

# Pal's problem for continuous curves

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In 1940 J.F. Pal posed the following problem [2]: Let  $(X, \rho)$  be a metric space and let  $c : [0, 1] \rightarrow X$  be a continuous curve and  $n$  a positive integer. Is there a partition  $0 < s_1 < s_2 < \dots < s_n < 1$  such that  $\rho(c(s_{i-1}), c(s_i)) = \rho(c(s_i), c(s_{i+1}))$ , for  $i = 1, 2, \dots, n$ , where  $s_0 = 0$  and  $s_{n+1} = 1$ ? One can generalize the problem by taking  $X$  to be merely a topological space and  $\rho : X \times X \rightarrow \mathbb{R}^+$  a continuous function such that  $\rho(x, y) = 0$  if and only if  $x = y$  (not necessarily a distance function). Moreover, one can require  $\lambda_{i-1}\rho(c(s_{i-1}), c(s_i)) = \lambda_i\rho(c(s_i), c(s_{i+1}))$ , for  $i = 1, 2, \dots, n$ , where  $\lambda_0 > 0, \lambda_1 > 0, \dots, \lambda_n > 0$  are given real numbers.

Of course the case  $n = 1$  is trivial, because if  $F : [0, 1] \rightarrow \mathbb{R}$  is the continuous function

$$F(s) = \lambda_1\rho(c(s), c(1)) - \lambda_0\rho(c(0), c(s)),$$

then, if  $c(0) \neq c(1)$ , we have  $F(0) > 0$  and  $F(1) < 0$  and by the Intermediate Value Theorem there exists some  $0 < s_1 < 1$  such that  $F(s_1) = 0$ . Every point in  $F^{-1}(0)$  is a solution to the problem.

In 1954 K. Urbanik gave a proof of the existence of a solution to Pal's problem based on the Brouwer fixed point theorem [6]. Another proof based again on the Brouwer fixed point theorem was presented recently in [4]. We present a new proof of the existence of solution to Pal's problem. The core of the argument is a generalization of the above simple idea involving the Intermediate Value Theorem from  $n = 1$  to the case of arbitrary  $n$ . The fact that  $F$  changes signs on the boundary points of  $[0, 1]$  is equivalent to saying that the restriction of  $F$  to the boundary  $\{0, 1\}$  of  $[0, 1]$  with values in  $\mathbb{R} \setminus \{0\}$  is not homotopic to a constant. For  $n > 1$  this is replaced by Proposition 1 below, which says that the restriction of a certain continuous function to the boundary of  $\Delta_n$  is not homotopic to a constant.

Computer scientists have shown lately interest in the problem, since some problems in computer science like signal summarization can be reduced to the partition of a curve in an appropriate space. In particular, computer scientists would like to have an existence proof, which could possibly lead to an algorithm that constructs a solution (approximately). The proof leads to an approximate algorithm, which is described in [5] along with applications in computer science and experimental results.

We proceed now to the proof. Let  $X$  be a topological space and  $\rho : X \times X \rightarrow \mathbb{R}^+$  be a continuous function such that  $\rho(x, y) = 0$  if and only if  $x = y$ . Let

$$\Delta_n = \{(s_1, s_2, \dots, s_n) \in \mathbb{R}^n : 0 \leq s_1 \leq s_2 \leq \dots \leq s_n \leq 1\}$$

be the  $n$ -dimensional simplex. Its boundary is  $\partial\Delta_n = B_1 \cup B_2 \cup \dots \cup B_{n+1}$ , where

$$B_j = \{(s_1, s_2, \dots, s_n) \in \Delta_n : s_{j-1} = s_j\},$$

$1 \leq j \leq n + 1$ , are the  $(n - 1)$ -faces of  $\Delta_n$ , putting  $s_0 = 0$  and  $s_{n+1} = 1$ .

Let  $c : [0, 1] \rightarrow X$  be a continuous curve such that  $c(0) \neq c(1)$ , and let  $d : \Delta_2 \rightarrow \mathbb{R}^+$  be the continuous function defined by  $d(s_1, s_2) = \rho(c(s_1), c(s_2))$ . Let also  $\lambda_0 > 0$ ,  $\lambda_1 > 0, \dots$ ,  $\lambda_n > 0$  be real numbers.

For each  $1 \leq j \leq n$  let  $f_j : \Delta_n \rightarrow \mathbb{R}$  be the function defined by

$$f_j(s_1, s_2, \dots, s_n) = \lambda_j d(s_j, s_{j+1}) - \lambda_{j-1} d(s_{j-1}, s_j),$$

where again we have set  $s_0 = 0$  and  $s_{n+1} = 1$ .

If  $F_n = (f_1, f_2, \dots, f_{n-1}, f_n)$ , then  $F_n$  vanishes nowhere on  $\partial\Delta_n$ , since  $c(0) \neq c(1)$ .

**Proposition 1.** *For every integer  $n \geq 2$ , the continuous map  $F_n|_{\partial\Delta_n} : \partial\Delta_n \rightarrow \mathbb{R}^n \setminus \{0\}$  has Brouwer degree  $(-1)^n$ .*

**Proof.** Examining the signs of  $f_j$ ,  $1 \leq j \leq n$ , on  $\partial\Delta_n$ , we observe that the vector  $F_n(s_1, s_2, \dots, s_n)$  at  $(s_1, s_2, \dots, s_n) \in \partial\Delta_n$  points inward  $\Delta_n$  for every  $(s_1, s_2, \dots, s_n) \in \partial\Delta_n$ . Indeed, the  $(n-1)$ -face  $B_j$ ,  $1 \leq j \leq n+1$ , is contained in the affine hyperplane  $g_j^{-1}(0)$ , where  $g_j : \mathbb{R}^n \rightarrow \mathbb{R}$  is the affine map  $g_j(s_1, s_2, \dots, s_n) = s_j - s_{j-1}$ . As above,  $s_0 = 0$  and  $s_{n+1} = 1$ . If  $(s_1, s_2, \dots, s_n) \in B_j$ , then

$$f_{j-1}(s_1, s_2, \dots, s_n) = -\lambda_{j-2} d(s_{j-2}, s_j) \leq 0,$$

$$f_j(s_1, s_2, \dots, s_n) = \lambda_j d(s_j, s_{j+1}) \geq 0$$

and therefore

$$\langle F_n(s_1, s_2, \dots, s_n), \nabla g_j(s_1, s_2, \dots, s_n) \rangle = \lambda_{j-2} d(s_{j-2}, s_j) + \lambda_j d(s_j, s_{j+1}),$$

for  $1 < j < n+1$ , while

$$\langle F_n(s_1, s_2, \dots, s_n), \nabla g_j(s_1, s_2, \dots, s_n) \rangle = \begin{cases} \lambda_1 d(0, s_2), & \text{if } j = 1, \text{ and} \\ \lambda_{n-1} d(s_{n-1}, 1), & \text{if } j = n+1. \end{cases}$$

Since  $\langle F_n(s_1, s_2, \dots, s_n), \nabla g_j(s_1, s_2, \dots, s_n) \rangle \geq 0$  for every  $(s_1, s_2, \dots, s_n) \in B_j$  and for every  $1 \leq j \leq n+1$ , we conclude that the vector  $F_n(s_1, s_2, \dots, s_n)$  at  $(s_1, s_2, \dots, s_n) \in \partial\Delta_n$  points inward for every  $(s_1, s_2, \dots, s_n) \in \partial\Delta_n$ .

It follows now from Hopf's formula that

$$(-1)^n \deg F_n = \chi(\Delta_n) = 1,$$

where  $\deg$  denotes the Brouwer degree and  $\chi$  the Euler characteristic (see Example 4.8 on page 269 and Proposition 4.9 on page 270 of [3]).  $\square$

**Theorem 2.** *Let  $X$  be a topological space and  $c : [0, 1] \rightarrow X$  such that  $c(s) \neq c(0)$  for  $1 < s \leq 1$ . Then for every continuous function  $\rho : X \times X \rightarrow \mathbb{R}^+$  with the property  $\rho(x, y) = 0$ , if and only if,  $x = y$ , and every positive integer  $n$  and real numbers  $\lambda_0 > 0$ ,  $\lambda_1 > 0, \dots$ ,  $\lambda_n > 0$ , there exist  $0 < s_1 < s_2 < \dots < s_n < 1$  such that*

$$\lambda_0 \rho(c(0), c(s_1)) = \lambda_1 \rho(c(s_1), c(s_2)) = \dots = \lambda_{n-1} \rho(c(s_{n-1}), c(s_n)) = \lambda_n \rho(c(s_n), c(1)).$$

**Proof.** It follows from Proposition 1 that  $F_n|_{\partial\Delta_n} : \partial\Delta_n \rightarrow \mathbb{R}^n \setminus \{0\}$  is not homotopic to a constant. Since  $F_n|_{\partial\Delta_n}$  has a continuous extension to  $\Delta_n$  with values in  $\mathbb{R}^n$ , it follows that  $F_n$  must vanish at some point in  $\text{int}\Delta_n$ .  $\square$

The approximate algorithm described in [5] is based on the further observation that if  $c(s) \neq c(0)$  for  $0 < s \leq 1$  and  $f = (f_1, f_2, \dots, f_{n-1})$ , then the connected component of  $f^{-1}(0)$  which contains  $(0, 0, \dots, 0)$  has non-empty intersection with  $\text{int}B_{n+1}$ .

A natural question that arises is whether Pal's problem has a solution for simple closed curves. More precisely, Let  $X$  be a topological space and  $\rho : X \times X \rightarrow \mathbb{R}^+$  be a continuous function such that  $\rho(x, y) = 0$  if and only if  $x = y$ . Let  $c : S^1 \rightarrow X$  be a topological embedding and  $n \geq 3$ . Do there exist distinct  $s_0, s_1, \dots, s_n \in S^1$  such that

$$\rho(c(s_0), c(s_1)) = \rho(c(s_1), c(s_2)) = \dots = \rho(c(s_{n-1}), c(s_n)) = \rho(c(s_n), c(s_0))?$$

Although this problem has been resolved positively for various classes of curves, it remains open in full generality. We mention only two recent positive results. In [7] the following theorem is proved.

**Theorem 3.** *Let  $c : S^1 \rightarrow \mathbb{R}^m$  be a continuous map. Let  $s_0 \in S^1$  be such that (i)  $c$  has nonzero continuous derivative in a neighbourhood of  $s_0$ , and (ii)  $c(s) \neq c(s_0)$  for all  $s \neq s_0$ .*

*Then for any  $n \geq 3$  there is a regular polygon of  $n$  edges inscribing  $c$ , containing  $c(s_0)$  as a vertex.  $\square$*

The condition (i) in Theorem 4.1 cannot be removed, if we insist that  $c(s_0)$  be a vertex of the inscribed  $n$ -gon. An easy example is a triangle with two equal edges joined at  $c(s_0)$  making an angle less than  $\pi/3$ . In this case there is no inscribed equilateral triangle with  $c(s_0)$  as a vertex.

Recently, the existence of inscribed equilateral triangles for planar Jordan curves has been proved [1].

**Theorem 4.** *Let  $c : S^1 \rightarrow \mathbb{R}^2$  be a Jordan curve. Suppose that  $\rho : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^+$  is an arbitrary distance function on  $\mathbb{R}^2$ . Then there exist distinct points  $s_0, s_1, s_2 \in S^1$  such that*

$$\rho(c(s_0), c(s_1)) = \rho(c(s_1), c(s_2)) = \rho(c(s_2), c(s_0)). \quad \square$$

## References

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