\theta) := \|f_\theta\|_{L^{\psi_2}(\mu_n)}$ has the same property. So,

$$\|f_\theta\|_{L^{\psi_2}(\mu_n)} \ge \|\theta\|_\infty\|f_{e_1}\|_{L^{\psi_2}(\mu_n)}. \tag{5.3.7}$$

Since $g_1 := f_1/\|f_1\|_{L^{\psi_2}(\mu_n)}$ has ψ_2-norm equal to 1, we have

$$2 \geq \int \exp\big([g_1(x)]^2\big) d\mu_n(x) \geq \frac{1}{n!} \frac{1}{\|f_{e_1}\|^{2n}_{L^{\psi_2}(\mu_n)}} \int |f_{e_1}(x)|^{2n} d\mu_n(x). \tag{5.3.8}$$

Taking into account (5.3.4), we get

$$\|f_{e_1}\|^{2n}_{L^{\psi_2}(\mu_n)} \geq \frac{1}{2n!} \frac{n!(2n)!}{(3n)!} \geq \frac{1}{2(3n)^n}. \tag{5.3.9}$$

This shows that

$$\sqrt{n}\|f_\theta\|_{L^{\psi_2}(\mu_n)} \geq \frac{\sqrt{n}}{\sqrt[2n]{2}\sqrt{3n}} \|\theta\|_\infty, \tag{5.3.10}$$

which is the left hand side inequality of the Theorem. □

Note that the proof of Theorem 5.3.2 gives the following estimate for $\|f_\theta\|_{L^p(\mu_n)}$, $p \geq 1$.

Proposition 5.3.3. *For every $\theta \in \mathbb{R}^n$ and every $p \geq 1$,*

$$\sqrt{n}\|f_\theta\|_p \leq c\sqrt{p}\|\theta\|_\infty, \tag{5.3.11}$$

where $c = 2\sqrt{2}$.

Proof. For $p = 2q$ this follows immediately (with $c = 2$) from the estimate

$$\int |f_\theta(x)|^{2q} d\mu_n(x) \leq \frac{4^q q! \alpha^{2q}}{n^{2q}} \tag{5.3.12}$$

where $\alpha = \sqrt{n}\|\theta\|_\infty$. It is then easily extended to all values of $p \geq 1$. □

We will prove a stronger statement. Define

$$C_n(\theta) = \|\theta\|_\infty \sqrt{n/\log n} \tag{5.3.13}$$

for $\theta \in \mathbb{R}^n$ and $n \geq 2$. Since the expectation of $\|\theta\|_\infty$ on S^{n-1} is of the order of $\sqrt{\log n/n}$, for a random $\theta \in S^{n-1}$ we have $C_n(\theta) \simeq 1$.

Theorem 5.3.4. *For every $\theta \in S^{n-1}$ and every $p \geq 2$,*

$$n\|f_\theta\|_{L^p(\mu_n)} \leq C \max\big\{\sqrt{p}, C_n(\theta)\sqrt{p\log p}\big\}, \tag{5.3.14}$$

where $C > 0$ is an absolute constant.

Remark 1: Observe that if $\theta \in S^{n-1}$ then

$$\max\big\{\sqrt{p}, C_n(\theta)\sqrt{p\log p}\big\} \leq \sqrt{np}\|\theta\|_\infty \tag{5.3.15}$$

for all $1 \leq p \leq n$. This shows that Theorem 5.3.4 gives stronger information than Proposition 5.3.3 in the range $p \in [1, n]$.

Lemma 5.3.5. *Let $q \geq 1$ be an integer and set*

$$P_q(y) = \sum_{q_1+\cdots+q_n=q} y_1^{q_1}\cdots y_n^{q_n} \tag{5.3.16}$$

on $\mathbb{R}^n_+$ (as always in this Section, the summation is over all non-negative integers $q_1,\dots,q_n$ with $q_1+\cdots+q_n=q$). If $y\in\mathbb{R}^n_+$ and $y_1+\cdots+y_n=1$, then

$$P_q(y) \leq (2e\max\{1/q,\|y\|_\infty\})^q\,. \tag{5.3.17}$$

Proof. Let $y\in\mathbb{R}^n_+$ with $y_1+\cdots+y_n=1$ and let $0\leq t\leq 1/(2\|y\|_\infty)$. If $a=ty$, then $0\leq a_i\leq 1/2$ and hence, $1/(1-a_i)\leq e^{2a_i}$ for every $i=1,\dots,n$. Consequently,
(5.3.18)

$$P_q(a) = \sum_{q_1+\cdots+q_n=q} a_1^{q_1}\cdots a_n^{q_n} \leq \sum_{q_i\geq 0} a_1^{q_1}\cdots a_n^{q_n} = \prod_{i=1}^n \frac{1}{(1-a_i)} \leq e^{2(a_1+\cdots+a_n)}.$$

In other words, for every $t\in[0,1/(2\|y\|_\infty)]$ we have

$$P_q(y) \leq \frac{e^{2t}}{t^q}. \tag{5.3.19}$$

Since $t^{-q}e^{2t}$ has a minimum at $q/2$, we get

$$P_q(y) \leq \left(\frac{2e}{q}\right)^q \tag{5.3.20}$$

if $\|y\|_\infty \leq 1/q$, and

$$P_q(y) \leq (2\|y\|_\infty)^q \exp(1/\|y\|_\infty) \tag{5.3.21}$$

if $\|y\|_\infty \geq 1/q$. $\square$

Proof of Theorem 5.3.4: Define $y=(\theta_1^2,\dots,\theta_n^2)$. From Lemma 5.3.1 and Lemma 5.3.5, for every $q\geq 1$ we have

$$\|f_\theta\|_{L^{2q}(\mu_n)}^{2q} \leq \frac{n!(2q)!}{(n+2q)!}\left(2e\max\{1/q,\|\theta\|_\infty^2\}\right)^q\,. \tag{5.3.22}$$

Since $(n+2q)!\geq n!n^{2q}$ and $(2q)!\leq (2q)^{2q}$, this gives

$$n\|f_\theta\|_{L^{2q}(\mu_n)} \leq 2\sqrt{2e}\max\Big\{\frac{1}{\sqrt{q}},\|\theta\|_\infty\Big\} = 2\sqrt{2e}\max\{\sqrt{q},q\|\theta\|_\infty\}. \tag{5.3.23}$$

Therefore, if $p\geq 2$ we find an integer $q\geq 1$ such that $q\leq p\leq 2q$ and write

$$\begin{aligned} n\|f_\theta\|_{L^p(\mu_n)} &\leq n\|f_\theta\|_{L^{2q}(\mu_n)} \leq 2\sqrt{2e}\max\{\sqrt{q},q\|\theta\|_\infty\} \\ &\leq 2\sqrt{2e}\max\{\sqrt{p},p\|\theta\|_\infty\} \\ &= 2\sqrt{2e}\max\{\sqrt{p},C_n(\theta)p\sqrt{\log n/n}\}. \end{aligned}$$

Now, it is easy to check that if $2\leq p\leq n$ we have $\frac{\log n}{n}\leq 2\frac{\log p}{p}$, and hence,

$$n\|f_\theta\|_{L^p(\mu_n)} \leq 4\sqrt{2e}\max\{\sqrt{p},C_n(\theta)\sqrt{p\log p}\}, \tag{5.3.24}$$

while, if $p>n$ then

$$n\|f_\theta\|_{L^p(\mu_n)} \leq n\|f_\theta\|_\infty = n\|\theta\|_\infty \leq C_n(\theta)\sqrt{n\log n} \leq C_n(\theta)\sqrt{p\log p}. \tag{5.3.25}$$

This proves the Theorem. $\square$

5.4 The comparison theorem

In Section 5.1 we saw that there exists an absolute constant $C > 0$ with the following property: if K is an isotropic 1-unconditional convex body in $\mathbb{R}^n$, then

$$|\{x \in K^+ : x_1 \geq \alpha_1, \dots, x_n \geq \alpha_n\}| \leq \left(1 - \frac{\alpha_1 + \cdots + \alpha_n}{Cn}\right)^n, \tag{5.4.1}$$

for all $(\alpha_1, \dots, \alpha_n) \in K^+$. In particular, $K \subseteq (C/2)nB_1^n$ (the constant $C = \sqrt{6}$ works). We fix such a constant C and define $V = (C/2)nB_1^n$. We also denote by μ_T the uniform distribution on a convex body T.

We say that a function $F : \mathbb{R}^n \to \mathbb{R}^+$ belongs to the class $\mathcal{F}_n$ if it satisfies the following:

1. F is symmetric with respect to each coordinate: we have

$$F(x_1, \dots, x_n) = F(\varepsilon_1 x_1, \dots, \varepsilon_n x_n)$$

for every $x \in \mathbb{R}^n$ and all choices of signs ε_i.

2. There exists a positive Borel measure ν on $\mathbb{R}_+^n$ which is finite on all compact sets, such that

$$F(x) = \nu([0, x_1] \times \cdots \times [0, x_n]) \tag{5.4.2}$$

for every $x = (x_1, \dots, x_n) \in \mathbb{R}_+^n$. If F is absolutely continuous with respect to Lebesgue measure, this means that there exists a measurable function $q : \mathbb{R}_+^n \to \mathbb{R}^+$ such that

$$F(x) = \int_0^{x_1} \cdots \int_0^{x_n} q(z)dz \tag{5.4.3}$$

for every $x = (x_1, \dots, x_n) \in \mathbb{R}_+^n$.

In this Section we prove the following comparison theorem.

Theorem 5.4.1. *Let $F \in \mathcal{F}_n$. For every isotropic 1-unconditional convex body K in $\mathbb{R}^n$ and every $t \geq 0$,*

$$\mu_K(\{x \in \mathbb{R}^n : F(x) \geq t\}) \leq \mu_V(\{x \in \mathbb{R}^n : F(x) \geq t\}). \tag{5.4.4}$$

In particular,

$$\int F(x) d\mu_K(x) \leq \int F(x) d\mu_V(x). \tag{5.4.5}$$

Proof. We will prove that for every $\alpha_1, \dots, \alpha_n \geq 0$,

$$\mu_K(|x_1| \geq \alpha_1, \dots, |x_n| \geq \alpha_n) \leq \mu_V(|x_1| \geq \alpha_1, \dots, |x_n| \geq \alpha_n). \tag{5.4.6}$$

We may assume that $\alpha = (\alpha_1, \dots, \alpha_n)$ is in the interior of K (otherwise, the quantity on the left is equal to zero). Then, we also have $\alpha \in V$.

Observe that (5.4.1) is equivalent to

$$\mu_K(|x_1| \geq \alpha_1, \dots, |x_n| \geq \alpha_n) \leq \left(1 - \frac{2(\alpha_1 + \cdots + \alpha_n)}{Cn}\right)^n. \tag{5.4.7}$$

On the other hand,

$$\begin{aligned}\mu_V(|x_1|\geq\alpha_1,\dots,|x_n|\geq\alpha_n) &= \frac{|\{x\in V:|x_1|\geq\alpha_1,\dots,|x_n|\geq\alpha_n\}|}{|V|}\\ &= \frac{|[(C/2)n-(\alpha_1+\cdots+\alpha_n)]B_1^n|}{|(C/2)nB_1^n|}\\ &= \left(1-\frac{2(\alpha_1+\ldots+\alpha_n)}{Cn}\right)^n.\end{aligned}$$

This proves (5.4.6). Observe that this is equivalent to (5.4.4) for the function $F_\alpha=\chi_{\{|x_1|\geq\alpha_1,\dots,|x_n|\geq\alpha_n\}}$. Note that F_α corresponds to the Dirac measure ν_α which gives a unit mass to the point α. Since the class $\{F_\alpha:\alpha\in\mathbb{R}^n_+\}$ coincides with the extreme points of the cone $\mathcal{F}_n$, the result follows. $\square$

5.5 Ψ_2-behavior of linear functionals

Let K be an isotropic 1-unconditional convex body in $\mathbb{R}^n$. For every $\theta=(\theta_1,\dots,\theta_n)\in\mathbb{R}^n$, consider the linear functional $f_\theta(x)=\theta_1x_1+\cdots+\theta_nx_n$. Note that for every $q\in\mathbb{N}$,

$$(5.5.1)\qquad \int|f_\theta(x)|^{2q}d\mu_K(x)=(2q)!\sum\frac{\theta_1^{2q_1}\dots\theta_n^{2q_n}}{(2q_1)!\cdots(2q_n)!}\int x_1^{2q_1}\cdots x_n^{2q_n}d\mu_K(x),$$

where the summation is over all non-negative integers $q_1,\dots,q_n$ with $q_1+\cdots+q_n=q$. From (5.4.6) it is clear that

$$(5.5.2)\qquad \int|x_1|^{p_1}\cdots|x_n|^{p_n}d\mu_K(x)\leq\int|x_1|^{p_1}\cdots|x_n|^{p_n}d\mu_V(x)$$

for all $p_1,\dots,p_n\geq 0$. This gives immediately the following.

Proposition 5.5.1. *Let K be an isotropic 1-unconditional convex body in $\mathbb{R}^n$. For every $\theta\in\mathbb{R}^n$ and every integer $q\geq 1$,*

$$(5.5.3)\qquad \int|f_\theta(x)|^{2q}d\mu_K(x)\leq\int|f_\theta(x)|^{2q}d\mu_V(x)$$

and

$$(5.5.4)\qquad \|f_\theta\|_{L^{\psi_2}(\mu_K)}\leq\|f_\theta\|_{L^{\psi_2}(\mu_V)}.$$

Proof. The second inequality follows from the first if we take the Taylor expansion of $\exp\big((f_\theta/r)^2\big)$ with $r=\|f_\theta\|_{L^{\psi_2}(\mu_V)}$ and integrate with respect to μ_K. $\square$

Observe that

$$(5.5.6)\qquad \|f_\theta\|_{L^{\psi_2}(\mu_V)}=\frac{Cn}{2}\|f_\theta\|_{L^{\psi_2}(\mu_n)}.$$

Then, the next Theorem follows from Theorem 5.3.2.

Theorem 5.5.2. *Let K be an isotropic 1-unconditional convex body in $\mathbb{R}^n$. For every $\theta \in \mathbb{R}^n$,*

$$\|f_\theta\|_{L^{\psi_2}(\mu_K)} \leq c\sqrt{n}\|\theta\|_\infty, \tag{5.5.7}$$

where $c > 0$ is an absolute constant. □

We also have a generalization of Theorem 5.3.4.

Theorem 5.5.3. *Let K be an isotropic 1-unconditional convex body in $\mathbb{R}^n$. For every $\theta \in \mathbb{R}^n$ and every $p \geq 2$,*

$$\|f_\theta\|_{L^p(\mu_K)} \leq c\max\{1, C_n(\theta)\}\sqrt{p\log p}, \tag{5.5.8}$$

where $c > 0$ is an absolute constant and $C_n(\theta) = \|\theta\|_\infty\sqrt{n/\log n}$. □

Notes and References

The results of this Chapter were proved by Bobkov and Nazarov (see [7] and [8]).

Chapter 6

Concentration of volume on isotropic convex bodies

6.1 Formulation of the problem

In this Chapter we discuss the following question about the concentration of volume on isotropic convex bodies.

QUESTION: *Do there exist an absolute constant $c > 0$ and a function $\phi : \mathbb{N} \to \mathbb{R}^+$ with $\phi(n) \to \infty$ as $n \to \infty$, such that for every isotropic convex body K in $\mathbb{R}^n$ the inequality*

$$\text{Prob}\left(\{x \in K : |x| \geq c\sqrt{n}L_K t\}\right) \leq \exp\left(-\phi(n)t\right) \tag{6.1.1}$$

holds true for every $t \geq 1$?

If we restrict ourselves to the class of isotropic 1-unconditional convex bodies, then Theorem 5.2.3 shows that the answer is positive, and one can even take $\phi(n) = \sqrt{n}$:

FACT 1: (Theorem 5.2.3) *There exists an absolute constant $c > 0$ such that if K is an isotropic 1-unconditional convex body in $\mathbb{R}^n$, then*

$$\text{Prob}\left(\{x \in K : |x| \geq c\sqrt{n}t\}\right) \leq \exp\left(-\sqrt{n}t\right) \tag{6.1.2}$$

for every $t \geq 1$. $\square$

Let us recall some general results of this kind. If K is an isotropic convex body in $\mathbb{R}^n$, then

$$\int_K |x|^2 dx = nL_K^2. \tag{6.1.3}$$

Applying Markov's inequality we see that $|K \cap (3\sqrt{n}L_K)B_2^n| \geq 8/9$, and Borell's lemma proves the following:

FACT 2: *If K is an isotropic convex body in $\mathbb{R}^n$, then*

$$\text{Prob}\left(\{x \in K : |x| \geq 3\sqrt{n}L_K t\}\right) \leq \exp(-t) \tag{6.1.4}$$

for every $t \geq 1$.

Alesker (see Theorem 2.2.4) showed that if K is isotropic, then the Euclidean norm $f(x) = |x|$ satisfies the ψ_2-estimate

$$\|f\|_{\psi_2} \leq c\sqrt{n}L_K, \tag{6.1.5}$$

where $c > 0$ is an absolute constant. In particular, we have the following improvement of the estimate in Fact 2:

FACT 3: *There exists an absolute constant $c > 0$ such that if K is an isotropic convex body in $\mathbb{R}^n$, then*

$$\mathrm{Prob}\left(\{x \in K : |x| \geq c\sqrt{n}L_K t\}\right) \leq 2\exp(-t^2) \tag{6.1.6}$$

for every $t \geq 1$.

Note that $L_K \simeq 1$ in the case of 1-unconditional convex bodies. Since the circumradius $R(K)$ of an isotropic convex body K in $\mathbb{R}^n$ is always bounded by cnL_K, the estimate of Bobkov and Nazarov is stronger than the previous ones for all $t \geq 1$.

6.2 A first reduction

Proposition 6.2.1. *Let K be an isotropic convex body in $\mathbb{R}^n$ which satisfies*

$$\mathrm{Prob}\left(\{x \in K : |x| \geq \gamma\sqrt{n}L_K t\}\right) \leq \exp\left(-\phi(n)t\right) \tag{6.2.1}$$

for every $t \geq 1$, where $\gamma \geq 1$ and $\phi(n) \gg 1$ are two constants. Then,

$$I_q(K) \leq 2\gamma\sqrt{n}L_K \tag{6.2.2}$$

for all $2 \leq q \leq c_1\phi(n)$, where $c_1 > 0$ is an absolute constant.

Proof. We write

$$\begin{aligned} I_q^q(K) &= \int_K |x|^q dx = \int_0^\infty qs^{q-1}\mathrm{Prob}\,(x \in K : |x| \geq s)\, ds \\ &\leq \int_0^{\gamma\sqrt{n}L_K} qs^{q-1}ds + \int_{\gamma\sqrt{n}L_K}^\infty qs^{q-1}\exp\left(-\frac{\phi(n)s}{\gamma\sqrt{n}L_K}\right) ds \\ &= \left(\gamma\sqrt{n}L_K\right)^q + \left(\frac{\gamma\sqrt{n}L_K}{\phi(n)}\right)^q \int_{\phi(n)}^\infty qt^{q-1}e^{-t}dt \\ &\leq \gamma^q n^{q/2} L_K^q \left(1 + \left(\frac{cq}{\phi(n)}\right)^q\right), \end{aligned}$$

where $c > 0$ is an absolute constant. If $cq \leq \phi(n)$, then

$$I_q(K) \leq 2^{1/q}\gamma\sqrt{n}L_K, \tag{6.2.3}$$

so we get the result for all $2 \leq q \leq c_1\phi(n)$, where $c_1 := 1/c$. $\square$

Proposition 6.2.2. *Let K be an isotropic convex body in $\mathbb{R}^n$ and let $\gamma \geq 1$, $\psi(n) > 2$ be two constants. If*

$$I_q(K) \leq \gamma\sqrt{n}L_K \tag{6.2.4}$$

for all $2 \leq q \leq \psi(n)$, then

$$\mathrm{Prob}\big(x \in K : |x| \geq c\gamma\sqrt{n}L_K t\big) \leq \exp\big(-\psi(n)t\big) \tag{6.2.5}$$

for every $t \geq 1$, where $c > 0$ is an absolute constant.

Proof. From Proposition 2.1.7 we have

$$\mathrm{Prob}\,(x \in K : |x| \geq 3CI_q(K)t) \leq e^{-qt} \tag{6.2.6}$$

for every $t \geq 1$, where $C > 0$ is an absolute constant. Setting $q = \psi(n)$ and using (6.2.4) we get

$$\mathrm{Prob}\big(x \in K : |x| \geq 3C\gamma\sqrt{n}L_K t\big) \leq \exp(-\psi(n)t) \tag{6.2.7}$$

for every $t \geq 1$, and the result follows with $c := 3C$. □

From Proposition 3.1.4 we have $w_q(Z_q(K)) \simeq \sqrt{q/n}I_q(K)$ for every isotropic convex body K in $\mathbb{R}^n$ and every $q \leq n$. Taking into account the previous two Propositions we may state the following.

Theorem 6.2.3. *Let $\phi(n) \gg 1$ and let K be an isotropic convex body in $\mathbb{R}^n$. For every $\gamma \geq 1$ the following are equivalent:*
(i) *For every $t \geq 1$,*

$$\mathrm{Prob}\big(x \in K : |x| \geq \gamma\sqrt{n}L_K t\big) \leq \exp\big(-\phi(n)t\big). \tag{6.2.8}$$

(ii) *For every $2 \leq q \leq c_1\phi(n)$,*

$$I_q(K) \leq c_2(\gamma)\sqrt{n}L_K, \tag{6.2.9}$$

where $c_2(\gamma) \simeq \gamma$.
(iii) *For every $2 \leq q \leq c_3\phi(n)$,*

$$w_q(Z_q(K)) \leq c_4(\gamma)\sqrt{q}L_K, \tag{6.2.10}$$

where $c_4(\gamma) \simeq \gamma$. □

6.3 Average decay of hyperplane sections

Let K be a convex body in $\mathbb{R}^n$ with volume 1 and center of mass at the origin. Recall that

$$f_{K,\theta}(t) = |K \cap (\theta^{\perp} + t\theta)| \tag{6.3.1}$$

for every $\theta \in S^{n-1}$ and $t \geq 0$. We define a function f_K, which measures the average decay of the volume of hyperplane sections of K, by

$$f_K(t) = \int_{S^{n-1}} f_{K,\theta}(t)\sigma(d\theta). \tag{6.3.2}$$

The next Proposition gives an integral formula for f_K (this proof is from [6]; see also [16]).

Proposition 6.3.1. *Let K be a convex body in $\mathbb{R}^n$ with volume 1 and center of mass at the origin. For every $t \geq 0$,*

$$f_K(t) = c_n \int_{U_K(t)} \frac{1}{|x|}\left(1 - \frac{t^2}{|x|^2}\right)^{\frac{n-3}{2}} dx, \tag{6.3.3}$$

where $U_K(t) = \{x \in K : |x| \geq t\}$ and

$$c_n = \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)}. \tag{6.3.4}$$

Proof. We denote by $\lambda_{\theta,t}$ the Lebesgue measure on the hyperplane $N_\theta(t) = \{x : \langle x, \theta\rangle = t\}$. Consider the measure

$$\lambda_t = \int_{S^{n-1}} \lambda_{\theta,t}\sigma(d\theta). \tag{6.3.5}$$

Then, λ_t is a positive measure on $\mathbb{R}^n$ and

$$f_K(t) = \int_{S^{n-1}}\int_{N_\theta(t)} \chi_K(x)d\lambda_{\theta,t}(x)\sigma(d\theta) = \lambda_t(K). \tag{6.3.6}$$

The density of λ_t is invariant under orthogonal transformations, therefore

$$\frac{d\lambda_t}{dx} = p_t(|x|) \tag{6.3.7}$$

where $p_t : [t, +\infty) \to [0, +\infty)$. In order to find p_t, for every $r > t$ we compute $\lambda_t(B(0,r))$ in two different ways. First, we have

$$\lambda_t(B(0,r)) = \int_{B(0,r)} p_t(|x|)dx = n\omega_n \int_t^r p_t(s)s^{n-1}ds. \tag{6.3.8}$$

On the other hand, since the intersection of $B(0,r)$ with the hyperplane $N_\theta(t)$ is an $(n-1)$-dimensional ball of radius $\sqrt{r^2 - t^2}$ for every $\theta \in S^{n-1}$, we get

$$\lambda_t(B(0,r)) = \int_{S^{n-1}} \lambda_{\theta,t}(B(0,r))\sigma(d\theta) = \omega_{n-1}(r^2 - t^2)^{\frac{n-1}{2}}. \tag{6.3.9}$$

Differentiating with respect to $r \geq t$, we see that

$$\omega_{n-1}\frac{n-1}{2}(r^2 - t^2)^{\frac{n-3}{2}}2r = n\omega_n p_t(r)r^{n-1}. \tag{6.3.10}$$

This shows that

$$p_t(r) = \frac{(n-1)\omega_{n-1}}{n\omega_n}\frac{(r^2 - t^2)^{\frac{n-3}{2}}}{r^{n-2}}. \tag{6.3.11}$$

Therefore,

$$f_K(t) = \int_{U_K(t)} p_t(|x|)dx = \frac{(n-1)\omega_{n-1}}{n\omega_n}\int_{U_K(t)} \frac{(|x|^2 - t^2)^{\frac{n-3}{2}}}{|x|^{n-2}}dx, \tag{6.3.12}$$

and the result follows. □

Corollary 6.3.2. *f_K is a decreasing function.* □

6.4 Reduction to the average sectional decay

In this Section we present one more reduction of the question, this time to the behavior of the function f_K. Our main tool will be some inequalities relating f_K to the generalized mean widths of the L_q-centroid bodies. It will be convenient to simplify the notation of Chapter 3.

Definition. Let K be an isotropic convex body in $\mathbb{R}^n$. For every $q > 0$ and $t > 0$ we define

$$Z(q) = w_q(Z_q(K)) = \left(\int_{S^{n-1}} \int_K |\langle x, \theta\rangle|^q dx\, \sigma(d\theta)\right)^{1/q} \tag{6.4.1}$$

and

$$Z(q,t) = \left(\int_{S^{n-1}} \int_{B_{K,\theta}(t)} |\langle x, \theta\rangle|^q dx\, \sigma(d\theta)\right)^{1/q} \tag{6.4.2}$$

where

$$B_{K,\theta}(t) = \{x \in K : |\langle x, \theta\rangle| \leq t\}. \tag{6.4.3}$$

Note that $Z(q,t) \leq Z(q)$ for every $t > 0$. Also, $Z(q)$ is a continuous and increasing function of q.

Lemma 6.4.1. *Let K be a convex body in $\mathbb{R}^n$ with volume 1 and center of mass at the origin. For every $t \geq 0$ we have the identity*

$$Z^q(q,t) = 2\int_0^t r^q f_K(r)dr. \tag{6.4.4}$$

Proof. It is an immediate consequence of Fubini's theorem:

$$\begin{aligned} Z^q(q,t) &= 2\int_{S^{n-1}} \int_0^t r^q f_{K,\theta}(r) dr \sigma(d\theta) = 2\int_0^t r^q \int_{S^{n-1}} f_{K,\theta}(r)\sigma(d\theta) dr \\ &= 2\int_0^t r^q f_K(r) dr, \end{aligned}$$

by the definition of f_K. □

Since $Z(q) = w_q(Z_q(K))$, Proposition 3.1.4 takes the following form.

Lemma 6.4.2. *Let K be a convex body in $\mathbb{R}^n$ with volume 1 and center of mass at the origin. Then,*

$$Z(q) \simeq \sqrt{\frac{q}{q+n}} I_q(K) \tag{6.4.5}$$

for every $q \geq 1$. □

For every $\theta \in S^{n-1}$ and $q \geq 1$ we write $H_q(\theta) = \|\langle \cdot, \theta\rangle\|_q$. Let $C \geq 1$ be the absolute constant from Theorem 2.1.5. The next Lemma shows that integration of the function $|\langle \cdot, \theta\rangle|^q$ on the strip $B_{K,\theta}(3\alpha H_q(\theta)s)$ - for suitable $s \simeq 1$ - essentially captures the value of $H_q^q(\theta)$.

Lemma 6.4.3. *Let K be a convex body in $\mathbb{R}^n$ with volume 1 and center of mass at the origin. Then, for every $\theta \in S^{n-1}$ and every $q, s \geq 1$,*

$$\left(1 - e^{-qs/2}(2C)^q\right) H_q^q(\theta) \leq \int_{B_{K,\theta}(3CH_q(\theta)s)} |\langle x, \theta\rangle|^q dx, \tag{6.4.6}$$

where $C > 0$ is the constant in Theorem 2.1.5.

Proof. Proposition 2.1.7 shows that

$$|K \setminus B_{K,\theta}(3CH_q(\theta)s)| \leq \exp(-qs) \tag{6.4.7}$$

for all $q, s \geq 1$. We write

$$\begin{aligned} H_q^q(\theta) &= \int_{B_{K,\theta}(3CH_q(\theta)s)} |\langle x, \theta\rangle|^q dx + \int_{K \setminus B_{K,\theta}(3CH_q(\theta)s)} |\langle x, \theta\rangle|^q dx \\ &\leq \int_{B_{K,\theta}(3CH_q(\theta)s)} |\langle x, \theta\rangle|^q dx + \exp(-qs/2) \left(\int_K |\langle x, \theta\rangle|^{2q} dx\right)^{1/2} \\ &\leq \int_{B_{K,\theta}(3CH_q(\theta)s)} |\langle x, \theta\rangle|^q dx + \exp(-qs/2)(2C)^q H_q^q(\theta), \end{aligned}$$

where we have used (6.4.7), Cauchy-Schwarz inequality and Theorem 2.1.5 (for the pair $q, 2q$). $\square$

Our main Lemma is the next one: it shows that $Z(q) \simeq Z(q,t)$ when t becomes of the order of $Z(q)$.

Proposition 6.4.4. *There exists an absolute constant $\beta > 0$ with the following property: if K is a convex body K in $\mathbb{R}^n$ with volume 1 and center of mass at the origin, then for every $q \geq 1$,*

$$Z^q(q) \leq 2Z^q(q, \beta Z(q)). \tag{6.4.8}$$

Proof. For every $t > 0$ we set $U_t = \{\theta \in S^{n-1} : H_q(\theta) \geq tZ(q)\}$. Markov's inequality shows that $\sigma(U_t) \leq t^{-q}$. Using Lemma 6.4.3, for every $s \geq 1$ we write

$$\begin{aligned} (1 - e^{-qs/2}(2C)^q) Z^q(q) &\leq \int_{S^{n-1} \setminus U_t} \int_{B_{K,\theta}(3CH_q(\theta)s)} |\langle x, \theta\rangle|^q dx\sigma(d\theta) \\ &\quad + \int_{U_t} \int_{B_{K,\theta}(3CH_q(\theta)s)} |\langle x, \theta\rangle|^q dx\sigma(d\theta) \\ &\leq \int_{S^{n-1}} \int_{B_{K,\theta}(3CtsZ(q))} |\langle x, \theta\rangle|^q dx\sigma(d\theta) \\ &\quad + \sigma(U_t)^{1/2} \left(\int_{S^{n-1}} \int_K |\langle x, \theta\rangle|^{2q} dx\sigma(d\theta)\right)^{1/2} \\ &\leq Z^q(q, 3CtsZ(q)) + t^{-q/2} Z^q(2q) \\ &\leq Z^q(q, 3CtsZ(q)) + (2C)^q t^{-q/2} Z^q(q), \end{aligned}$$

because $Z(2q) \leq 2CZ(q)$ (this follows from the fact that $I_{2q}(K,\theta) \leq 2CI_q(K,\theta)$ for every $\theta \in S^{n-1}$, by Theorem 2.1.5 applied to the functional $f(x) = \langle x, \theta\rangle$).

We now choose s, t so that $\sqrt{t} = 8C$ and $e^{s/2} = 8C$. Then,

$$(1 - 4^{-q})Z^q(q) \leq Z^q(q, 3CtsZ(q)) + 4^{-q}Z^q(q). \tag{6.4.9}$$

Inserting the values of t, s in (6.4.9) we compute the value of β. □

Remark 1: The argument gives

$$\big(1 - (2C)^q(e^{-qs/2} + t^{-q/2})\big)Z^q(q) \leq Z^q(q, 3CtsZ(q)) \tag{6.4.10}$$

for all $t, s \geq 1$.

Proposition 6.4.5. *There exists an absolute constant $c > 0$ with the following property: if K is a convex body in $\mathbb{R}^n$ with volume 1 and center of mass at the origin, then for every $q \geq 1$,*

$$f_K(cZ(q)) \leq \frac{1}{Z(q)} \exp(-q). \tag{6.4.11}$$

Proof. We set

$$G(t) = Z^q(q, t). \tag{6.4.12}$$

From (6.4.10) and the observation that $Z(q, t) \leq Z(q)$ we have

$$\big(1 - (2C)^q(e^{-qs/2} + t^{-q/2})\big)Z^q(q) \leq G(3CtsZ(q)) \leq Z^q(q) \tag{6.4.13}$$

for all $t, s \geq 1$. Lemma 6.4.1 and the fact that f_K is decreasing show that, if $u < v$ then

$$G(v) - G(u) = 2\int_u^v r^q f_K(r)dr \geq 2f_K(v)\frac{v^{q+1} - u^{q+1}}{q+1}. \tag{6.4.14}$$

It follows that for every $t, s \geq 1$ we have

$$f_K(6CtsZ(q)) \leq \frac{q+1}{2} \frac{(2C)^q(e^{-qs/2} + t^{-q/2})}{(6CtsZ(q))^{q+1} - (3CtsZ(q))^{q+1}} Z^q(q). \tag{6.4.15}$$

Choosing $s = 1$ and $t = e$ we have

$$f_K(6CeZ(q)) \leq \frac{q+1}{2^{q+1} - 1} \frac{1}{Z(q)} (3e/2)^{-q} e^{-q/2} \leq \frac{1}{Z(q)} e^{-q}, \tag{6.4.16}$$

which proves the Proposition with $c := 6Ce$. □

We can now see the exact relation of our question to the behavior of f_K.

Theorem 6.4.6. *Let $\gamma \geq 1$ and let K be an isotropic convex body in $\mathbb{R}^n$. If $1 \ll \phi(n) \ll n$ and*

$$\mathrm{Prob}\left(\{x \in K : |x| \geq \gamma\sqrt{n}L_K t\}\right) \leq \exp\left(-\phi(n)t\right) \tag{6.4.17}$$

for every $t \geq 1$, then

$$f_K(t) \leq \frac{c_1}{L_K} \exp\left(-c_2 t^2/(\gamma^2 L_K^2)\right) \tag{6.4.18}$$

for all $0 < t \leq c_3\gamma\sqrt{\phi(n)}L_K$.

Proof. We will use the integral formula for f_K. We assume that $n > 3$ and write

$$f_K(t) = c_n \int_{U_K(t)} g_t(|x|)dx \tag{6.4.19}$$

for all $t > 0$, where g_t is defined by

$$g_t(s) = \frac{1}{s}\left(1 - \frac{t^2}{s^2}\right)^{\frac{n-3}{2}} \tag{6.4.20}$$

on $[t, \infty)$, and $c_n \simeq \sqrt{n}$. Differentiating g_t we see that it is increasing on $[t, t\sqrt{n-2}]$ and then decreasing. Let $0 < t \leq c_3\gamma\sqrt{\phi(n)}L_K$, where the absolute constant $c_3 > 0$ is to be chosen. Assume first that $\gamma\sqrt{n}L_K \leq t\sqrt{n-2}$ (this is satisfied if $t \geq \sqrt{2}\gamma L_K$). Then, we write

$$\begin{aligned}
f_K(t) &= c_n \int_{K\cap\{t\leq|x|\leq\gamma\sqrt{n}L_K\}} g_t(|x|)dx + c_n \int_{U_K(\gamma\sqrt{n}L_K)} g_t(|x|)dx \\
&\leq c_n g_t(\gamma\sqrt{n}L_K) + \exp(-\phi(n))c_n g_t(t\sqrt{n-2}) \\
&= \frac{c_n}{\gamma\sqrt{n}L_K}\left(1 - \frac{t^2}{\gamma^2 n L_K^2}\right)^{\frac{n-3}{2}} + \exp(-\phi(n))\frac{c_n}{t\sqrt{n-2}}\left(1 - \frac{1}{n-2}\right)^{\frac{n-3}{2}} \\
&\leq \frac{c_1'}{L_K}\exp(-c_2t^2/\gamma^2L_K^2) + \frac{c_1''}{L_K}\exp(-\phi(n)) \\
&\leq \frac{c_1}{L_K}\exp(-c_2t^2/\gamma^2L_K^2),
\end{aligned}$$

because $\phi(n) \geq c_2t^2/\gamma^2L_K^2$ if we choose $c_3 = 1/\sqrt{c_2}$.
If $0 < t \leq \min\{\sqrt{2}\gamma L_K, c_3\gamma\sqrt{\phi(n)}L_K\}$, then

$$f_K(t) \leq \frac{c_5}{L_K} \leq \frac{2c_5}{L_K}\exp(-c_7t^2/\gamma^2L_K^2), \tag{6.4.21}$$

because $f_{K,\theta}(t) \leq c_5/L_K$ for all $\theta \in S^{n-1}$ and $\exp(-c_7t^2/\gamma^2L_K^2) \geq \exp(-2c_7) \geq 1/2$ if $c_7 > 0$ is suitably chosen. It follows that (6.4.18) holds true for all $0 < t \leq c_3\gamma\sqrt{\phi(n)}L_K$. $\square$

Second proof: From Theorem 6.2.3 we have

$$Z(q) = w_q(Z_q(K)) \leq c_2\gamma\sqrt{q}L_K \tag{6.4.22}$$

for all $2 \leq q \leq \psi(n) := c_1\phi(n)$.
Let $c > 0$ be the constant in Proposition 6.4.5. The function $D : [2, \psi(n)] \to \mathbb{R}$ defined by $D(q) = cZ(q)$ is continuous and increasing. Since $D(q) = cZ(q) \simeq \sqrt{q/n}\, I_q(K) \geq c_3\sqrt{q}L_K$ for every $2 \leq q \leq n$, we have

$$D([2, \psi(n)]) \supseteq [cL_K, c_3\sqrt{\psi(n)}L_K]. \tag{6.4.23}$$

Then, for every $t \in [cL_K, c_3\sqrt{\psi(n)}L_K]$ there exists $q(t) \in [2, \psi(n)]$ such that $cZ(q(t)) = D(q(t)) = t$. Moreover, $Z(q(t)) \leq c_2\gamma\sqrt{q(t)}L_K$ from (6.4.22). Therefore,

$$t \leq c_4\gamma\sqrt{q(t)}L_K. \tag{6.4.24}$$

Now, Proposition 6.4.5 shows that

$$f_K(t) \leq \frac{1}{Z(q(t))}\exp(-q(t)) \leq \frac{c}{t}\exp(-t^2/c_4^2\gamma^2 L_K^2). \tag{6.4.25}$$

Since $c/t \leq 1/L_K$, we obtain

$$f_K(t) \leq \frac{c_5}{L_K}\exp(-c_6 t^2/\gamma^2 L_K^2). \tag{6.4.26}$$

The same inequality is clearly correct for $0 < t < cL_K$: it suffices to change the value of the (absolute) constants c_5, c_6 if needed. □

The other direction is a consequence of Proposition 6.4.1.

Theorem 6.4.7. *Let $\gamma \simeq 1$ and let K be an isotropic convex body in $\mathbb{R}^n$. Assume that*

$$f_K(t) \leq \frac{c_1}{L_K}\exp\left(-t^2/\gamma^2 L_K^2\right) \tag{6.4.27}$$

for all $0 < t \leq \gamma\psi(n)L_K$. Then, for every $2 \leq q \leq c_2\psi^2(n)$ we have

$$I_q(K) \leq c_3\gamma\sqrt{n}L_K. \tag{6.4.28}$$

Proof. Note that $Z(2) = L_K$ and $Z(n) \geq c_0 R(K)$ for some absolute constant $c_0 > 0$.

We first assume that $\beta \leq \gamma\psi(n) < \beta c_0 R(K)/L_K$. Then, there exists $2 \leq s \leq n$ such that $\beta Z(s) = \gamma\psi(n)L_K$. From Lemma 6.4.1 and Proposition 6.4.4 we see that

$$\begin{aligned}
Z^s(s) &\leq 2Z^s(s, \beta Z(s)) = 4\int_0^{\beta Z(s)} r^s f_K(r)dr \\
&= 4\int_0^{\gamma\psi(n)L_K} r^s f_K(r)dr \\
&\leq \frac{4c_1}{L_K}\int_0^{\gamma\psi(n)L_K} r^s \exp(-r^2/\gamma^2 L_K^2)dr \\
&\leq \frac{4c_1}{L_K}\int_0^{\infty} r^s \exp(-r^2/\gamma^2 L_K^2)dr \\
&\leq (c_4\gamma\sqrt{s}L_K)^s.
\end{aligned}$$

In other words,

$$Z(s) \leq c_4\gamma\sqrt{s}L_K. \tag{6.4.29}$$

Lemma 6.4.2 implies that

$$I_s(K) \leq c_5\sqrt{\frac{n}{s}}Z(s) \leq c_3\gamma\sqrt{n}L_K, \tag{6.4.30}$$

and Hölder's inequality gives

$$I_q(K) \leq I_s(K) \leq c_3\gamma\sqrt{n}L_K \tag{6.4.31}$$

for all $q \leq s$. On the other hand, by the definition of s and (6.4.29),

$$s \geq \frac{Z^2(s)}{c_4^2\gamma^2 L_K^2} = \frac{\psi^2(n)}{c_4^2\beta^2} =: c_2\psi^2(n), \tag{6.4.32}$$

which gives the result in this case.

To conclude the proof, observe that the range $\beta \leq \gamma\psi(n) \leq \beta c_0 R(K)/L_K$ is the interesting one for the parameter $\psi(n)$. If $0 < \gamma\psi(n) < \beta$, then the conclusion of the Theorem is trivially true. If $\gamma\psi(n) \geq \beta c_0 R(K)/L_K$, then we have (6.4.27) for every $t > 0$. Following the previous argument, we check that $I_n(K) \simeq Z(n) \leq c\gamma\sqrt{n}L_K$. But $I_n(K) \simeq R(K)$, and this implies (6.4.28) for every $q \geq 2$. $\square$

Theorems 6.4.6 and 6.4.7, combined with the reduction in §6.2 prove the following.

Theorem 6.4.8. *Let $1 \ll \phi(n) \ll n$ and let K be an isotropic convex body in $\mathbb{R}^n$. For every $\gamma \simeq 1$ the following are equivalent:*

(i) *For every $t \geq 1$,*

$$\mathrm{Prob}(x \in K : |x| \geq \gamma\sqrt{n}L_K t) \leq \exp(-\phi(n)t). \tag{6.4.33}$$

(ii) *For every $0 < t \leq c_1(\gamma)\sqrt{\phi(n)}L_K$,*

$$f_K(t) \leq \frac{c_2}{L_K}\exp\big(-t^2/(c_3(\gamma)L_K)^2\big), \tag{6.4.34}$$

where $c_i(\gamma) \simeq \gamma$. $\square$

NOTES AND REFERENCES

The results of this Chapter come from the thesis of Paouris: see also [49].

Chapter 7

Random points and random polytopes

7.1 Random points in isotropic convex bodies

The purpose of this Section is to prove the following Theorem about independent random points which are uniformly distributed in an isotropic convex body.

Theorem 7.1.1. *Let K be an isotropic convex body in $\mathbb{R}^n$ and let $\delta, \varepsilon \in (0,1)$. If $m \geq c(\delta,\varepsilon)\varepsilon^{-2}n(\log n)^2$, then m random points $x_1, \dots, x_m$ which are chosen independently and uniformly from K satisfy with probability $> 1-\delta$ the following: For every $\theta \in S^{n-1}$,*

$$(1-\varepsilon)L_K^2 \leq \frac{1}{m}\sum_{j=1}^{m}\langle x_j, \theta\rangle^2 \leq (1+\varepsilon)L_K^2. \tag{7.1.1}$$

Remark 1: The constant $c(\delta,\varepsilon)$ may be assumed to be bounded by a fixed power of $\log\left(\frac{2}{\varepsilon\delta}\right)$.

We will use the following facts from Chapter 2: First, K satisfies a ψ_1-*estimate with constant* C for some absolute constant $C > 0$. That is,

$$\int_K \exp\bigl(|\langle x, \theta\rangle|/CL_K\bigr)dx \leq 2 \tag{7.1.2}$$

for every $\theta \in S^{n-1}$. Equivalently, there exists an absolute constant $c_1 > 0$ such that

$$L_K \leq \left(\int_K |\langle x, \theta\rangle|^p dx\right)^{1/p} \leq c_1 p L_K, \tag{7.1.3}$$

for every $p \geq 2$ and $\theta \in S^{n-1}$. Second, we have a ψ_2-estimate for the Euclidean norm: there exists an absolute constant $c_2 > 0$ such that

$$\int_K \exp\left(\frac{|x|^2}{c_2^2 n L_K^2}\right) dx \leq 2. \tag{7.1.4}$$

The important step for the proof is done in the next Theorem.

Theorem 7.1.2. *Let $\delta \in (0,1)$ and let $x_1, \ldots, x_m$ be random points independently and uniformly distributed in K. With probability greater than $1-\delta$ we have*

$$|x_j| \leq c_1(\delta)\sqrt{n}L_K\sqrt{\log m} \tag{7.1.5}$$

for all $j \in \{1, \ldots, m\}$, and

$$\Big|\sum_{i\in E} x_i\Big| \leq c_2(\delta)L_K\sqrt{\log m}\sqrt{|E|}\sqrt{n} + c_3(\delta)L_K(\log m)|E| \tag{7.1.6}$$

for all $E \subseteq \{1, \ldots, m\}$, where $c_i(\delta) \simeq \sqrt{\log \frac{2}{\delta}}$, $i = 1, 2, 3$.

Proof. From (7.1.4) we have

$$\mathrm{Prob}\left(\max_{1\leq j\leq m} |x_j| \geq c_2 t\sqrt{n}L_K\right) \leq m\exp(-t^2) < \frac{\delta}{2} \tag{7.1.7}$$

provided that $t \geq \sqrt{\log(2m/\delta)}$. So, with probability greater than $1-(\delta/2)$ we have

$$|x_j| \leq c_1(\delta)\sqrt{n}L_K\sqrt{\log m} \tag{7.1.8}$$

for every $j = 1, \ldots, m$.
Let $E \subseteq \{1, \ldots, m\}$. We write
(7.1.9)

$$\Big|\sum_{i\in E} x_i\Big|^2 = \sum_{i\in E}|x_i|^2 + \sum_{i\neq j\in E}\langle x_i, x_j\rangle \leq c_1^2(\delta)L_K^2 n(\log m)|E| + \sum_{i\neq j\in E}\langle x_i, x_j\rangle.$$

If δ_j takes the values 0 or 1 with probability $1/2$, then

$$\mathbb{E}_{\vec{\delta}}\Big\langle \sum_{i=1}^m \delta_i x_i, \sum_{j=1}^m (1-\delta_j)x_j\Big\rangle = \frac{1}{4}\sum_{i\neq j\in E}\langle x_i, x_j\rangle. \tag{7.1.10}$$

Therefore, we can find $E_1, E_2 \subset E$ with $|E_1| \geq |E_2|$, $E_1 \cap E_2 = \emptyset$, $E_1 \cup E_2 = E$, such that

$$\begin{aligned}\sum_{i\neq j\in E}\langle x_i, x_j\rangle &\leq 4\Big\langle \sum_{i\in E_1} x_i, \sum_{j\in E_2} x_j\Big\rangle \\ &\leq 4\sum_{i\in E_1}\Big|\Big\langle x_i, \sum_{j\in E_2} x_j\Big\rangle\Big|.\end{aligned}$$

Rewrite this last sum in the form

$$\sum_{i\in E_1}\Big|\Big\langle x_i, \sum_{j\in E_2} x_j\Big\rangle\Big| = \Big|\sum_{j\in E_2} x_j\Big| \cdot \sum_{i\in E_1}|\langle x_i, y_{E_2}\rangle|, \tag{7.1.11}$$

where

$$y_{E_2} = \frac{\sum_{j\in E_2} x_j}{\big|\sum_{j\in E_2} x_j\big|}, \tag{7.1.12}$$

and $|y_{E_2}| = 1$. Observe that the set $\{x_i\}_{i\in E_1}$ is independent from y_{E_2}, since $E_1 \cap E_2 = \emptyset$. If we fix $|E_1| = k$, applying (7.1.2) and Fubini's theorem we see that

$$\int_{K^k} \exp\left(\frac{1}{CL_K}\sum_{i\in E_1}|\langle x_i, y_{E_2}\rangle|\right)dx \leq 2^k. \tag{7.1.13}$$

Therefore,

$$\mathrm{Prob}\left(\vec{x}\in K^m : \sum_{i\in E_1}|\langle x_i, y_{E_2}\rangle| > CtkL_K\right) < 2^k e^{-kt} \leq e^{-kt/2} \tag{7.1.14}$$

if $t \geq 2$. The number of possible E_1's is bounded by m^k, and hence, for every $k \leq m$

(7.1.15)

$$\mathrm{Prob}\left(\vec{x}\in K^m : \exists E_1\subset E, |E_1| = k, \sum_{i\in E_1}|\langle x_i, y_{E_2}\rangle| > CtkL_K\right) < m^k e^{-kt/2}.$$

This probability will be smaller than δ/m if $t \simeq_\delta \log m$. Doing this for $k = \left[\frac{m}{2}\right]+1,\dots,m$, we see that $(x_1,\dots,x_m)\in K^m$ satisfies with probability greater than $1-\frac{\delta}{2}$ the following: For every $E\subseteq\{1,\dots,m\}$,

$$\sum_{i\neq j\in E}\langle x_i, x_j\rangle \leq c_2(\delta)L_K(\log m)\max_{E_1\subset E}\left\{|E_1|\left|\sum_{j\in E\setminus E_1}x_j\right|\right\}. \tag{7.1.16}$$

To finish the proof, consider any m-tuple which satisfies (7.1.9) and (7.1.16), and for every $s\leq m$ write

$$A_s = \max\Big|\sum_{j\in F}x_j\Big|, \tag{7.1.17}$$

where the maximum is over all $F\subseteq\{1,\dots,m\}$ with $|F|\leq s$. For every $E\subseteq\{1,\dots,m\}$ we have

$$\Big|\sum_{i\in E}x_i\Big|^2 \leq c_1^2(\delta)L_K^2 n(\log m)|E| + c_2(\delta)L_K(\log m)|E|A_{|E|}, \tag{7.1.18}$$

therefore

$$A_{|E|}^2 \leq c_1^2(\delta)L_K^2 n(\log m)|E| + c_2(\delta)L_K(\log m)|E|A_{|E|}, \tag{7.1.19}$$

which implies

$$A_{|E|} \leq c_2(\delta)L_K\sqrt{n}\sqrt{\log m}\sqrt{|E|} + c_3(\delta)L_K(\log m)|E|. \tag{7.1.20}$$

From the argument we easily check that $c_i(\delta)\simeq\sqrt{\log\frac{2}{\delta}}$, $i=1,2,3$. □

We will also use a version of Bernstein's inequality:

Lemma 7.1.3. *Let $\{f_j\}_{j\leq m}$ be independent random variables with mean 0 on some probability space (Ω,μ). If $\|f_j\|_1 \leq 2$ and $\|f_j\|_\infty \leq M$, then, for every $\varepsilon\in(0,1)$,*

$$\text{Prob}\left(\Big|\sum_{j=1}^m f_j\Big| > \varepsilon m\right) \leq 2\exp(-\varepsilon^2 m/8M). \tag{7.1.21}$$

□

Lemma 7.1.4. *Let K be an isotropic convex body in $\mathbb{R}^n$. Fix $\delta,\zeta,\varepsilon\in(0,1)$. If*

$$B \geq c_4 \log\left(\frac{2}{\varepsilon}\right) L_K \tag{7.1.22}$$

and

$$m \geq c_5\varepsilon^{-2} n\left(\frac{B}{L_K}\right)^2 \log\left(\frac{2}{\delta\zeta}\right), \tag{7.1.23}$$

then m random points $x_1,\ldots,x_m$ which are chosen independently and uniformly from K satisfy with probability greater than $1-\delta$ the following: for all θ in a ζ-net for S^{n-1},

$$\left(1-\frac{\varepsilon}{2}\right)L_K^2 \leq \frac{1}{m}\sum_{\{j:|\langle x_j,\theta\rangle|\leq B\}}\langle x_j,\theta\rangle^2 \leq \left(1+\frac{\varepsilon}{2}\right)L_K^2 \tag{7.1.24}$$

Proof. There exists a ζ-net $\mathcal{N}$ for S^{n-1} with cardinality $|\mathcal{N}|\leq(3/\zeta)^n$.

We fix $\theta\in\mathcal{N}$ and define $f_\theta : K\to\mathbb{R}$ by

$$f_\theta(x) = \frac{1}{L_K^2}\langle x,\theta\rangle^2\chi_{\{z\in K:|\langle z,\theta\rangle|\leq B\}}(x). \tag{7.1.23}$$

From (7.1.2) and (7.1.3) we see that

$$\begin{aligned}\frac{1}{L_K^2}\int_{\{|\langle\cdot,\theta\rangle|\geq B\}}\langle x,\theta\rangle^2 dx &\leq |\{z\in K : |\langle z,\theta\rangle|\geq B\}|^{1/2}\cdot\frac{1}{L_K^2}\left(\int_K\langle x,\theta\rangle^4 dx\right)^{1/2}\\ &\leq c_3\exp(-B/CL_K),\end{aligned}$$

where $c_3>0$ is an absolute constant. This will be less than $\varepsilon/4$ if

$$B \geq c_4\log\left(\frac{2}{\varepsilon}\right)L_K \tag{7.1.25}$$

where $c_4>0$ is an absolute constant. If B satisfies (7.1.25), then

$$1-\mathbb{E}f_\theta = \frac{1}{L_K^2}\int_{\{z\in K:|\langle z,\theta\rangle|>B\}}\langle x,\theta\rangle^2 dx \leq \frac{\varepsilon}{4}, \tag{7.1.26}$$

for every $\theta\in S^{n-1}$. We define $f_{\theta,j}(x_1,\ldots,x_m) = f_\theta(x_j)-\mathbb{E}f_\theta$ on K^m. We easily check that

$$\|f_{\theta,j}\|_1\leq 2\ ,\ \mathbb{E}f_{\theta,j}=0 \text{ and } \|f_{\theta,j}\|_\infty \leq \left(\frac{B}{L_K}\right)^2. \tag{7.1.27}$$

Then, Lemma 7.1.3 gives

$$\text{(7.1.28)} \quad \text{Prob}\left(\Big|\frac{1}{m}\sum_{j=1}^{m} f_\theta(x_j) - \mathbb{E}f_\theta\Big| > \frac{\varepsilon}{4}\right) \leq 2\exp\left(-\frac{\varepsilon^2 m}{c_3'(B/L_K)^2}\right) < \frac{\delta}{|\mathcal{N}|}$$

if we choose

$$\text{(7.1.29)} \qquad m \geq c_5\varepsilon^{-2} n \left(\frac{B}{L_K}\right)^2 \log\left(\frac{2}{\delta\zeta}\right).$$

Therefore, with probability greater than $1-\delta$ the points $x_1,\dots,x_m$ satisfy

$$\text{(7.1.30)} \qquad \left|\frac{1}{m}\sum_{\{j:|\langle x_j,\theta\rangle|\leq B\}} \langle x_j,\theta\rangle^2 - L_K^2\mathbb{E}f_\theta\right| \leq \frac{\varepsilon L_K^2}{4},$$

for every $\theta \in \mathcal{N}$. Taking into account (7.1.26) we get the assertion of the Lemma. $\square$

Proof of Theorem 7.1.1: We may assume that $x_1,\dots,x_m$ satisfy the conclusion of Theorem 7.1.2 with probability $> 1-\frac{\delta}{2}$. We fix $\zeta \in (0,1)$ (which will be later chosen of the order of ε). Let

$$\text{(7.1.31)} \qquad B = 4c_3(\delta)L_K\log m,$$

where $c_3(\delta)$ is the constant in Theorem 7.1.2. If $\log m > c_6\log\left(\frac{2}{\varepsilon}\right)/\log^{1/2}\left(\frac{2}{\delta}\right)$ and

$$\text{(7.1.32)} \qquad m \geq c_7\varepsilon^{-2}n(\log m)^2\log\left(\frac{2}{\delta}\right)\log\left(\frac{2}{\delta\zeta}\right),$$

then, we know that with probability greater than $1-\frac{\delta}{2}$, the random points $x_1,\dots,x_m \in K$ satisfy

$$\text{(7.1.33)} \qquad \left(1-\frac{\varepsilon}{2}\right)L_K^2 \leq \frac{1}{m}\sum_{\{j:|\langle x_j,\theta\rangle|\leq B\}}\langle x_j,\theta\rangle^2 \leq \left(1+\frac{\varepsilon}{2}\right)L_K^2$$

for all θ in a ζ-net $\mathcal{N}$ of S^{n-1}.

For every $\beta \geq B$ and every $\theta \in S^{n-1}$ we define

$$\text{(7.1.35)} \qquad E_\beta(\theta) = \{j \leq m : |\langle x_j,\theta\rangle| > \beta\}.$$

Using Theorem 7.1.2 we can estimate the cardinality of $E_\beta(\theta)$ as follows:

$$\begin{aligned}
\beta|E_\beta| &\leq \sum_{j\in E_\beta}|\langle x_j,\theta\rangle| \\
&\leq \max_{\varepsilon_j=\pm1}\Big|\sum_{j\in E_\beta}\varepsilon_j x_j\Big| \\
&\leq 2\max_{F\subseteq E_\beta}\Big|\sum_{j\in F}x_j\Big| \\
&\leq 2c_2(\delta)L_K\sqrt{\log m}\sqrt{n}\sqrt{|E_\beta|} + 2c_3(\delta)L_K(\log m)|E_\beta| \\
&\leq 2c_2(\delta)L_K\sqrt{\log m}\sqrt{n}\sqrt{|E_\beta|} + \frac{\beta}{2}|E_\beta|.
\end{aligned}$$

This gives: for every $\theta \in S^{n-1}$,

$$\beta^2|E_\beta(\theta)| \leq 16c_2^2(\delta)L_K^2 n\log m. \tag{7.1.36}$$

It follows that

$$\begin{aligned}
\sum_{\{j:|\langle x_j,\theta\rangle|>B\}} \langle x_j,\theta\rangle^2 &= \sum_{k=0}^{k_0-1} \sum_{\{j:2^kB<|\langle x_j,\theta\rangle|\leq 2^{k+1}B\}} \langle x_j,\theta\rangle^2 \\
&\leq \sum_{k=0}^{k_0-1} |E_{2^kB}|(2^{k+1}B)^2 \\
&\leq c\log\left(\frac{2}{\delta}\right) L_K^2 n(\log m)k_0,
\end{aligned}$$

where the summation is over all non-empty E, and k_0 is the least integer for which $R(K) \leq 2^{k_0}B$. Since K is isotropic, we have $R(K) \leq (n+1)L_K$. So, recalling the definition of B, we get

$$k_0 \leq c\log\left(\frac{nL_K}{B}\right) \leq c'\log n. \tag{7.1.37}$$

Now, if m satisfies (7.1.32), we have

$$\frac{1}{m}\sum_{\{j:|\langle x_j,\theta\rangle|>B\}} \langle x_j,\theta\rangle^2 \leq \frac{c\log\left(\frac{2}{\delta}\right) L_K^2 n(\log n)(\log m)}{m} < \frac{\varepsilon L_K^2}{20}. \tag{7.1.38}$$

Combining with (7.1.33) we conclude the proof for every $\theta \in \mathcal{N}$. Finally, choosing $\zeta = \varepsilon/10$ and using a standard successive approximation argument we get a similar estimate for every $\theta \in S^{n-1}$. □

7.2 Random polytopes in isotropic convex bodies

Let K be a convex body in $\mathbb{R}^n$ with volume 1. We fix $N \geq n+1$ and consider random points $x_1,\dots,x_N$ independently and uniformly distributed in K. Let $C(x_1,\dots,x_N)$ be their convex hull. For every $p>0$ we consider the quantity

$$\mathbb{E}_p(K,N) = \left(\int_K \cdots \int_K |C(x_1,\dots,x_N)|^p dx_N \dots dx_1\right)^{1/pn}. \tag{7.2.1}$$

Observe that $\mathbb{E}_p(K,N)$ is an affinely invariant quantity, so we may also assume that K has its center of mass at the origin. When $N=n+1$, these quantities are exact functions of the isotropic constant of K. To see this, recall that

$$S_2^2(K) := \int_K \cdots \int_K |\mathrm{co}(0,x_1,\dots,x_n)|^2 dx_n \dots dx_1 \tag{7.2.2}$$

satisfies the identity

$$L_K^{2n} = n!S_2^2(K) \tag{7.2.3}$$

and

$$S_2^2(K) \leq \mathbb{E}_2^{2n}(K, n+1) \leq (n+1)^2 S_2^2(K). \tag{7.2.4}$$

It follows that

$$c_1 \frac{L_K}{\sqrt{n}} \leq \mathbb{E}_2(K, n+1) \leq c_2 \frac{L_K}{\sqrt{n}} \tag{7.2.5}$$

where $c_1, c_2 > 0$ are absolute constants. Moreover, using Khintchine type inequalities for linear functionals on convex bodies (see §2.1) one can show that $\mathbb{E}_p(K, n+1) \geq cL_K/\sqrt{n}$ for every $p > 0$, where $c > 0$ is an absolute constant.

In this Section, we give estimates for the volume radius $\mathbb{E}_{1/n}(K, N)$ of a random N-tope $C(x_1, \ldots, x_N)$ in K.

1. Upper bounds: As it turns out, a generalization of the upper bound in (7.2.5) is possible.

Theorem 7.2.1. *Let K be a convex body in $\mathbb{R}^n$ with volume 1. For every $N \geq n+1$, we have*

$$\mathbb{E}_{1/n}(K, N) \leq cL_K \frac{\log(2N/n)}{\sqrt{n}} \tag{7.2.6}$$

where $c > 0$ is an absolute constant.

Remark 1: Let $\alpha \in [1, 2]$. We will say that a convex body K in $\mathbb{R}^n$ is a *ψ_α-body with constant b_α* if

$$\left(\int_K |\langle x, y \rangle|^p dx \right)^{1/p} \leq b_\alpha p^{1/\alpha} \int_K |\langle x, y \rangle| dx \tag{7.2.7}$$

for every $y \in \mathbb{R}^n$ and $p \geq 1$. Our method shows that if K is a *ψ_2-body* then one has the stronger estimate $\mathbb{E}_{1/n}(K, N) \leq cL_K \sqrt{\log(2N/n)}/\sqrt{n}$. This is optimal and might be the right dependence for every convex body K in $\mathbb{R}^n$.

We shall prove the following.

Theorem 7.2.2. *Let K be an isotropic convex body in $\mathbb{R}^n$. Assume that for some $\alpha \in [1, 2]$, K is a ψ_α-body with constant b_α. Then, for every $N \geq n+1$,*

$$\mathbb{E}_{1/n}(K, N) \leq cb_\alpha L_K \frac{\big(\log(2N/n)\big)^{1/\alpha}}{\sqrt{n}}. \tag{7.2.8}$$

This implies Theorem 7.2.1 because every convex body of volume 1 has an affine image which is isotropic, and every isotropic convex body is a ψ_1-body with constant $b_1 \leq C$. For the proof, we will use a result of Ball and Pajor on the volume of symmetric convex bodies which are intersections of symmetric strips in $\mathbb{R}^n$.

Lemma 7.2.3. *Let $x_1, \ldots, x_N \in \mathbb{R}^n \setminus \{0\}$ and let $1 \leq q < \infty$. If*

$$W = \{z \in \mathbb{R}^n : |\langle z, x_j \rangle| \leq 1,\ j = 1, \ldots, N\}, \tag{7.2.9}$$

then

$$|W|^{1/n} \geq 2 \left(\frac{n+q}{n} \sum_{j=1}^N \frac{1}{|B_q^n|} \int_{B_q^n} |\langle z, x_j \rangle|^q dz \right)^{-1/q}. \tag{7.2.10}$$

□

Proof of Theorem 7.2.2: In the sequel we will write K_N for the absolute convex hull $\mathrm{co}\{\pm x_1, \dots, \pm x_N\}$ of N random points from K. By the Blaschke-Santaló inequality,

$$\mathbb{E}_{1/n}(K,N) \leq \mathbb{E}|K_N|^{1/n} \leq \omega_n^{2/n} \cdot \mathbb{E}|K_N^\circ|^{-1/n} \tag{7.2.11}$$

where K_N° is the polar body of K_N. Lemma 7.2.3 shows that

$$|K_N^\circ|^{-1/n} \leq \frac{1}{2}\left(\frac{n+q}{n}\sum_{j=1}^N \frac{1}{|B_q^n|}\int_{B_q^n} |\langle z, x_j\rangle|^q dz\right)^{1/q} \tag{7.2.12}$$

for every $q \geq 1$. Consider the convex body $W = K \times \cdots \times K$ (N times) in $\mathbb{R}^{Nn}$. We apply Hölder's inequality, change the order of integration and use the ψ_α-property of K:

$$\begin{aligned}
\mathbb{E}|K_N^\circ|^{-1/n} &\leq \int_W \frac{1}{2}\left(\frac{n+q}{n}\sum_{j=1}^N \frac{1}{|B_q^n|}\int_{B_q^n} |\langle z, x_j\rangle|^q \, dz\right)^{1/q} dx_N \dots dx_1 \\
&\leq \frac{1}{2}\left(\frac{n+q}{n}\sum_{j=1}^N \frac{1}{|B_q^n|}\int_{B_q^n}\int_W |\langle z, x_j\rangle|^q \, dx_N \dots dx_1 \, dz\right)^{1/q} \\
&\leq \frac{1}{2}\left(\frac{n+q}{n}\left(q^{1/\alpha} b_\alpha L_K\right)^q N \frac{1}{|B_q^n|}\int_{B_q^n} |z|^q \, dz\right)^{1/q}.
\end{aligned}$$

Since $\omega_n^{2/n} \leq c_1/n$ and $|z| \leq n^{1/2-1/q}$ for all $z \in B_q^n$ when $q \geq 2$, we get

$$\mathbb{E}_{1/n}(K,N) \leq \frac{c}{\sqrt{n}} b_\alpha L_K q^{1/\alpha}\left(\frac{N}{n}\right)^{1/q}\left(\frac{n+q}{n}\right)^{1/q} \tag{7.2.13}$$

for every $q \geq 2$. Choosing $q = \log(e^2 N/n)$ we complete the proof. □

2. The 1-unconditional case: Let K be an isotropic 1-unconditional convex body in $\mathbb{R}^n$. We will prove an optimal upper bound.

Theorem 7.2.4. *Let K be an isotropic 1-unconditional convex body in $\mathbb{R}^n$. Then, for every $N \geq n+1$,*

$$\mathbb{E}_{1/n}(K,N) \leq C\frac{\sqrt{\log(2N/n)}}{\sqrt{n}}, \tag{7.2.14}$$

where $C > 0$ is an absolute constant.

We will use the following properties of K (which were proved in Chapter 5):

(i) For every $y \in \mathbb{R}^n$ and every $p \geq 1$,

$$\left(\int_K |\langle x, y\rangle|^p dx\right)^{1/p} \leq c_1\sqrt{p}\sqrt{n}\|y\|_\infty. \tag{7.2.15}$$

In other words,

$$\|\langle\cdot,y\rangle\|_{\psi_2}\leq c\sqrt{n}\|y\|_\infty. \tag{7.2.16}$$

(ii) There exists an absolute constant $c_2>0$ such that for every $t\geq 1$,

$$\left|\left\{x\in K : |x|\geq c_2t\sqrt{n}\right\}\right|\leq \exp\left(-t\sqrt{n}\right). \tag{7.2.17}$$

We will also use a result of Bárány and Füredi which is in the spirit of Lemma 7.2.3.

Lemma 7.2.5. *There exists an absolute constant $c_3>0$ such that: if $N\geq n+1$ and $x_1,\dots,x_N\in\mathbb{R}^n$, then*

$$|C(x_1,\dots,x_N)|^{1/n}\leq c_3\cdot\max_{i\leq N}|x_i|\cdot\frac{\sqrt{\log(2N/n)}}{n}. \tag{7.2.18}$$

Proof of Theorem 7.2.4: We distinguish two cases (small and large N):
Case 1: $N\leq n^2$: Fix $t\geq 1$, which will be suitably chosen. We know that

$$\mathrm{Prob}\left((x_1,\dots,x_N):\ \exists i\leq N \text{ s.t } |x_i|\geq c_2t\sqrt{n}\right)\leq N\cdot\exp\left(-t\sqrt{n}\right). \tag{7.2.19}$$

If A is the event in (7.2.19), using Lemma 7.2.5 we write

$$\begin{aligned}\mathbb{E}_{1/n}(K,N) &= \int_A |C(x_1,\dots,x_N)|^{1/n}+\int_{A^c}|C(x_1,\dots,x_N)|^{1/n}\\ &\leq \mathrm{Prob}(A)+\mathrm{Prob}(A^c)\cdot c_3c_2t\sqrt{n}\cdot\frac{\sqrt{\log(2N/n)}}{n}\\ &\leq N\cdot\exp\left(-c_1t\sqrt{n}\right)+c_4t\cdot\frac{\sqrt{\log(2N/n)}}{\sqrt{n}}.\end{aligned}$$

We have assumed that $N\leq n^2$, which implies

$$\exp\left(c_1t\sqrt{n}\right)\geq\frac{c_1^6t^6n^3}{6!}\geq\frac{n\cdot N}{c_4t\sqrt{\log 2}} \tag{7.2.20}$$

if $t\geq 1$ is chosen large enough (independently from n and N). Then,

$$\mathbb{E}_{1/n}(K,N)\leq(2c_4t)\cdot\frac{\sqrt{\log(2N/n)}}{\sqrt{n}}. \tag{7.2.21}$$

Case 2: $N\geq n^2$: Repeating the argument of the proof of Theorem 7.2.2, we arrive at

$$\mathbb{E}_{1/n}(K,N)\leq\frac{\omega_n^{2/n}}{2}\left(\frac{N(n+q)}{n}\frac{1}{|B_q^n|}\int_{B_q^n}\int_K|\langle z,x\rangle|^q dx\,dz\right)^{1/q}. \tag{7.2.22}$$

Now,

$$\int_K|\langle z,x\rangle|^q dx\leq\left(C\sqrt{q}\sqrt{n}\|z\|_\infty\right)^q \tag{7.2.23}$$

for every $z \in B_q^n$. Therefore,

$$(7.2.24)\qquad \mathbb{E}_{1/n}(K,N) \leq \frac{\omega_n^{2/n}}{2} C\sqrt{q}\sqrt{n}\left(\frac{N(n+q)}{n}\frac{1}{|B_q^n|}\int_{B_q^n}\|z\|_\infty^q dz\right)^{1/q}.$$

Observe that $\|z\|_\infty \leq \|z\|_q$ and

$$\begin{aligned}\int_{B_q^n}\|z\|_q^q dz &= \int_0^1 qt^{q-1}|\{z\in B_q^n : \|z\|_q \geq t\}|dt\\ &= \int_0^1 qt^{q-1}|B_q^n \setminus tB_q^n|dt\\ &= |B_q^n|\int_0^1 qt^{q-1}(1-t^n)dt\\ &= \frac{n}{n+q}|B_q^n|.\end{aligned}$$

It follows that

$$(7.2.25)\qquad \frac{1}{|B_q^n|}\int_{B_q^n}\|z\|_\infty^q dz \leq \frac{1}{|B_q^n|}\int_{B_q^n}\|z\|_q^q dz = \frac{n}{n+q}.$$

Combining the above we get

$$(7.2.26)\qquad \mathbb{E}_{1/n}(K,N) \leq C_2\frac{\sqrt{q}}{\sqrt{n}}\cdot N^{1/q}$$

for every $q \geq 1$. Choose $q = \log(2N/n)$. Since $N \geq n^2$, we have

$$(7.2.27)\qquad N^{1/q} = \exp\left(\frac{\log N}{\log(2N/n)}\right) \leq \exp\left(\frac{\log N}{\log(2\sqrt{N})}\right) \leq e^2.$$

Therefore,

$$(7.2.28)\qquad \mathbb{E}_{1/n}(K,N) \leq C_3\frac{\sqrt{\log(2N/n)}}{\sqrt{n}}.$$

□

3. Lower bounds: Our lower bound is based on an extension of a result of Groemer.

Theorem 7.2.6. *Let K be a convex body in $\mathbb{R}^n$ with volume 1. Write B for a ball in $\mathbb{R}^n$ with volume 1. Then,*

$$(7.2.29)\qquad \mathbb{E}_p(K,N) \geq \mathbb{E}_p(B,N)$$

for every $p > 0$. In particular, the expected volume radius $\mathbb{E}_{1/n}(K,N)$ of a random N-tope in K is minimal when $K = B$.

The method is to show that Steiner symmetrization decreases $\mathbb{E}_p(K,N)$: Let H be an $(n-1)$-dimensional subspace of $\mathbb{R}^n$. We identify H with $\mathbb{R}^{n-1}$ and write

$x = (y, t)$, $y \in H$, $t \in \mathbb{R}$ for a point $x \in \mathbb{R}^n$. If K is a convex body in $\mathbb{R}^n$ with $|K| = 1$ and $P(K)$ is the orthogonal projection of K onto H, then

$$\mathbb{E}_p^{pn}(K, N) = \int_{P(K)} \cdots \int_{P(K)} M_{p,K}(y_1, \ldots, y_N) dy_N \ldots dy_1 \tag{7.2.30}$$

where

$$M_{p,K}(y_1, \ldots, y_N) = \int_{\ell(K,y_1)} \cdots \int_{\ell(K,y_N)} |C((y_1, t_1), \ldots, (y_N, t_N))|^p dt_N \ldots dt_1 \tag{7.2.31}$$

and $\ell(K, y) = \{t \in \mathbb{R} : (y, t) \in K\}$.

We fix $y_1, \ldots, y_N \in H$ and consider the function $F_Y : \mathbb{R}^N \to \mathbb{R}$ defined by

$$F_Y(t_1, \ldots, t_N) = |C((y_1, t_1), \ldots, (y_N, t_N))|, \tag{7.2.32}$$

where $Y = (y_1, \ldots, y_N)$. The key observation is the following:

Lemma 7.2.7. *For any $y_1, \ldots, y_N \in P(K)$, the function F_Y is convex.* □

We now also fix $r_1, \ldots, r_N > 0$ and define $Q = \{U = (u_1, \ldots, u_N) : |u_i| \leq r_i,\ i = 1, \ldots, N\}$. For every N-tuple $W = (w_1, \ldots, w_N) \in \mathbb{R}^N$ we set

$$G_W(u_1, \ldots, u_N) = F_Y(w_1 + u_1, \ldots, w_N + u_N), \tag{7.2.33}$$

and write

$$G_W(U) = F_Y(W + U).$$

This is the volume of the polytope which is generated by the points $(y_i, w_i + u_i)$. Finally, for every $W \in \mathbb{R}^N$ and $\alpha > 0$, we define

$$A(W, \alpha) = \{U \in Q : G_W(U) \leq \alpha\}. \tag{7.2.34}$$

With this notation, we have

Lemma 7.2.8. *Let $\alpha > 0$ and $\lambda \in (0, 1)$. If $W, W' \in \mathbb{R}^N$, then*

$$|A(\lambda W + (1 - \lambda)W', \alpha)| \geq |A(W, \alpha)|^\lambda |A(W', \alpha)|^{1-\lambda}. \tag{7.2.35}$$

Proof. Let $U \in A(W, \alpha)$ and $U' \in A(W', \alpha)$. Then, using the convexity of F_Y we see that

$$\begin{aligned} G_{\lambda W+(1-\lambda)W'}(\lambda U + (1 - \lambda)U') &= F_Y(\lambda(W + U) + (1 - \lambda)(W' + U')) \\ &\leq \lambda F_Y(W + U) + (1 - \lambda)F_Y(W' + U') \\ &= \lambda G_W(U) + (1 - \lambda)G_{W'}(U') \\ &\leq \alpha. \end{aligned}$$

Therefore,

$$A(\lambda W + (1 - \lambda)W') \supseteq \lambda A(W, \alpha) + (1 - \lambda)A(W', \alpha) \tag{7.2.36}$$

and the result follows from the Brunn-Minkowski inequality. □

Observe that the polytopes $C((y_i, w_i + u_i)_{i\leq N})$ and $C((y_i, -w_i - u_i)_{i\leq N})$ have the same volume since they are reflections of each other with respect to H. It follows that

$$A(-W,\alpha) = -A(W,\alpha) \tag{7.2.37}$$

for every $\alpha > 0$. Taking $W' = -W$ and $\lambda = 1/2$ in Lemma 7.2.8, we obtain the following:

Lemma 7.2.9. *Let $y_1, \dots, y_N \in H$. For every $W \in \mathbb{R}^N$ and every $\alpha > 0$,*

$$|A(O,\alpha)| \geq |A(W,\alpha)|, \tag{7.2.38}$$

where O is the origin in $\mathbb{R}^N$. □

For every $y \in P(K)$, we denote by $w(y)$ the midpoint and by $2r(y)$ the length of $\ell(K,y)$. Let $S(K)$ be the Steiner symmetral of K. By definition, $P(S(K)) = P(K) = P$ and for every $y \in P$ the midpoint and length of $\ell(S(K),y)$ are $w'(y) = 0$ and $2r'(y) = 2r(y)$ respectively.

Lemma 7.2.10. *Let $y_1, \dots, y_N \in P(K) = P(S(K))$. Then,*

$$M_{p,K}(y_1,\dots,y_N) \geq M_{p,S(K)}(y_1,\dots,y_N) \tag{7.2.39}$$

for every $p > 0$.

Proof. In the notation of the previous Lemmas, we have

$$\begin{aligned} M_{p,K}(y_1,\dots,y_N) &= \int_Q [G_W(U)]^p dU \\ &= \int_0^\infty |\{U \in Q : G_W(U) \geq t^{1/p}\}| dt \\ &= \int_0^\infty \Big(|Q| - |A(W,t^{1/p})|\Big)\, dt. \end{aligned}$$

By the definition of $S(K)$,

$$M_{p,K}(y_1,\dots,y_N) = \int_Q [G_O(U)]^p dU = \int_0^\infty \Big(|Q| - |A(O,t^{1/p})|\Big)\, dt, \tag{7.2.40}$$

and the result follows from Lemma 7.2.9. □

It is now clear that $\mathbb{E}_p(K,N)$ decreases under Steiner symmetrization.

Theorem 7.2.11. *Let K be a convex body with volume $|K| = 1$ and let H be an $(n-1)$-dimensional subspace of $\mathbb{R}^n$. If $S_H(K)$ is the Steiner symmetral of K with respect to H, then*

$$\mathbb{E}_p(S(K),N) \leq \mathbb{E}_p(K,N) \tag{7.2.41}$$

for every $p > 0$.

Proof. We may assume that $H = \mathbb{R}^{n-1}$. Since $P(S(K)) = P(K)$, Lemma 7.2.10 and (7.2.30) show that

$$\begin{aligned} \mathbb{E}_p^{pn}(K,N) &= \int_{PK} \cdots \int_{PK} M_{p,K}(y_1,\ldots,y_N)dy_N \ldots dy_1 \\ &\geq \int_{P(SK)} \cdots \int_{P(SK)} M_{p,SK}(y_1,\ldots,y_N)dy_N \ldots dy_1 \\ &= \mathbb{E}_p^{pn}(S(K),N). \end{aligned}$$

□

Proof of Theorem 7.2.6: Since the ball B of volume 1 is the Hausdorff limit of a sequence of successive Steiner symmetrizations of K, Theorem 7.2.11 shows that the expected volume radius is minimal in the case of B. □

Remark 2: The argument shows that a more general fact holds true:

Theorem 7.2.12. *Let K be a convex body in $\mathbb{R}^n$ with volume 1. Write B for a ball in $\mathbb{R}^n$ with volume 1. Then,*

(7.2.42)
$$\int_K \cdots \int_K f(|C(x_1,\ldots,x_N)|)dx_N \ldots dx_1 \geq \int_B \cdots \int_B f(|C(x_1,\ldots,x_N)|)dx_N \ldots dx_1$$

for every increasing function $f : \mathbb{R}^+ \to \mathbb{R}^+$. □

Next, we give a lower bound for $\mathbb{E}_{1/n}(B,N)$. We will actually prove that the convex hull of N random points from $K = B$ contains a ball of radius $c\sqrt{\log(2N/n)}/\sqrt{n}$.

Lemma 7.2.13. *Let $B = rB_2^n$ be the centered ball of volume 1 in $\mathbb{R}^n$. If $\theta \in S^{n-1}$, then*

$$\mathrm{Prob}\,(x \in B : \langle x,\theta\rangle \geq \varepsilon r) \geq \exp(-4\varepsilon^2 n) \tag{7.2.43}$$

for every $\varepsilon \in (c_1/\sqrt{n}, 1/4)$, where $c_1 > 0$ is an absolute constant.

Proof. A simple calculation shows that

$$\begin{aligned} \mathrm{Prob}\,(x \in B : \langle x,\theta\rangle \geq \varepsilon r) &= \omega_{n-1}r^n \int_\varepsilon^1 (1-t^2)^{(n-1)/2}dt \\ &\geq \frac{\omega_{n-1}}{\omega_n}\varepsilon(1-4\varepsilon^2)^{(n-1)/2} \\ &\geq \exp(-4(n-1)\varepsilon^2) \geq \exp(-4\varepsilon^2 n) \end{aligned}$$

since $\sqrt{n}\omega_n \leq c_1\omega_{n-1}$ for some absolute constant $c_1 > 0$. □

Lemma 7.2.14. *There exist $c > 0$ and $n_0 \in \mathbb{N}$ such that: if $n \geq n_0$ and $n(\log n)^2 \leq N \leq \exp(cn)$, then*

$$C(x_1,\ldots,x_N) \supseteq \frac{\sqrt{\log(N/n)}}{6\sqrt{n}}B \tag{7.2.44}$$

with probability greater than $1 - \exp(-n)$.

Proof. By Lemma 7.2.13, for every $\theta \in S^{n-1}$ we have

$$\begin{aligned}\mathrm{Prob}\left((x_1,\ldots,x_N) : \max_{j\leq N}\langle x_j,\theta\rangle \leq \varepsilon r\right) &\leq (1-\exp(-4\varepsilon^2 n))^N \\ &\leq \exp(-N\exp(-4\varepsilon^2 n))\end{aligned}$$

for every $\varepsilon \in (c_1/\sqrt{n}, 1/4)$. Let $\mathcal{N}$ be a ρ-net for S^{n-1} with cardinality $|\mathcal{N}| \leq \exp(\log(1+2/\rho)n)$. If

$$\exp\left(n\log(1+2/\rho) - N\exp(-4\varepsilon^2 n)\right) \leq \exp(-n), \tag{7.2.45}$$

then with probability greater than $1-\exp(-n)$ we have $\max_{j\leq N}\langle x_j,\theta\rangle > \varepsilon r$ for all $\theta \in \mathcal{N}$. For every $u \in S^{n-1}$ we find $\theta \in \mathcal{N}$ with $|\theta - u| < \rho$. Then,

$$\max_{j\leq N}\langle x_j,u\rangle \geq \max_{j\leq N}\langle x_j,\theta\rangle - \max_{j\leq N}\langle x_j,\theta-u\rangle \geq (\varepsilon-\rho)r. \tag{7.2.46}$$

We choose $\varepsilon = 2a\left((\log(N/n)/n\right)^{1/2}$ ($a > 0$ is an absolute constant to be determined) and $\rho = \varepsilon/2$. Then,

$$n\log(1+2/\rho) + n \leq 2n\log(3/\rho) \leq n\log\left(\frac{9n}{a^2\log(N/n)}\right) \leq n\log n, \tag{7.2.47}$$

if $a^2 \geq 9/\log(N/n)$. Therefore, (7.2.45) will be a consequence of

$$\exp(16a^2\log(N/n)) \leq \frac{N}{n\log n}, \tag{7.2.48}$$

which can be written equivalently in the form

$$\left(\frac{N}{n}\right)^{1-16a^2} \geq \log n. \tag{7.2.49}$$

If $N \geq n(\log n)^2$ and $a = 1/6$, then (7.2.49) is clearly satisfied. Finally, our choice of a should be such that $a^2 \geq 9/(2\log\log n)$, which is also satisfied when $n \geq n_0$, for a suitable (absolute) $n_0 \in \mathbb{N}$. □

Theorem 7.2.15. *Let B be a ball of volume 1 in $\mathbb{R}^n$. If $n(\log n)^2 \leq N \leq \exp(cn)$, then*

$$\mathbb{E}_{1/n}(B,N) \geq c\frac{\sqrt{\log(N/n)}}{\sqrt{n}} \tag{7.2.50}$$

where $c > 0$ is an absolute constant.

Proof. Let $f(N,n) = \sqrt{\log(N/n)}/(6\sqrt{n})$ and

$$A = \{(x_1,\ldots,x_N) : C(x_1,\ldots,x_n) \supseteq f(N,n)B\}.$$

By Lemma 7.2.14 we have $\mathrm{Prob}(A) \geq 1-\exp(-n)$, and hence

$$\begin{aligned}\mathbb{E}_{1/n}(B,N) &\geq \int_A |C(x_1,\ldots,x_N)|^{1/n}dx_1\ldots dx_N \\ &\geq (1-\exp(-n))f(N,n)|B| \\ &\geq f(N,n)/2\end{aligned}$$

if n exceeds some (absolute) $n_0 \in \mathbb{N}$. This proves the Theorem. □

Combining Theorem 7.2.15 with Theorem 7.2.6 we have:

Theorem 7.2.16. *Let K be a convex body of volume 1 in $\mathbb{R}^n$. If $n(\log n)^2 \leq N \leq \exp(cn)$, then*

$$\mathbb{E}_{1/n}(K,N) \geq c\frac{\sqrt{\log(N/n)}}{\sqrt{n}} \tag{7.2.51}$$

where $c > 0$ is an absolute constant. □

Notes and References

Theorem 7.1.1 was proved by Bourgain with $n(\log n)^3$ dependence of m on n (see [11]). Rudelson [52] gave a different proof with $n(\log n)^2$ dependence. The proof we present follows Bourgain's ideas and comes from [25] (where several extensions of this result are obtained). The material of §7.2 comes from [26] (Theorem 7.2.7 is an observation which appears in [28]).

Chapter 8

The central limit problem

8.1 Concentration property for p-balls

Let $1 \leq p \leq \infty$ and let $r_{p,n} > 0$ be a constant such that $|r_{p,n}B_p^n| = 1$. We write $L_{p,n}$ for the isotropic constant of B_p^n and $\mu_{p,n}$ for the Lebesgue measure on $r_{p,n}B_p^n$.
As the next Theorem shows, most of the volume of the normalized ℓ_p^n-ball lies in a very thin spherical shell around the radius $\sqrt{n}L_{p,n}$:

Theorem 8.1.1. *For every $t > 0$,*

$$\mu_{p,n}\left(\left|\frac{|x|^2}{n} - L_{p,n}^2\right| \geq t\right) \leq \frac{CL_{p,n}^4}{nt^2}, \tag{8.1.1}$$

where $C > 0$ is an absolute constant.

The proof is based on the fact that normalized ℓ_p^n-balls have the following subindependence property.

Theorem 8.1.2 (Subindependence). *Let $K := r_{p,n}B_p^n$ and $P := \mu_{p,n}$. If $t_1,\ldots,t_n$ are non-negative numbers, then*

$$P\left(\bigcap_{i=1}^{n}\{|x_i| \geq t_i\}\right) \leq \prod_{i=1}^{n} P\big(\{|x_i| \geq t_i\}\big). \tag{8.1.2}$$

Proof. The Theorem will follow by induction if we show that

$$P\left(\bigcap_{i=1}^{n}\{|x_i| \geq t_i\}\right) \leq P\left(|x_1| \geq t_1\right) P\left(\bigcap_{i=2}^{n}\{|x_i| \geq t_i\}\right). \tag{8.1.3}$$

Set

$$S = \bigcap_{i=2}^{n}\{|x_i| \geq t_i\}. \tag{8.1.4}$$

Then, we need to prove that

$$\frac{|K \cap S \cap \{|x_1| \geq t_1\}|}{|K|} \leq \frac{|K \cap \{|x_1| \geq t_1\}|}{|K|} \cdot \frac{|K \cap S|}{|K|}. \tag{8.1.5}$$

We will apply the following simple fact: if μ is a positive measure on $[0,1]$ and $f:[0,1]\to\mathbb{R}$ is increasing, then

$$\mu([0,1])\int_0^s f d\mu \leq \mu([0,s])\int_0^1 f d\mu \tag{8.1.6}$$

for all $s\in[0,1]$. If

$$f(u)=\frac{|K\cap S\cap\{|x_1|=1-u\}|}{|K\cap\{|x_1|=1-u\}|}, \tag{8.1.7}$$

it is not hard to check that f is increasing. Let μ be the probability measure with density

$$g(u)=\frac{|K\cap\{|x_1|=1-u\}|}{|K|}. \tag{8.1.8}$$

Then,

$$\int_0^1 f(u)d\mu=\int_0^1\frac{|K\cap S\cap\{|x_1|=1-u\}|}{|K\cap\{|x_1|=1-u\}|}\frac{|K\cap\{|x_1|=1-u\}|}{|K|}du=\frac{|K\cap S|}{|K|} \tag{8.1.9}$$

and

$$\int_0^{1-t} f(u)d\mu=\frac{|K\cap S\cap\{|x_1|\geq t\}|}{|K|}. \tag{8.1.10}$$

Applying (8.1.6) for $s=1-t_1$ we get the result. $\square$

Theorem 8.1.2 immediately implies an anti-correlation inequality for the coordinate functions.

Corollary 8.1.3. *Let $K:=r_{p,n}B_p^n$. Then,*

$$\int_K x_i^2x_j^2dx\leq\int_K x_i^2dx\cdot\int_K x_j^2dx \tag{8.1.11}$$

for all $i\neq j$ in $\{1,\ldots,n\}$.

Proof. We write

$$\begin{aligned}\int_K x_i^2x_j^2dx &= 4\int_{K\cap\{x_i\geq 0,x_j\geq 0\}}x_i^2x_j^2dx\\ &= 4\int_0^\infty\int_0^\infty 4t_it_jP(x_i\geq t_i,x_j\geq t_j)dt_idt_j\\ &\leq 4\int_0^\infty\int_0^\infty 4t_it_jP(x_i\geq t_i)P(x_j\geq t_j)dt_idt_j\\ &\leq 4\left(\int_0^\infty 2t_iP(x_i\geq t_i)dt_i\right)\left(\int_0^\infty 2t_jP(x_j\geq t_j)dt_j\right)\\ &= 4\int_{K\cap\{x_i\geq 0\}}x_i^2dx\int_{K\cap\{x_j\geq 0\}}x_j^2dx\\ &= \int_K x_i^2dx\int_K x_j^2dx.\end{aligned}$$

Proof of Theorem 8.1.1: From the Cauchy-Schwarz inequality we have

$$n^2 L_{p,n}^4 = \left(\int_K |x|^2 dx\right)^2 \leq \int_K |x|^4 dx. \tag{8.1.12}$$

On the other hand, using Corollary 8.1.3 we have

$$\begin{aligned}
\int_K |x|^4 dx &= \int_K \left(\sum_{i=1}^n x_i^2\right)^2 dx = \sum_{i=1}^n \int_K x_i^4 dx + \sum_{i\neq j} \int_K x_i^2 x_j^2 dx \\
&\leq n \int_K x_1^4 dx + \sum_{i\neq j} \int_K x_i^2 dx \int_K x_j^2 dx \\
&= n \int_K x_1^4 dx + n(n-1) L_{p,n}^4.
\end{aligned}$$

Since

$$\int_K x_1^4 dx \leq C\left(\int_K x_1^2 dx\right)^2 = C L_{p,n}^4 \tag{8.1.13}$$

for some absolute constant $C > 0$, we get

$$L_{p,n}^4 \leq \frac{1}{n^2}\int_K |x|^4 dx \leq \left(1 + \frac{C}{n}\right) L_{p,n}^4. \tag{8.1.14}$$

This implies that

$$\int_K \left(\frac{|x|^2}{n} - L_{p,n}^2\right)^2 dx = \frac{1}{n^2}\int_K |x|^4 dx - L_{p,n}^4 \leq \frac{C}{n} L_{p,n}^4. \tag{8.1.15}$$

Then, Chebyshev's inequality gives

$$t^2 \mu_{p,n}\left(\left|\frac{|x|^2}{n} - L_{p,n}^2\right| \geq t\right) \leq \int_K \left(\frac{|x|^2}{n} - L_{p,n}^2\right)^2 dx \leq \frac{C}{n} L_{p,n}^4 \tag{8.1.16}$$

for every $t > 0$, which is exactly the assertion of the Theorem. □

Corollary 8.1.4. *For every $t > 0$,*

$$\mu_{p,n}\left(\left|\frac{|x|}{\sqrt{n}} - L_{p,n}\right| \geq t\right) \leq \frac{C L_{p,n}^2}{n t^2}. \tag{8.1.17}$$

Proof. Let $t > 0$. We have

$$\begin{aligned}
\mu_{p,n}\left(\big|\, |x| - \sqrt{n} L_{p,n}\, \big| \geq t\sqrt{n}\right) &\leq \mu_{p,n}\left(\big|\, |x|^2 - n L_{p,n}^2 \big| \geq t n L_{p,n}\right) \\
&\leq \frac{C L_{p,n}^4}{t^2 n L_{p,n}^2} = \frac{C L_{p,n}^2}{t^2 n}
\end{aligned}$$

by Theorem 8.1.1. □

8.2 The ε-concentration hypothesis

Let K be an isotropic convex body in $\mathbb{R}^n$. We view K as a probability space (with the Lebesgue measure μ_K on K) and for every $\theta \in S^{n-1}$ we consider the random variable $X_\theta(x) = \langle x, \theta\rangle$. Since K is isotropic, we have

$$\mathbb{E}X_\theta = 0 \ \text{ and } \ \mathrm{Var}(X_\theta) = L_K^2 \tag{8.2.1}$$

for every $\theta \in S^{n-1}$.

It is conjectured that most of these random variables have to be very close to a Gaussian random variable γ with mean 0 and variance L_K^2. In this Section we will see that this is true, at least for isotropic symmetric convex bodies, under the following general hypothesis which states that the Euclidean norm concentrates near the value $\sqrt{n}L_K$ as a function on K.

Concentration hypothesis: Let $0 < \varepsilon < \frac{1}{2}$. We say that K satisfies the *ε-concentration hypothesis* if

$$\mu_K\left(\left|\frac{|x|}{\sqrt{n}} - L_K\right| \geq \varepsilon L_K\right) \leq \varepsilon. \tag{8.2.2}$$

Remark 1: The results of §8.1 (see Corollary 8.1.4) show that the class of l_p^n balls satisfies the ε-concentration hypothesis with $\varepsilon \simeq \frac{1}{n^{1/3}}$.

Before stating the Theorem, we need to introduce some notation. We denote by $g(s)$ the density of the Gaussian random variable γ with variance L_K^2 and for simplicity we write $g_\theta(s)$ for the density of X_θ. Note that

$$g_\theta(s) = f_{K,\theta}(s) = |K \cap (\theta^\perp + s\theta)| \tag{8.2.3}$$

and

$$g(s) = \frac{1}{\sqrt{2\pi}L_K}\exp\left(-\frac{s^2}{2L_K^2}\right). \tag{8.2.4}$$

Theorem 8.2.1. *Let K be an isotropic symmetric convex body in $\mathbb{R}^n$ which satisfies the ε-concentration hypothesis for some $0 < \varepsilon < \frac{1}{2}$. Then, for every $\delta > 0$*

$$\begin{aligned}\sigma\left(\left\{\theta : \left|\int_{-t}^t g_\theta(s)\,ds - \int_{-t}^t g(s)\,ds\right| \leq \delta + 4\varepsilon + \frac{c_1}{\sqrt{n}} \ \textit{for every } \ t \in \mathbb{R}\right\}\right)& \\ \geq 1 - n\,e^{-c_2\delta^2 n},&\end{aligned} \tag{8.2.5}$$

where $c_1, c_2 > 0$ are absolute constants.

The proof is divided three steps. We first consider the average function

$$A(t) = \int_{S^{n-1}}\int_{-t}^t g_\theta(s)\,ds\,\sigma(d\theta) \tag{8.2.6}$$

and show that, under the ε-concentration hypothesis,

$$\left|A(t) - \int_{-t}^t g(s)\,ds\right| \leq 4\varepsilon + \frac{c_1}{\sqrt{n}} \tag{8.2.7}$$

for every $t > 0$.

Lemma 8.2.2. *Let K be an isotropic convex body in $\mathbb{R}^n$. For every $t > 0$,*

$$\left| A(t) - \frac{2}{\sqrt{2\pi}} \int_K \int_0^{\frac{t\sqrt{n}}{|x|}} e^{-\frac{u^2}{2}}\, du\, dx \right| \leq \frac{c_1}{\sqrt{n}}, \tag{8.2.8}$$

where $c_1 > 0$ is an absolute constant.

Proof. From Proposition 6.3.1 we have

$$f_K(s) = \int_{S^{n-1}} g_\theta(s)\sigma(d\theta) = c_n \int_{\{x\in K:|x|\geq s\}} \frac{1}{|x|}\left(1 - \frac{s^2}{|x|^2}\right)^{\frac{n-3}{2}} dx, \tag{8.2.9}$$

where $c_n = \Gamma\left(\frac{n}{2}\right)/\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)$. Note that

$$\frac{1}{2c_n} = \int_0^1 (1-u^2)^{\frac{n-3}{2}} du. \tag{8.2.10}$$

Now, Fubini's theorem gives

$$\begin{aligned} A(t) &= 2\int_0^t f_K(s)ds = 2c_n \int_0^t \int_{\{x\in K:|x|\geq t\}} \frac{1}{|x|}\left(1 - \frac{s^2}{|x|^2}\right)^{\frac{n-3}{2}} dx\, ds \\ &= 2c_n \int_K \int_0^{\min\left\{1,\frac{t}{|x|}\right\}} (1-u^2)^{\frac{n-3}{2}} du\, dx. \end{aligned}$$

We will prove that

$$\left| 2c_n \int_0^{\min\left\{1,\frac{t}{|x|}\right\}} (1-u^2)^{\frac{n-3}{2}} du - \frac{2}{\sqrt{2\pi}} \int_0^{\frac{t\sqrt{n}}{|x|}} e^{-\frac{u^2}{2}}\, du \right| \leq \frac{c_1}{\sqrt{n}} \tag{8.2.11}$$

for every $t > 0$ and $x \in K$, where $c_1 > 0$ is an absolute constant. This will prove the Lemma since the volume of K is equal to 1.

Case 1: If $|x| \leq t$, then by (8.2.10) we have to show that

$$\left| 1 - \frac{2}{\sqrt{2\pi}} \int_0^{\frac{t\sqrt{n}}{|x|}} e^{-\frac{u^2}{2}}\, du \right| \leq \frac{c_1}{\sqrt{n}}. \tag{8.2.12}$$

But the left hand side is equal to

$$\frac{2}{\sqrt{2\pi}} \int_{\frac{t\sqrt{n}}{|x|}}^\infty e^{-\frac{u^2}{2}} du \leq \frac{2}{\sqrt{2\pi}} \int_{\sqrt{n}}^\infty e^{-\frac{u^2}{2}} du \leq \frac{2}{\sqrt{2\pi n}} e^{-\frac{n}{2}} \leq \frac{2}{\sqrt{2\pi n}}. \tag{8.2.13}$$

Case 2: If $|x| \geq t$, we write

$$2c_n \int_0^{\frac{t}{|x|}} (1-u^2)^{\frac{n-3}{2}} du = \frac{2c_n}{\sqrt{n}} \int_0^{\frac{t\sqrt{n}}{|x|}} \left(1 - \frac{u^2}{n}\right)^{\frac{n-3}{2}} du. \tag{8.2.14}$$

Then, the left hand side of (8.2.11) is bounded by

$$\left| \frac{2c_n}{\sqrt{n}} - \frac{2}{\sqrt{2\pi}} \right| \int_0^{\frac{t\sqrt{n}}{|x|}} e^{-\frac{u^2}{2}} du + \frac{2c_n}{\sqrt{n}} \int_0^{\frac{t\sqrt{n}}{|x|}} \left| \left(1 - \frac{u^2}{n}\right)^{\frac{n-3}{2}} - e^{-\frac{u^2}{2}} \right| du. \tag{8.2.15}$$

Using the asymptotic formula $\Gamma(x) = x^{x-1}e^{-x}\sqrt{2\pi x}\left(1+\frac{1}{12x}+O(x^{-2})\right)$ as $x \to +\infty$, we see that

$$\frac{2c_n}{\sqrt{n}} = \frac{2}{\sqrt{2\pi}}\left(\frac{n}{n-1}\right)^{\frac{n}{2}-1}\frac{1}{\sqrt{e}}\left(1+O\left(\frac{1}{n}\right)\right). \tag{8.2.16}$$

It follows that the first term in (8.2.15) is bounded by c/n as $n \to \infty$.

For the second term, consider the function $h(u) = e^{-\frac{u^2}{2}} - \left(1-\frac{u^2}{n}\right)^{\frac{n-3}{2}}$ on $[0,\sqrt{n}]$. Note that $h(0) = 0$ and $h(\sqrt{n}) = \exp(-n/2)$. If there is a point $v \in [0,\sqrt{n}]$ such that $h'(v) = 0$, then $\left(1-\frac{v^2}{n}\right)^{\frac{n-5}{2}} = \frac{n}{n-3}e^{-\frac{v^2}{2}}$, and hence, $h(v) = \frac{v^2-3}{n-3}e^{-\frac{v^2}{2}}$ Since this last expression is $O(1/n)$ on $[0,\sqrt{n}]$, we have

$$\int_0^{\frac{t\sqrt{n}}{|x|}}\left|\left(1-\frac{u^2}{n}\right)^{\frac{n-3}{2}} - e^{-\frac{u^2}{2}}\right|du \leq \frac{c'}{\sqrt{n}}. \tag{8.2.17}$$

On the other hand we have $c_n \simeq \sqrt{n}$, which shows that the second term in (8.2.15) is bounded by $c/\sqrt{n}$ as $n \to \infty$. □

Theorem 8.2.3. *Let K be an isotropic convex body in $\mathbb{R}^n$. If K satisfies the ε-concentration hypothesis, then*

$$\left|A(t) - \int_{-t}^{t} g(s)ds\right| \leq 4\varepsilon + \frac{c_1}{\sqrt{n}} \tag{8.2.18}$$

for every $t > 0$.

Proof. Let $t > 0$ and set

$$F_t(s) = \frac{2}{\sqrt{2\pi}}\int_0^{\frac{t}{s}} e^{-\frac{u^2}{2}}\,du. \tag{8.2.19}$$

In this notation, Lemma 8.2.2 states that

$$\left|A(t) - \int_K F_t\left(\frac{|x|}{\sqrt{n}}\right)dx\right| \leq \frac{c_1}{\sqrt{n}}. \tag{8.2.20}$$

Note that

$$F_t(L_K) = \frac{2}{\sqrt{2\pi}}\int_0^{\frac{t}{L_K}} e^{-\frac{u^2}{2}}du = \frac{1}{\sqrt{2\pi}L_K}\int_{-t}^{t} e^{-\frac{s^2}{2L_K^2}}ds = \int_{-t}^{t} g(s)ds. \tag{8.2.21}$$

So, the Theorem states that

$$|A(t) - F_t(L_K)| \leq 4\varepsilon + \frac{c_1}{\sqrt{n}}. \tag{8.2.22}$$

Because of (8.2.20), the Theorem will be proved if we check that

$$\left|\int_K F_t\left(\frac{|x|}{\sqrt{n}}\right)dx - F_t(L_K)\right| \leq 4\varepsilon. \tag{8.2.23}$$

We divide K into two subsets:

$$K_1 = K \cap \left\{ \left| \frac{|x|}{\sqrt{n}} - L_K \right| \leq \varepsilon L_K \right\} \tag{8.2.24}$$

and

$$K_2 = K \cap \left\{ \left| \frac{|x|}{\sqrt{n}} - L_K \right| \geq \varepsilon L_K \right\}. \tag{8.2.25}$$

Then,

$$\left| \int_K F_t\left(\frac{|x|}{\sqrt{n}}\right) dx - F_t(L_K) \right| \leq \sum_{i=1}^{2} \int_{K_i} \left| F_t\left(\frac{|x|}{\sqrt{n}}\right) - F_t(L_K) \right| dx. \tag{8.2.26}$$

To estimate the integral on K_2, we just use the fact that $F_t\left(\frac{|x|}{\sqrt{n}}\right)$ and $F_t(L_K)$ are bounded by 1. Since K satisfies the ε-concentration hypothesis,

$$\int_{K_2} \left| F_t\left(\frac{|x|}{\sqrt{n}}\right) - F_t(L_K) \right| dx \leq 2|K_2| \leq 2\varepsilon. \tag{8.2.27}$$

For the integral on K_1 we shall use a Lipschitz estimate for F_t. Observe that

$$|F'(s)| = \frac{2}{\sqrt{2\pi}} \frac{t}{s^2} e^{-\frac{t^2}{2s^2}} \leq \frac{1}{s}, \tag{8.2.28}$$

since $x \exp(-x^2/2) \leq 1/\sqrt{e}$ on $[0, +\infty)$. Also, since $\varepsilon < 1/2$, for every $x \in K_1$ we have

$$\frac{|x|}{\sqrt{n}} > \frac{L_K}{2}. \tag{8.2.29}$$

It follows that

$$\int_{K_1} \left| F_t\left(\frac{|x|}{\sqrt{n}}\right) - F_t(L_K) \right| dx \leq \int_{K_1} \frac{2}{L_K} \left| \frac{|x|}{\sqrt{n}} - L_K \right| dx \leq \int_{K_1} 2\varepsilon \, dx \leq 2\varepsilon. \tag{8.2.30}$$

Combining the above we get (8.2.23), and taking into account (8.2.20) we conclude the proof. $\square$

In the second step we use the estimate of Theorem 8.2.3 for the average $A(t)$ to obtain a similar estimate for "most directions" $\theta \in S^{n-1}$. The idea is to show that $\int_{-t}^{t} g_\theta(s) ds$ is the (restriction on S^{n-1} of the) radial function of a symmetric convex body in $\mathbb{R}^n$ and then use the spherical isoperimetric inequality in the context of Lipschitz continuous functions on the sphere.

Here, we make use of Busemann's inequality (see §4.3) which states the following:

Lemma 8.2.4. *Let W be a symmetric convex body in $\mathbb{R}^m$ and define $r(u) = |W \cap u^\perp|$ for all $u \in S^{m-1}$. Then, r is the (the restriction of the) radial function of a symmetric convex body in $\mathbb{R}^m$.* $\square$

Proposition 8.2.5. *Let K be a symmetric convex body in $\mathbb{R}^n$. Fix $t > 0$ and define*

$$\|x\|_t = \frac{|x|}{\int_{-t}^{t} g_{\frac{x}{|x|}}(s)ds}. \tag{8.2.31}$$

Then, $\|\cdot\|_t$ is a norm on $\mathbb{R}^n$.

Proof. Recall that $g_\theta(s) = \left|K \cap (\theta^\perp + s\theta)\right|$ for every $\theta \in S^{n-1}$. We define

$$v(x,t) = \int_{-t}^{t} g_{\frac{x}{|x|}}(s)ds \tag{8.2.32}$$

and prove that for all $x, y \in \mathbb{R}^n$,

$$\frac{1}{2}\left(\frac{|x|}{v(x,t)} + \frac{|y|}{v(y,t)}\right) \geq \frac{\left|\frac{x+y}{2}\right|}{v\left(\frac{x+y}{2}, t\right)}. \tag{8.2.33}$$

We may clearly assume that x and y are linearly independent. Consider the convex body $K' = K \times [-1,1]$ in $\mathbb{R}^{n+1}$. Lemma 8.2.4 shows that $\frac{|\theta|}{|K' \cap \theta^\perp|}$ defines a norm on $\mathbb{R}^{n+1}/\{0\}$. That is,

$$\frac{1}{2}\left(\frac{|\theta|}{|K' \cap \theta^\perp|} + \frac{|\phi|}{|K' \cap \phi^\perp|}\right) \geq \frac{\left|\frac{\theta+\phi}{2}\right|}{\left|K' \cap \left(\frac{\theta+\phi}{2}\right)^\perp\right|} \tag{8.2.34}$$

for all linearly independent $\theta, \phi \in \mathbb{R}^{n+1}$.

Let $r \in (0,1)$ be defined by the equation $tr = \sqrt{1-r^2}$. We observe that if $z \in \mathbb{R}^n \setminus \{0\}$ and

$$u(z) = \left(r\,\frac{z}{|z|}, \sqrt{1-r^2}\right), \tag{8.2.35}$$

then the projection of $K' \cap u(z)^\perp$ onto the first n coordinates is $\{w \in K : |\langle w, z\rangle| \leq t|z|\}$. It follows that

$$v(z,t) = \sqrt{1-r^2}\,|K' \cap u(z)^\perp|. \tag{8.2.36}$$

We define $\eta(z) = |z|\,u(z)$. Then, $|K' \cap u(z)^\perp| = |K' \cap \eta(z)^\perp|$ and $|\eta(z)| = |z|$. If we set $\theta = \eta(x)$ and $\phi = \eta(y)$, Lemma 8.2.4 shows that

$$\frac{1}{2}\left(\frac{|x|}{v(x,t)} + \frac{|y|}{v(y,t)}\right) \geq \frac{1}{\sqrt{1-r^2}}\frac{\left|\frac{\theta+\phi}{2}\right|}{\left|K' \cap \left(\frac{\theta+\phi}{2}\right)^\perp\right|}. \tag{8.2.37}$$

Observe that

$$\begin{aligned}
\frac{\theta+\phi}{2} &= \left(r\,\frac{x+y}{2}, \sqrt{1-r^2}\,\frac{(|x|+|y|)}{2}\right) \\
&= \frac{|x+y|}{2}\left(r\frac{x+y}{|x+y|}, \sqrt{1-r^2}\frac{|x|+|y|}{|x+y|}\right).
\end{aligned}$$

Then, the projection of $K' \cap \left(\frac{\theta+\phi}{2}\right)^{\perp}$ onto the first n coordinates is a strip perpendicular to $\frac{x+y}{2}$, with width

$$s = \frac{|x|+|y|}{|x+y|}\, t, \tag{8.2.38}$$

and this gives

$$\frac{v\left(\frac{x+y}{2}, s\right)}{\left|K' \cap \left(\frac{\theta+\phi}{2}\right)^{\perp}\right|} = \frac{\sqrt{1-r^2}}{2}\, \frac{|x|+|y|}{\left|\frac{\theta+\phi}{2}\right|}. \tag{8.2.39}$$

Then, (8.2.37) takes the form

$$\frac{1}{2}\left(\frac{|x|}{v(x,t)} + \frac{|y|}{v(y,t)}\right) \geq \frac{1}{2}\, \frac{|x|+|y|}{v\left(\frac{x+y}{2}, s\right)}. \tag{8.2.40}$$

Observe that if $a \geq 1$, then for every $z \in \mathbb{R}^n$ we have

$$v(z, at) \leq a\, v(z,t) \tag{8.2.41}$$

(this follows from the fact that $g_\theta(as) \leq g_\theta(s)$ for all $\theta \in S^{n-1}$ and $s > 0$). So,

$$\frac{|x|+|y|}{2v\left(\frac{x+y}{2}, s\right)} = \frac{|x|+|y|}{2v\left(\frac{x+y}{2}, \frac{|x|+|y|}{|x+y|}t\right)} \geq \frac{|x|+|y|}{2\frac{|x|+|y|}{|x+y|} v\left(\frac{x+y}{2}, t\right)} = \frac{\left|\frac{x+y}{2}\right|}{v\left(\frac{x+y}{2}, t\right)}. \tag{8.2.42}$$

Going back to (8.2.40) we conclude the proof. $\square$

Lemma 8.2.6. *Let K be a symmetric convex body in $\mathbb{R}^n$ with volume 1 . For every $\theta \in S^{n-1}$ and every $t > 0$,*

$$\int_t^\infty g_\theta(s)ds \leq \frac{1}{2} e^{-2g_\theta(0)t}. \tag{8.2.43}$$

Proof. Consider the function

$$H(t) = \int_t^\infty g_\theta(s)ds = \int_0^\infty \chi_{[t,\infty)}(s) g_\theta(s)ds. \tag{8.2.44}$$

Using the fact that g_θ is log-concave and applying the Prékopa-Leindler inequality, we may easily check that H is log-concave. It follows that

$$(\log H)(t) - (\log H)(0) \leq (\log H)'(0)t \tag{8.2.45}$$

for every $t > 0$. Observe that $H(0) = 1/2$ by the symmetry of K, and

$$(\log H)'(0) = -\frac{g_\theta(0)}{H(0)} = -2g_\theta(0). \tag{8.2.46}$$

It follows that

$$H(t) \leq H(0) \exp\big((\log H)'(0)t\big) = \frac{1}{2}\exp(-2g_\theta(0)t), \tag{8.2.47}$$

as stated in the Lemma. $\square$

Lemma 8.2.7. *Let K be an isotropic symmetric convex body in $\mathbb{R}^n$. For every $t > 0$, the norm*

$$\|x\|_t = \frac{|x|}{\int_{-t}^{t} g_{\frac{x}{|x|}}(s)\,ds} \tag{8.2.48}$$

satisfies

$$a|x| \leq \|x\|_t \leq b|x| \tag{8.2.49}$$

for every $x \in \mathbb{R}^n$, where a, b are two positive constants such that $a \geq 1$ and $b/a \leq c$ for some absolute constant $c > 0$.

Proof. Since K is isotropic, we know that $g_\theta(0) \simeq L_K^{-1}$ for every $\theta \in S^{n-1}$. Then, by the symmetry of K we have

$$\int_{-t}^{t} g_\theta(s)\,ds \leq \min\{2t\, g_\theta(0), 1\} \leq \min\left\{\frac{c_1 t}{L_K}, 1\right\}. \tag{8.2.50}$$

Also, Lemma 8.2.6 shows that

$$\int_{-t}^{t} g_\theta(s)\,ds = 1 - 2\int_{t}^{\infty} g_\theta(s)\,ds \geq 1 - e^{-2g_\theta(0)t} \geq 1 - e^{-\frac{c_2 t}{L_K}}. \tag{8.2.51}$$

We easily check that

$$1 - e^{-\frac{c_5 t}{L_K}} \geq \frac{c_2 t}{2L_K} \tag{8.2.52}$$

if $c_2 t \leq L_K$. In any case,

$$\int_{-t}^{t} g_\theta(s)\,ds \geq \min\left\{\frac{c_3 t}{L_K}, 1 - e^{-1}\right\}. \tag{8.2.53}$$

In other words

$$a := \max\left\{\frac{L_K}{c_4 t}, 1\right\} \leq \|\theta\|_t \leq b := \max\left\{\frac{L_K}{c_3 t}, \frac{e}{e-1}\right\} \tag{8.2.54}$$

for every $\theta \in S^{n-1}$. Note that $a \geq 1$ and b/a is bounded independently of t and L_K. □

Theorem 8.2.8. *Let K be an isotropic symmetric convex body in $\mathbb{R}^n$. If K satisfies the ε-concentration hypothesis, then for every $t > 0$ and $\delta > 0$,*

$$\sigma\left(\left\{\theta : \left|\int_{-t}^{t} g_\theta(s)ds - \int_{-t}^{t} g(s)ds\right| \geq \delta + 4\varepsilon + \frac{c_3}{\sqrt{n}}\right\}\right) \leq 2e^{-c_4\delta^2 n}, \tag{8.2.55}$$

where $c_3, c_4 > 0$ are absolute constants.

Proof. Let $t > 0$ and $\delta > 0$ be fixed. We will use the spherical isoperimetric inequality through the following fact: If $f : S^{n-1} \to \mathbb{R}$ is d-Lipschitz and M(f) is its mean, then

$$\sigma\left(\left\{\theta : |f - M(f)| \geq \delta + \frac{c_2}{\sqrt{n}}\right\}\right) \leq 2e^{-\frac{\delta^2 n}{2d^2}}. \tag{8.2.56}$$

We shall apply this to the function $f(\theta) = \int_{-t}^{t} g_{\theta(s)}\, ds$, Observe that

$$\begin{aligned}\left|\int_{-t}^{t} g_\theta(s)\,ds - \int_{-t}^{t} g_\phi(s)\,ds\right| &= \left|\frac{1}{||\theta||} - \frac{1}{||\phi||}\right| \\ &\leq \frac{||\theta-\phi||}{||\theta||\;||\phi||} \\ &\leq \frac{b}{a^2}|\theta-\phi| \\ &\leq c|\theta-\phi|,\end{aligned}$$

where c is the absolute constant in Lemma 8.2.7. Also, note that $M(f) = A(t)$. It follows that

$$\sigma\left(\left\{\theta : \left|\int_{-t}^{t} g_\theta(s)\,ds - A(t)\right| \geq \delta + \frac{c_2}{\sqrt{n}}\right\}\right) \leq 2\exp\left(-\frac{\delta^2 n}{2c^2}\right). \tag{8.2.57}$$

Combining this with Theorem 8.2.3, we get

$$\sigma\left(\left\{\theta : \left|\int_{-t}^{t} g_\theta(s)ds - \int_{-t}^{t} g(s)ds\right| \geq \delta + 4\varepsilon + \frac{c_3}{\sqrt{n}}\right\}\right) \leq 2\,\exp\left(-c_4\delta^2 n\right), \tag{8.2.58}$$

where $c_3 = c_1 + c_2$ (c_1 is the constant in Theorem 8.2.3) and $c_4 = 1/(2c^2)$. $\square$

Proof of Theorem 8.2.1: First, fix some $\theta \in S^{n-1}$. Since

$$g_\theta(s) \leq g_\theta(0) \leq \frac{c_1}{L_K} \tag{8.2.59}$$

and

$$g(s) = \frac{1}{\sqrt{2\pi}L_K}\exp(-s^2/(2L_K^2)) \leq \frac{1}{\sqrt{2\pi}L_K} \tag{8.2.60}$$

for every $s > 0$, the function

$$H(t) = \left|\int_{-t}^{t} g_\theta(s)ds - \int_{-t}^{t} g(s)ds\right| \tag{8.2.61}$$

is Lipschitz continuous with constant $d \leq \frac{c_2}{L_K}$, where c_2 is an absolute constant.

Also, there is an absolute constant $c_3 > 0$ such that $H(t) \leq 1/\sqrt{n}$ for every $t \geq c_3 L_K \log n$. This is a consequence of the equality

$$H(t) = 2\left|\int_{t}^{\infty} g_\theta(s)ds - \int_{t}^{\infty} g(s)ds\right| \tag{8.2.62}$$

and of Lemma 8.2.6: if $c_3 > 0$ is chosen large enough, when $t \geq c_3 L_K \log n$ we have

$$\max\left\{\int_{t}^{\infty} g_\theta(s)ds, \int_{t}^{\infty} g(s)ds\right\} < \frac{1}{2\sqrt{n}}. \tag{8.2.63}$$

Define $t_k = k\alpha$, where $\alpha = L_K/\sqrt{n}$ and $k = 1, \ldots, k_0 = [c_3\sqrt{n}\log n] + 1$. From Theorem 8.2.8, for every $\delta > 0$ we have

$$\sigma(A) \leq 2c_3\sqrt{n}(\log n)e^{-c_6\delta^2 n}, \tag{8.2.64}$$

where

$$A = \left\{ \theta : \exists k \le k_0 \text{ s.t. } \left| \int_{-t_k}^{t_k} g_\theta(s)ds - \int_{-t_k}^{t_k} g(s)ds \right| \ge \delta + 4\varepsilon + \frac{c_5}{\sqrt{n}} \right\} \tag{8.2.65}$$

and $c_5, c_6 > 0$ are absolute constants. If θ is not in A, then

$$H(t_k) \le \delta + 4\varepsilon + \frac{c_5}{\sqrt{n}} \tag{8.2.66}$$

for all $k = 1, \dots, k_0$. Since H is $\frac{c_2}{L_K}$-Lipschitz, we get a similar estimate for $H(t)$, $t \in [0, c_3 L_K \log n]$. Finally, if $t > c_3 L_K \log n$, we know that $H(t) < 1/\sqrt{n}$. $\square$

8.3 The variance hypothesis

Let K be an isotropic convex body in $\mathbb{R}^n$. In this Section we study the parameter σ_K of K which is defined by

$$\sigma_K^2 = \frac{\mathrm{Var}(|x|^2)}{nL_K^4} \tag{8.3.1}$$

and its connections with the problems we discussed in this Chapter as well as in Chapter 6.

It is useful to write σ_K in the form

$$\sigma_K^2 = \frac{n\mathrm{Var}(|x|^2)}{\left(\mathbb{E}|x|^2\right)^2}. \tag{8.3.2}$$

In this way the quantity becomes invariant under homotheties, and hence, easier to compute.

A simple computation shows that if $K = B_2^n$ then

$$\mathbb{E}|x|^4 = \frac{n}{n+4} \ \text{ and } \ \mathbb{E}|x|^2 = \frac{n}{n+2}. \tag{8.3.3}$$

Therefore,

$$\sigma_{B_2^n}^2 = n\left(\frac{\mathbb{E}|x|^4}{\left(\mathbb{E}|x|^2\right)^2} - 1 \right) = \frac{4}{n+4}. \tag{8.3.4}$$

Actually, Euclidean balls have minimal σ_K as the next Theorem shows.

Theorem 8.3.1. *Let K be an isotropic convex body in $\mathbb{R}^n$. Then,*

$$\sigma_K \ge \sigma_{B_2^n}. \tag{8.3.5}$$

Proof. Let x be uniformly distributed in K. The distribution function $F(r) = |\{x \in K : |x| \le r\}|$ has density

$$F'(r) = n\omega_n r^{n-1}\sigma\Big(\frac{1}{r}K\Big) \tag{8.3.6}$$

for $r > 0$. We define $q(r) = n\omega_n \sigma\left(\frac{1}{r}K\right)$. Observe that q is increasing and can be assumed absolutely continuous. Therefore, we can write q in the form

$$q(r) = n\int_r^\infty \frac{p(s)}{s^n}ds, \tag{8.3.7}$$

where $p : (0, +\infty) \to \mathbb{R}$ is a non-negative measurable function. Then, Fubini's theorem shows that

$$\int_0^\infty p(s)ds = n\int_0^\infty \frac{p(s)}{s^n}\left(\int_0^s r^{n-1}dr\right)ds = \int_0^\infty r^{n-1}q(r)dr = 1, \tag{8.3.8}$$

which means that p is the density of some positive random variable ξ.

Also, for every $\alpha > -n$,

$$\mathbb{E}|x|^\alpha = \int_0^\infty r^{\alpha+n-1}q(r)dr = \frac{n}{n+\alpha}\int_0^\infty s^\alpha p(s)ds = \frac{n}{n+\alpha}\mathbb{E}\xi^\alpha. \tag{8.3.9}$$

We can now compute

$$\begin{aligned} \mathrm{Var}(|x|^2) &= \frac{n}{n+4}\mathbb{E}\xi^4 - \left(\frac{n}{n+2}\mathbb{E}\xi^2\right)^2 \\ &= \frac{4n}{(n+4)(n+2)^2}\left(\mathbb{E}\xi^2\right)^2 + \frac{n}{n+4}\mathrm{Var}(\xi^2) \\ &\geq \frac{4n}{(n+4)(n+2)^2}\left(\mathbb{E}\xi^2\right)^2. \end{aligned}$$

It follows that

$$\sigma_K^2 = n\frac{\mathrm{Var}(|x|^2)}{\left(\mathbb{E}|x|^2\right)^2} \geq n\frac{\frac{4n}{(n+4)(n+2)^2}\left(\mathbb{E}\xi^2\right)^2}{\left(\frac{n}{n+2}\mathbb{E}\xi^2\right)^2} = \frac{4}{n+4}, \tag{8.3.10}$$

and the Theorem follows from (8.3.4). □

Remark 1: Simple computations show that

$$\sigma_{B_1^n}^2 = 1 - \frac{2(n+1)}{(n+3)(n+4)} \to 1 \ \text{ as } \ n \to \infty \tag{8.3.11}$$

and

$$\sigma_{B_\infty^n} = \frac{4}{5} \ \text{ for every } n. \tag{8.3.12}$$

In the next Section we briefly discuss various consequences of the following hypothesis.

Variance hypothesis: There exists an absolute constant $C > 0$ such that $\sigma_K^2 \leq C$ for every isotropic convex body.

Let us first note that σ_K^2 is uniformly bounded for all ℓ_p^n. This follows by the subindependence theorem of §8.1. Actually, the argument is inside the proof of Theorem 8.1.1.

Proposition 8.3.2. *There exists an absolute constant $C>0$ such that $\sigma_{B_p^n}^2 \leq C$ for every n and every $p\in[1,\infty]$.*

Proof. In the proof of Theorem 8.1.1 we saw that

$$n^2 L_{p,n}^4 \leq \int_K |x|^4 dx \leq (n^2+Cn)L_{p,n}^4 \tag{8.3.13}$$

for some absolute constant $C>0$. Then,

$$\sigma_{B_p^n}^2 = n\left(\frac{\mathbb{E}|x|^4}{n^2 L_{p,n}^4}-1\right) \leq C \tag{8.3.14}$$

for all p and n. □

8.4 Consequences of the variance hypothesis

8.4.1 The ε-concentration hypothesis

Theorem 8.4.1. *Let K be an isotropic convex body in $\mathbb{R}^n$ with $8\sigma_K^2 < n$. Then, K satisfies the ε-concentration hypothesis with $\varepsilon = \left(\frac{\sigma_K^2}{n}\right)^{1/3}$.*

Proof. For every $\varepsilon>0$ we have

$$\begin{aligned}\mu_K\left(x\in K : \big|\,|x|-\sqrt{n}L_K\,\big| \geq \varepsilon\sqrt{n}L_K\right) &\leq \mu_K\left(x\in K : \big|\,|x|^2-nL_K^2\big| \geq \varepsilon n L_K^2\right)\\ &\leq \frac{\mathrm{Var}(|x|^2)}{\varepsilon^2 n^2 L_K^4}.\end{aligned}$$

By the definition of σ_K^2,

$$\frac{\mathrm{Var}(|x|^2)}{\varepsilon^2 n^2 L_K^4} = \frac{\sigma_K^2 n L_K^4}{\varepsilon^2 n^2 L_K^4} \leq \frac{\sigma_K^2}{\varepsilon^2 n}. \tag{8.4.1}$$

If we define $\varepsilon = \left(\frac{\sigma_K^2}{n}\right)^{1/3}$, then $\frac{\sigma_K^2}{\varepsilon^2 n} = \varepsilon$ and the result follows. □

Corollary 8.4.2. *Assume that there is an absolute constant $C>0$ such that $\sigma_K^2 \leq C$ for every isotropic convex body K. Then, every isotropic convex body satisfies the ε-concentration hypothesis with $\varepsilon = \left(\frac{C}{n}\right)^{1/3}$.* □

Then, Theorem 8.1.1 has the following consequence.

Corollary 8.4.3. *Assume that there is an absolute constant $C>0$ such that $\sigma_K^2 \leq C$ for every isotropic convex body K. Then, for every isotropic symmetric convex body K in $\mathbb{R}^n$ we have*

$$\sigma\left(\left\{\theta : \left|\int_{-t}^t g_\theta(s)\,ds - \int_{-t}^t g(s)\,ds\right| \leq \frac{c_1\sqrt[3]{C}}{\sqrt[3]{n}} \text{ for all } t\in\mathbb{R}\right\}\right) \geq 1-e^{-c_2\sqrt[3]{n}}, \tag{8.4.2}$$

where $c_1,c_2>0$ are absolute constants. □

8.4.2 Tail estimates for the Euclidean norm

Let K be an isotropic convex body in $\mathbb{R}^n$. The proof of Theorem 8.4.1 shows that

$$\mu_K(x \in K : |x| \geq (1+s)\sqrt{n}L_K) \leq \frac{\sigma_K^2}{s^2 n}. \tag{8.4.3}$$

Therefore,

$$\mu_K(x \in K : |x| \geq (1+2\sigma_K)\sqrt{n}L_K) \leq \frac{1}{4n}. \tag{8.4.4}$$

Applying Borell's lemma we get:

Theorem 8.4.4. *Let K be an isotropic convex body in $\mathbb{R}^n$. Then,*

$$|\{x \in K : |x| \geq (1+2\sigma_K)\sqrt{n}L_K t\}| \leq n^{-\frac{t}{2}} \tag{8.4.5}$$

for every $t \geq 1$. □

Corollary 8.4.5. *Assume that there is an absolute constant $C > 0$ such that $\sigma_K^2 \leq C$ for every isotropic convex body K. Then, the Question of Chapter 6 has an affirmative answer with $\phi(n) \simeq \log n$.* □

8.4.3 Average sectional decay

Bobkov and Koldobsky (see [6]) proved the following.

Theorem 8.4.6. *Let K be an isotropic convex body in $\mathbb{R}^n$. Then,*

$$\left| f_K(t) - \frac{1}{\sqrt{2\pi}L_K} e^{-\frac{t^2}{2L_K^2}} \right| \leq c_1 \left(\frac{\sigma_K L_K}{t^2\sqrt{n}} + \frac{1}{n} \right) \tag{8.4.6}$$

for all $0 < t \leq c_2\sqrt{n}$, where $c_1, c_2 > 0$ are absolute constants. □

If we assume that there is an absolute constant $C > 0$ such that $\sigma_K^2 \leq C$ for every isotropic convex body K, this result shows that: for every isotropic convex body, f_K is very close to the Gaussian density with mean zero and variance L_K^2.

8.4.4 Further reductions

Let K be an isotropic convex body in $\mathbb{R}^n$. For every $x \in K$ we denote by $\chi(x)$ the length of the longest line segment which is contained in K and has x as its midpoint. Let

$$\chi(K) = \int_K \chi(x)dx. \tag{8.4.7}$$

Using the localization lemma, Kannan, Lovasz and Simonovits (see [33]) proved the following.

Theorem 8.4.7. *If K is an isotropic convex body, then $\sigma_K L_K \leq 8\chi(K)$.* □

One should also mention the following result of Voigt [58]:

Theorem 8.4.8. *Let $\varepsilon > 0$. Consider the quantity*

$$A(\varepsilon, n) := \sup\left\{\mu_K(x \in K : |\, |x|^2 - nL_K^2 \,| \geq \varepsilon n L_K^2)\right\}, \tag{8.4.8}$$

where the supremum is over all isotropic convex bodies in $\mathbb{R}^n$. If

$$\lim_{n\to\infty} A(\varepsilon, n) = 0 \tag{8.4.9}$$

for every $\varepsilon > 0$, then

$$\lim_{n\to\infty} \sup\left\{\int_K |x|^2 dx \left(\int_K \frac{1}{|x|} dx\right)^2\right\} = 1, \tag{8.4.10}$$

where the supremum is again over all isotropic convex bodies in $\mathbb{R}^n$. □

Proposition 6.3.1 shows that

$$f_K(0) = c_n \int_K \frac{1}{|x|} dx \tag{8.4.11}$$

and, in the proof of Lemma 8.2.2, we saw that

$$\lim_{n\to\infty} \frac{c_n\sqrt{2\pi}}{\sqrt{n}} = 1. \tag{8.4.12}$$

Therefore, (8.4.10) is equivalent to

$$\lim_{n\to\infty} \sup\left\{\frac{f_K(0)}{\sqrt{2\pi} L_K} : K \text{ isotropic in } \mathbb{R}^n\right\} = 1. \tag{8.4.13}$$

This has been verified only in special cases by Koldobsky and Lifshits (see [32]).

Notes and References

The results of Sections §8.1 and §8.2 are due to Anttila, Ball and Perissinaki (see [2] and [4]). The variance hypothesis is discussed in [6]. It appears implicitly in various articles (see the discussion in §8.4).

Appendix A

Entropy estimates

A.1 Subgaussian processes

Let (T, d) be a metric space, let $(\Omega, \mathcal{A}, P)$ be a probability space, and let $\mathcal{Y} = (Y_t)_{t \in T}$ be a family of real valued random variables with indices from T. We say that the process $\mathcal{Y} = (Y_t)_{t \in T}$ is subgaussian with respect to d if

$$\mathbb{E} Y_t = 0 \tag{A.1}$$

for all $t \in T$ and, for all $t, s \in T$ and every $u > 0$,

$$\operatorname{Prob}(|Y_t - Y_s| \geq u) \leq 2 \exp\left(- \frac{u^2}{d^2(t,s)} \right). \tag{A.2}$$

Typical example: Let $E_2^n = \{-1, 1\}^n$ be equipped with the uniform probability measure. Write $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ for the points of E_2^n. The Rademacher functions are the random variables $r_i : E_2^n \to \{-1, 1\}$ $(1 \leq i \leq n)$ defined by $r_i(\varepsilon) = \varepsilon_i$.

For every $t = (t_1, \dots, t_n) \in T \subseteq \mathbb{R}^n$ we define

$$Y_t(\varepsilon) = \langle t, \varepsilon \rangle = t_1 r_1(\varepsilon) + \cdots + t_n r_n(\varepsilon). \tag{A.3}$$

The inequality

$$\operatorname{Prob}(\varepsilon \in E_2^n : |t_1 r_1(\varepsilon) + \cdots + t_n r_n(\varepsilon)| \geq u) \leq 2 \exp\big(- u^2 / 2(t_1^2 + \cdots + t_n^2) \big) \tag{A.4}$$

which is - for example - a consequence of the classical Khintchine inequality, shows that $\mathcal{Y} = (Y_t)_{t \in \mathbb{R}^n}$ is subgaussian with respect to the Euclidean metric.

We write g for a standard Gaussian random variable and $G = (g_1, \dots, g_n)$ for the standard Gaussian random vector in $\mathbb{R}^n$. The distribution of G is the Gaussian measure γ_n, with density $(2\pi)^{-n/2} \exp(-|x|^2/2)$.

Let T be a non empty set. A family $\mathcal{Z} = (Z_t)_{t \in T}$ of real valued random variables on $(\Omega, \mathcal{A}, P)$ is called a Gaussian process if every linear combination $a_1 Z_{t_1} + \cdots + a_m Z_{t_m}$ of Z_t's is a Gaussian random variable with mean 0. We may view $\mathcal{Z}$ as a subset of $L^2(\Omega)$, and then it induces on T the metric

$$d(t, s) = \|Z_t - Z_s\|_{L^2(\Omega)}. \tag{A.5}$$

By the definition of a Gaussian process, for every $t, s \in T$, $Z_t - Z_s$ is a Gaussian random variable with mean 0 and variance $\mathbb{E}(Z_t - Z_s)^2 = d^2(t,s)$. Consequently, for every $u > 0$ we have

(A.6)

$$\operatorname{Prob}(|Z_t - Z_s| \geq u) = \frac{2}{d(t,s)\sqrt{2\pi}} \int_u^\infty \exp\left(- \frac{r^2}{2d^2(t,s)} \right) dr \leq 2 \exp\left(- \frac{u^2}{d^2(t,s)} \right),$$

which implies that $\mathcal{Z}$ is subgaussian with respect to the metric d it induces to T.

Typical examples: (i) If $g_1, \dots, g_N$ are independent standard Gaussian random variables on $(\Omega, \mathcal{A}, P)$, then $\mathcal{Z} = \{g_1, \dots, g_N\}$ is a Gaussian process.

(ii) Consider n independent standard Gaussian random variables $g_1, \dots, g_n$. For every non empty $T \subseteq \mathbb{R}^n$ we define a process $\mathcal{Z} = (Z_t)_{t\in T}$, by

$$Z_t(\omega) = \langle t, G(\omega)\rangle = \Big\langle t, \sum_{i=1}^n g_i e_i \Big\rangle = \sum_{i=1}^n \langle t, e_i\rangle g_i, \tag{A.7}$$

where $\{e_1, \dots, e_n\}$ is an orthonormal basis of $\mathbb{R}^n$ and $G = (g_1, \dots, g_n)$. Then, $\mathcal{Z}$ is a Gaussian process and the induced metric is the Euclidean metric on $\mathbb{R}^n$: for all $t, s \in T$,

$$d(t,s) = \|Z_t - Z_s\|_{L^2(\Omega)} = |t-s|. \tag{A.8}$$

If $\mathcal{Y} = (Y_t)_{t\in T}$ is a subgaussian process, we define

$$\mathbb{E} \sup_{t\in T} Y_t = \sup \big\{ \mathbb{E} \max_{t\in F} Y_t : F \subseteq T, 0 < |F| < \infty \big\}. \tag{A.9}$$

The results of the next three Sections concern the relation of $\mathbb{E}\sup Y_t$ with the geometry of (T,d).

A.2 Metric entropy - the case of Gaussian processes

Let (T,d) be a metric space. For every $\varepsilon > 0$ we define

$$N(T,d,\varepsilon) = \min \Big\{ N : \ t_1, \dots, t_N \in T : T \subseteq \bigcup_{i=1}^N B(t_i, \varepsilon) \Big\}, \tag{A.10}$$

where $B(t,\varepsilon) = \{s \in T : d(t,s) < \varepsilon\}$. The function $\varepsilon \mapsto H(T,d,\varepsilon) = \log N(T,d,\varepsilon)$ is the *metric entropy function* of T.

Consider as an example the Gaussian process $\mathcal{Z} = \{g_1, \dots, g_N\}$. We easily check that $\|g_i - g_j\|_2 = \sqrt{2}$ if $i \neq j$, and hence, $N(\varepsilon) = N$ if $0 < \varepsilon \leq \sqrt{2}$ and $N(\varepsilon) = 1$ if $\varepsilon > \sqrt{2}$. Also, using the fact that g_i are independent we may check that

$$\mathbb{E} \max_{1\leq i\leq N} g_i \simeq \sqrt{\log N}. \tag{A.11}$$

Let $\mathcal{Z} = (Z_t)_{t\in T}$ be a Gaussian process. We view T as a metric space with the induced metric d. The next Theorem gives upper and lower bounds for $\mathbb{E}\sup Z_t$ in terms of the metric entropy function of (T,d).

Theorem A.1 *There exist constants $c_1, c_2 > 0$ with the following property: if $\mathcal{Z} = (Z_t)_{t\in T}$ is a Gaussian process and d is the induced metric, then*

$$c_1 \sup_{\varepsilon>0} \varepsilon \sqrt{\log N(T,d,\varepsilon)} \leq \mathbb{E} \sup_{t\in T} Z_t \leq c_2 \int_0^\infty \sqrt{\log N(T,d,\varepsilon)}\, d\varepsilon. \tag{A.12}$$

The left hand side inequality is Sudakov's inequality [54], while the right hand side inequality is Dudley's inequality. In the example of $\mathcal{Z} = \{g_1, \dots, g_N\}$, both bounds give the right order of $\mathbb{E}\sup g_i$. The proof of Sudakov's inequality is based on a classical comparison lemma of Slepian.

Theorem A.2 *If $(X_1, \dots, X_N)$ and $(Y_1, \dots, Y_N)$ are two N-tuples of Gaussian random variables with mean 0 which satisfy the condition*

$$\|X_i - X_j\|_2 \leq \|Y_i - Y_j\|_2 \tag{A.13}$$

for all $i \neq j$, then

$$\mathbb{E} \max_{i \le N} X_i \le \mathbb{E} \max_{i \le N} Y_i. \tag{A.14}$$

We now use Slepian's lemma as follows: Let $\mathcal{Z} = (Z_t)_{t \in T}$ be a Gaussian process and let d be the induced metric. Given $\varepsilon > 0$ we consider a subset $\{t_1, \dots, t_N\}$ of T which is maximal with respect to the condition "$d(t,s) \ge \varepsilon$ if $t \neq s$". Then $T \subseteq \bigcup_{i=1}^N B(t_i, \varepsilon)$, which implies $N(T,d,\varepsilon) \le N$.

If $\delta = \min \|Z_{t_i} - Z_{t_j}\|_2$, we consider the N-tuple $\big(\frac{\delta g_1}{\sqrt{2}}, \dots, \frac{\delta g_N}{\sqrt{2}}\big)$, where g_i are independent standard Gaussian random variables. If $i \neq j$ then

$$\left\| \frac{\delta g_i}{\sqrt{2}} - \frac{\delta g_j}{\sqrt{2}} \right\|_2 = \delta \le \|Z_{t_i} - Z_{t_j}\|_2, \tag{A.15}$$

so we can apply Slepian's lemma. It follows that

$$\mathbb{E} \sup_{t \in T} Z_t \ge \mathbb{E} \max_{i \le N} Z_{t_i} \ge \frac{\delta}{\sqrt{2}} \mathbb{E} \max_{i \le N} g_i \ge c_1 \varepsilon \sqrt{\log N}. \tag{A.16}$$

Thus, $\mathbb{E} \sup_{t \in T} Z_t \ge c_1 \sup_{\varepsilon > 0} \varepsilon \sqrt{\log N(T,d,\varepsilon)}$.

A.3 Dudley's bound for subgaussian processes

Dudley's inequality is more generally valid for subgaussian processes $\mathcal{Y} = (Y_t)_{t \in T}$. The proof uses a successive approximation argument which we briefly describe:

We consider a non empty finite subset F of T and fix $t_0 \in F$. We set $R = \max\{d(t,t_0) : t \in F\}$ and $r_k = R/2^k$ for all $k \ge 0$.

We define $A_0 = \{t_0\}$ and for every $k \ge 1$ we find $A_k \subseteq F$ with cardinality $|A_k| = N(F,d,r_k)$ such that $F \subseteq \bigcup_{t \in A_k} B(t, r_k)$. Finally, for every $t \in F$ and $k \ge 0$ we choose $\pi_k(t) \in A_k$ with the property $d(t, \pi_k(t)) \le r_k$. Since F is finite, for every $t \in F$ we eventually have $\pi_k(t) = t$. Note also that

$$d(\pi_k(t), \pi_{k-1}(t)) \le r_k + r_{k-1} = 3r_k. \tag{A.17}$$

For every $t \in F$ we write

$$Y_t - Y_{t_0} = \sum_{k=1}^{\infty} \big(Y_{\pi_k(t)} - Y_{\pi_{k-1}(t)}\big) \tag{A.18}$$

and, using $\mathbb{E} Y_{t_0} = 0$,

$$\mathbb{E} \max_{t \in F} Y_t = \mathbb{E} \max_{t \in F} (Y_t - Y_{t_0}) = \int_0^\infty \mathrm{Prob}\left(\max_{t \in F} (Y_t - Y_{t_0}) \ge u \right) du. \tag{A.19}$$

We fix $\alpha_k > 0$ (which will be suitably chosen) with $S := \sum \alpha_k < \infty$ and set $B_k = \{(w,z) \in A_k \times A_{k-1} : d(w,z) \le 3r_k\}$. Using the subgaussian assumption we write

$$\begin{aligned}
\mathrm{Prob}\left(\max_{t \in F} (Y_t - Y_{t_0}) \ge uS \right) &\le \mathrm{Prob}\left(\sum_{k=1}^\infty \max_{t \in F} (Y_{\pi_k(t)} - Y_{\pi_{k-1}(t)}) \ge \sum_{k=1}^\infty u\alpha_k \right) \\
&\le \sum_{k=1}^\infty \mathrm{Prob}\left(\max_{(w,z) \in B_k} (Y_w - Y_z) \ge u\alpha_k \right) \\
&\le \sum_{k=1}^\infty \exp(-u^2 \alpha_k^2 / 9r_k^2) |A_k| \cdot |A_{k-1}|.
\end{aligned}$$

We now choose $\alpha_k = 3r_k\sqrt{\log(2^k|A_k|^2)}$. For every $u \geq 1$ and every k we have

$$\exp(-u^2\alpha_k^2/9r_k^2)|A_k|\cdot|A_{k-1}| \leq |A_k|^2\left(2^k|A_k|^2\right)^{-u^2} \leq 2^{-u^2k}, \tag{A.20}$$

and hence.

$$\text{Prob}\left(\max_{t\in F}(Y_t - Y_{t_0}) \geq uS\right) \leq \sum_{k=1}^{\infty} 2^{-u^2k} \leq c2^{-u^2}. \tag{A.21}$$

This shows that
(A.22)

$$\mathbb{E}\max_{t\in F} Y_t = S\int_0^{\infty}\text{Prob}\left(\max_{t\in F}(Y_t - Y_{t_0}) \geq uS\right)du \leq S + S\int_1^{\infty} c2^{-u^2}du \leq c'S.$$

Going back to the definition of S we see that

$$\begin{aligned} S &= \sum_{k=1}^{\infty}\frac{3R}{2^k}\left(k\sqrt{\log 2} + \sqrt{\log N(F,d,R/2^k)}\right) \simeq \sum_{k=1}^{\infty}\frac{R}{2^k}\sqrt{\log N(F,d,R/2^k)} \\ &\simeq \int_0^{\infty}\sqrt{\log N(F,d,\varepsilon)}d\varepsilon. \end{aligned}$$

Since $N(F,d,\varepsilon) \leq N(T,d,\varepsilon)$, the proof is complete.

A.4 Majorizing measures

Dudley's bound is not always sharp, as one can see by the following example: Consider an infinite sequence $\{g_n\}$ of independent standard Gaussian random variables, fix $a = (a_n) \in \ell_2$ and define the ellipsoid

$$\mathcal{E} = \left\{t = (t_n) \in \ell_2 : \sum_{n=1}^{\infty} t_n^2/a_n^2 \leq 1\right\}. \tag{A.23}$$

If we set $Z_t = \sum_n t_n g_n$, then $\mathcal{Z} = (Z_t)_{t\in\mathcal{E}}$ is a Gaussian process and

$$\mathbb{E}\sup_{t\in\mathcal{E}} Z_t \simeq \left(\sum_{n=1}^{\infty} a_n^2\right)^{1/2} < \infty. \tag{A.24}$$

On the other hand, one can choose $a \in \ell_2$ so that "Dudley's integral" will diverge.

A second approach to the problem is through majorizing measures: If (T,d) is a metric space, for every Borel probability measure μ on T we consider the quantity

$$\gamma(T,\mu) = \sup_{t\in T}\int_0^{\infty}\sqrt{\log\left(\frac{1}{\mu(B(t,\varepsilon))}\right)}dt. \tag{A.25}$$

In a sense, $1/\mu(B(t,\varepsilon))$ replaces the entropy number $N(T,d,\varepsilon)$ and the integral (which depends on $t \in T$) is the analogue of Dudley's integral.

Theorem A.3 *There exists a constant $C > 0$ with the following property. If (T,d) is a metric space and $\mathcal{Y} = (Y_t)_{t\in T}$ is a subgaussian (with respect to d) process, then*

$$\mathbb{E}\sup_{t\in T} Y_t \leq C\cdot\gamma(T), \tag{A.26}$$

where

$$\gamma(T) = \inf_{\mu}\gamma(T,\mu). \tag{A.27}$$

In the case of Gaussian processes, Talagrand [57] proved that the bound of Theorem A.3 gives always the right order for $\mathbb{E}\sup_{t\in T} Z_t$.

Theorem A.4 *There exists a constant $C > 0$ with the following property. If $\mathcal{Z} = (Z_t)_{t\in T}$ is a Gaussian process and d is the induced metric, then*

$$\frac{1}{C}\cdot \gamma(T) \leq \mathbb{E}\sup_{t\in T} Z_t. \tag{A.28}$$

A direct consequence of the above is the following comparison theorem.

Theorem A.5 *Let $\mathcal{Z} = (Z_t)_{t\in T}$ be a Gaussian process and let d be the induced metric. If the process $\mathcal{Y} = (Y_t)_{t\in T}$ is subgaussian with respect to d, then*

$$\mathbb{E}\sup_{t\in T} Y_t \leq C \cdot \mathbb{E}\sup_{t\in T} Z_t, \tag{A.29}$$

where $C > 0$ is an absolute constant.

A.5 Translation to the language of convex bodies

Let K be a convex body in $\mathbb{R}^n$. We assume that $0 \in \mathrm{int}(K)$ and denote by h_K the support function of K which is defined by

$$h_K(y) = \max_{t\in K}\langle t, y\rangle. \tag{A.30}$$

The width of K in the direction of $\theta \in S^{n-1}$ is the quantity $w(K,\theta) = h_K(\theta)+h_K(-\theta)$, and the mean width of K is defined by

$$w(K) = \frac{1}{2}\int_{S^{n-1}} w(K,\theta)\sigma(d\theta) = \int_{S^{n-1}} h_K(\theta)\sigma(d\theta). \tag{A.31}$$

We can associate to K the Gaussian process $\mathcal{Z} = (Z_t)_{t\in K}$, where $Z_t(\omega) = \langle t, G(\omega)\rangle$ ($G = (g_1,\dots,g_n)$ is the standard Gaussian random vector). We have already seen that the induced metric is the Euclidean metric on K: $\|Z_t - Z_s\|_2 = |t-s|$ for all $t,s \in K$.

A basic observation, which connects the preceding theory with convex geometric analysis, is that

$$\begin{aligned}
\mathbb{E}\sup_{t\in K} Z_t &= \mathbb{E}\sup_{t\in K}\langle t, G\rangle = \mathbb{E}\, h_K(G) \\
&= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} h_K(x)e^{-|x|^2/2}dx \\
&\simeq \sqrt{n}\int_{S^{n-1}} h_K(\theta)\sigma(d\theta) \simeq \sqrt{n}w(K).
\end{aligned}$$

We just used the fact that the distribution of G is the standard Gaussian measure on $\mathbb{R}^n$ and polar integration.

In this context, Sudakov's inequality takes the following form: if K is a convex body in $\mathbb{R}^n$, then for every $\varepsilon > 0$ we have

$$N(K,\varepsilon B_2^n) \leq \exp\left(cn\frac{w^2(K)}{\varepsilon^2}\right), \tag{A.32}$$

where $N(K,\varepsilon B_2^n)$ is the least number of balls of radius ε whose union covers K. Dudley's inequality takes an analogous form.

Mean width plays an important role in convex geometric analysis. From Urysohn's inequality, for every convex body K in $\mathbb{R}^n$ we have the inequality

$$w(K) \geq \left(\frac{|K|}{|B_2^n|} \right)^{1/n}. \tag{A.33}$$

(among all bodies of the same volume, Euclidean ball has minimal mean width). Fundamental results of Lewis [34], Figiel and Tomczak-Jaegermann [23], Pisier [50] establish the following "reverse Urysohn inequality".

Theorem A.6 *There exists an absolute constant $c > 0$ with the following property. For every convex body K in $\mathbb{R}^n$ we can find an affine image $T(K)$ such that $|T(K)| = 1$ and*

$$w(T(K)) \leq c\sqrt{n} \log n. \tag{A.34}$$

Notes and References

Complete proofs of the above can be found in the book of Ledoux and Talagrand [35].

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