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Denjoy C^1 diffeomorphisms of the circle and McDuff's question

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Abstract

In this expository article, after the basic theory of orientation preserving C^1 diffeomorphisms of the circle, we present D. McDuff's theorem on the lengths of the complementary intervals of the unique Cantor minimal set of a Denjoy C^1 diffeomorphism of the circle. This leads to a question posed by Dusa McDuff which is related to the solvability of a cohomological equation on the Cantor set.

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1. Introduction

The combinatorial, topological and statistical study of the dynamics of C^k diffeomorphisms $f : S^1 \to S^1$, $k \geq 0$, is an old but still active area of research in dynamical systems having its origins in the work of H. Poincaré. The case $k = 0$ means that f is merely a homeomorphism. The theory of circle diffeomorphisms gives insight and motivation for the creation of theories to study dynamical systems on higher dimensional phase spaces, apart from the fact that often the latter reduces to lower dimensional ones. The case of sufficiently smooth or even real analytic circle diffeomorphisms can be considered as one of the simple cases where methods resembling KAM-theory are used.

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Concerning the topological dynamics of f, there is a dichotomy: if f has periodic points, then every minimal set is finite, but the absence of periodic points implies the existence of a unique minimal set which is either S^1 itself or is a Cantor set topologically. The last case does not occur if $k \geq 2$, as was shown by A. Denjoy [4]. Actually, it cannot occur for orientation preserving C^1 diffeomorphisms such that the logarithm of the derivative has bounded variation (see Theorem 3.6 below) or satisfies the Zygmund condition. The latter was proved by J. Hu and D. Sullivan [6]. In the sequel we focus on the orientation preserving C^1 diffeomorphisms of S^1 which have a unique Cantor minimal set. They are usually called Denjoy C^1 diffeomorphisms. Their existence had been known to P. Bohl from his studies on differential equations on the 2-torus [2].

A Cantor set is always the unique minimal set of some orientation preserving homeomorphism of S^1. It may not be the minimal set of any Denjoy C^1 diffeomorphism of S^1, but its image under some orientation preserving C^1 diffeomorphism may be. In the late seventies M. Herman asked which Cantor subsets of S^1 can be minimal sets of Denjoy C^1 diffeomorphisms? The invariance of a Cantor set under a Denjoy C^1 diffeomorphism implies geometric constraints. In Section 5 we present D. McDuff's theorem [8], which gives a necessary condition in terms of the set of lengths of its complementary intervals. More precisely, if we arrange the lengths of the complementary intervals of the minimal Cantor set K of a Denjoy C^1 diffeomorphism f in decreasing order $\lambda_1 > \lambda_2 > \cdots > 0$, then $\liminf_{n\to+\infty} \frac{\lambda_n}{\lambda_{n+1}} = 1$. It follows that the standard ternary Cantor set cannot be the minimal set of any Denjoy C^1 diffeomorphism. More examples of this kind have been constructed by A. Norton [9] and A. Portela [10].

The question which arises from McDuff's result is whether the sequence of ratios $\left(\frac{\lambda_n}{\lambda_{n+1}}\right)_{n\in\mathbb{N}}$ actually tends to 1. This question was asked by D. McDuff in [8] and to the author's present knowledge remains still open. In Section 6 we describe a connection between this question and the behavior of the derivative on K. This observation can be traced in the work of A. Portela [11]. To be more precise, we show that the answer is affirmative if $\log f'$ is a continuous coboundary on K. As it is shown in the beginning of Section 6, $\log f'$ is never a continuous coboundary on the whole S^1. However there are examples where $\log f'$ is a continuous coboundary on K and examples where it is not. Hopefully the effort to characterize the class of Denjoy C^1 diffeomorphisms f for which $\log f'$ is a continuous coboundary on their Cantor minimal set will give some new insight for the resolution of McDuff's question. The result and question of D. McDuff and the question on the cohomological triviality of $\log f'$ on K are about geometric features of the Cantor set K and the unique invariant Borel probability measure of the f, respectively.

A general reference for basic notions and terminology used in the sequel is [7].

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2. Homeomorphisms of the circle

Let $f : S^1 \to S^1$ be a homeomorphism. There is then a homeomorphism $F : \mathbb{R} \to \mathbb{R}$ such that $f(e^{2\pi it}) = e^{2\pi iF(t)}$ for every $t \in \mathbb{R}$. Such an F is called a *lift* of f. Clearly, any two lifts of f differ by an integer. The original homeomorphism f is orientation preserving if and only if F is increasing, and orientation reversing if F is decreasing. It is easy to see that in the latter case $F(t+k) = F(t) - k$ for every $k \in \mathbb{Z}$, and f has exactly two fixed points. We shall be concerned exclusively with orientation preserving homeomorphisms f of S^1. Then $F(t+k) = F(t)+k$ for every $k \in \mathbb{Z}$ or equivalently $F - id$ is periodic with period 1. So we have a well defined continuous function $\psi : S^1 \to \mathbb{R}$ with $\psi(e^{2\pi it}) = F(t) - t$, the displacement function.

Lemma 2.1. *If $a = \min\{\psi(z) : z \in S^1\}$ and $b = \max\{\psi(z) : z \in S^1\}$, then $b - a < 1$.*

Proof. If $s, t \in \mathbb{R}$ and $s \le t < s + 1$, then

$$\psi(e^{2\pi is}) - \psi(e^{2\pi it}) = F(s) - s - F(t) + t \le t - s < 1,$$

because F is increasing. Therefore, $\psi(e^{2\pi is}) < 1 + \psi(e^{2\pi it})$ for every $t \in [s, s+1)$. Consequently, $\psi(e^{2\pi is}) < 1 + a$ for every $s \in \mathbb{R}$, and so $b < 1 + a$. $\square$

Proposition 2.2 (*Poincaré*)**.** *There exists a constant $\rho(F) \in \mathbb{R}$ such that*

$$\lim_{n\to+\infty} \frac{1}{n}(F^n - id) = \rho(F)$$

uniformly on $\mathbb{R}$.

Proof. Let $\mu \in \mathcal{M}_f(S^1)$, where $\mathcal{M}_f(S^1)$ denotes the set of f-invariant Borel probability measures. Let $\psi_n : S^1 \to \mathbb{R}$ be the continuous function

$$\psi_n(e^{2\pi it}) = \frac{1}{n}(F^n(t) - t).$$

Then, $\psi = \psi_1$ and

$$\begin{aligned} \frac{1}{n}\sum_{k=0}^{n-1}(\psi \circ f^k)(e^{2\pi it}) &= \frac{1}{n}\sum_{k=0}^{n-1}\psi(e^{2\pi iF^k(t)}) = \frac{1}{n}\sum_{k=0}^{n-1}(F - id)(F^k(t)) \\ &= \frac{1}{n}\sum_{k=0}^{n-1} F^{k+1}(t) - F^k(t) = \frac{1}{n}(F^n(t) - t) = \psi_n(e^{2\pi it}). \end{aligned}$$

Thus, the integral of ψ_n is equal to the integral of ψ and

$$\int_{S^1}\left(n\psi_n - n\int_{S^1}\psi d\mu\right)d\mu = 0.$$

Applying now Lemma 2.1 to f^n, which lifts to F^n with displacement function $n\psi_n$, we get

$$\left\|\psi_n - \int_{S^1}\psi d\mu\right\| < \frac{1}{n},$$

for every $n \in \mathbb{N}$. Hence

$$\lim_{n\to+\infty}\psi_n = \int_{S^1}\psi d\mu$$

uniformly on S^1. $\square$

Remarks 2.3. (a) For every $\mu \in \mathcal{M}_f(S^1)$ we have

$$\rho(F) = \int_{S^1} \psi d\mu.$$

(b) $\|F^n - id - n\rho(F)\| < 1$ for every $n \in \mathbb{N}$.

(c) If $a = \rho(F)$, there exists some $t_0 \in \mathbb{R}$ such that

$$F^n(t_0) - t_0 - na = n\psi_n(e^{2\pi i t_0}) - n \int_{S^1} \psi d\mu = 0.$$

So $F^n(t_0) = R_{na}(t_0)$, or in other words $R_{-na} \circ F^n$ has a fixed point t_0, where $R_{na} : \mathbb{R} \to \mathbb{R}$ is the translation $R_{na}(t) = t + na$.

(d) For every $a \in \mathbb{R}$ we have $\rho(R_a) = a$.

(e) Since $R_k \circ F = F \circ R_k$ for every $k \in \mathbb{Z}$, we have

$$\frac{(R_k \circ F)^n - id}{n} = \frac{R_{nk} \circ F^n - id}{n} = \frac{F^n - id + nk}{n} \to \rho(F) + k.$$

It follows that the number $\rho(f) = e^{2\pi i \rho(F)} \in S^1$ does not depend on the choice of the particular lift F of f. It is called the *Poincaré rotation number* of f.

Proposition 2.4. *An orientation preserving homeomorphism $f : S^1 \to S^1$ has a periodic orbit if and only if $\rho(f) \in \mathbb{Q}/\mathbb{Z}$.*

Proof. Let F be a lift of f. If $z_0 = e^{2\pi i t_0}$ is a periodic point of f of period q, then $z_0 = f^q(e^{2\pi i t_0}) = e^{2\pi i F^q(t_0)}$, and therefore $p = F^q(t_0) - t_0 \in \mathbb{Z}$. So we have

$$\rho(F) = \lim_{n\to+\infty} \frac{F^{nq}(t_0) - t_0}{nq} = \lim_{n\to+\infty} \frac{np}{nq} = \frac{p}{q}.$$

Conversely, if $\rho(F) = p/q \in \mathbb{Q}$, then $R_{-p} \circ F^q$ has a fixed point $t_0 \in \mathbb{R}$ or equivalently $F^q(t_0) = t_0 + p$. $\square$

If $f : X \to X$ is a homeomorphism of a metric space X, the set

$$L^+(x) = \{y \in X : f^{n_k}(x) \to y \text{ for some } n_k \to +\infty\}$$

is called the *positive limit set* of the point $x \in X$, and is a closed invariant set. Similarly, the *negative limit set* $L^-(x)$ is defined and has the same properties.

Proposition 2.5. *If the orientation preserving homeomorphism $f : S^1 \to S^1$ has irrational rotation number, then there exists a compact f-invariant set $K \subset S^1$ with the following properties.*

(i) $L^+(x) = L^-(x) = K$ *for every $x \in S^1$, and in particular K is minimal.*

(ii) *Either $K = S^1$ or K is a Cantor set.*

(iii) $\operatorname{supp}\mu = K$ *for every f-invariant Borel probability measure.*

Proof. Let $x \in S^1$ and $K = L^+(x)$. Since K is closed and invariant, we have $L^+(y) \cup L^-(y) \subset K$ for every $y \in K$. The connected components $I_n,\ n \in \mathbb{Z}$, of $S^1 \setminus K$ are permuted by f. Let now $y \in S^1 \setminus K$. If $L^+(y) \cap (S^1 \setminus K) \neq \varnothing$, there are some $n, k, l \in \mathbb{Z}$ with $k > l$ such that $f^k(y),\ f^l(y) \in I_n$. This means that $y \in f^{-k}(I_n) \cap f^{-l}(I_n)$ and therefore $f^{k-l}(I_n) \cap I_n \neq \varnothing$. Then, $f^{k-l}(\bar{I}_n) = \bar{I}_n$, and from the intermediate value theorem f^{k-l} must have a fixed point in $\bar{I}_n$. This contradicts Proposition 2.4, since f is supposed to have irrational rotation number. Hence $L^+(y) \subset K$ and similarly $L^-(y) \subset K$ for every $y \in S^1$. In other words, we have shown that $L^+(y) \cup L^-(y) \subset L^+(x)$ for every $x, y \in S^1$ and similarly $L^+(y) \cup L^-(y) \subset L^-(x)$. Thus $L^+(y) \cup L^-(y) \subset L^+(x) \cap L^-(x)$ for every $x, y \in S^1$, and symmetrically we get

$$L^+(x) \cup L^-(x) \subset L^+(y) \cap L^-(y) \subset L^+(y) \cup L^-(y) \subset L^+(x) \cap L^-(x)$$

for every $x, y \in S^1$. Hence $K = L^+(y) = L^-(y) = L^+(x) = L^-(x)$ for every $x, y \in S^1$. It is clear now that K is a perfect set. If K is not totally disconnected, it contains an open interval $J \subset S^1$. Then, for every $x \in S^1$ there exists $n \in \mathbb{Z}$ such that $f^n(x) \in J$, that is $x \in f^{-n}(J) \subset K$. This shows that $K = S^1$, if it is not a Cantor set. Obviously, $K = \{x \in S^1 : x \in L^+(x)\}$, and so from Poincaré's recurrence theorem we have $\mathrm{supp}\mu \subset K$ for every $\mu \in \mathcal{M}_f(S^1)$. Since K is minimal, we must have equality. $\square$

An important property of the rotation number is that it remains invariant under orientation preserving semi-conjugation. We use the term orientation preserving surjection for a continuous surjection of the circle onto itself which is induced by a nondecreasing map of the real line.

Proposition 2.6. *Let $f, g : S^1 \to S^1$ be two orientation preserving homeomorphisms and let $h : S^1 \to S^1$ be an orientation preserving surjection such that $h \circ f = g \circ h$. Then $\rho(f) = \rho(g)$.*

Proof. Let F, G and H be lifts of f, g and h, respectively, and let $\phi : S^1 \to \mathbb{R}$ be the displacement function of h, that is $\phi(e^{2\pi it}) = H(t) - t$. There exists some $k \in \mathbb{Z}$ such that $H(F(t)) = G(H(t)) + k$ for every $t \in \mathbb{R}$ and inductively $H \circ F^n = G^n \circ H + k,\ n \in \mathbb{Z}$. So,

$$F^n + \phi \circ F^n = G^n \circ H - H + id + \phi + nk$$

and therefore

$$\frac{1}{n}(F^n - id) + \frac{1}{n}\phi \circ F^n = \frac{1}{n}(G^n - id) \circ H + \frac{1}{n}\phi + k.$$

Taking the limit we get $\rho(F) = \rho(G) + k$. $\square$

Theorem 2.7. *If the orientation preserving homeomorphism $f : S^1 \to S^1$ has irrational rotation number $e^{2\pi ia}$, then there exists a non-decreasing continuous surjection $H : \mathbb{R} \to \mathbb{R}$ such that $H(0) = 0$ and $H(t + 1) = H(t) + 1$ for every $t \in \mathbb{R}$ which induces an orientation preserving continuous surjection $h : S^1 \to S^1$ of degree 1 such that $h \circ f = r_a \circ h$, where r_a is the rotation by $2\pi a$. The map h is a homeomorphism if and only if f is minimal.*

Proof. For convenience we put $\exp(t) = e^{2\pi i t}$. Let μ be an f-invariant Borel probability measure and $\nu = (\exp|[0,1))_*^{-1}\mu$ on $[0,1)$. Since f has no periodic points, by Proposition 2.4, μ has no atoms and so does ν. We extend ν to an infinite measure on $\mathbb{R}$ periodically. More precisely, on $[n, n+1)$ we set $\nu = (\exp|[n, n+1))_*^{-1}\mu$. If $A \subset \mathbb{R}$ is a Borel set, we have

$$\nu(A) = \sum_{n\in\mathbb{Z}} \nu(A \cap [n, n+1)) = \sum_{n\in\mathbb{Z}} \mu(\exp(A \cap [n, n+1)))$$

and $\nu(F(A)) = \nu(A)$, for every lift F of f, since μ is f-invariant. Let now $H : \mathbb{R} \to \mathbb{R}$ be defined by

$$H(t) = \int_0^t d\nu.$$

Obviously, $H(0) = 0$ and $H(t+1) = H(t) + 1$. Also, H is continuous, because ν has no atoms. In addition,

$$\begin{aligned} H(F(x)) - H(x) &= \int_0^{F(0)} d\nu + \int_{F(0)}^{F(t)} d\nu - \int_0^t d\nu \\ &= \int_0^{F(0)} d\nu = H(F(0)) - H(0). \end{aligned}$$

Consequently, H induces an orientation preserving surjection $h : S^1 \to S^1$ such that $h \circ f = r_b \circ h$, where $b = H(F(0)) - H(0)$. But from Proposition 2.6 we necessarily have $a = b \pmod 1$.

If f is minimal, then μ is positive on non-empty open sets and therefore H is an increasing homeomorphism. Conversely, if h is a homeomorphism, then the f-orbit of any point $z \in S^1$ is $h^{-1}(\{h(z)e^{2\pi i n a} : n \in \mathbb{Z}\})$, which is dense in S^1. $\square$

3. Denjoy's theory of C^1 diffeomorphisms of the circle

Let $a \in \mathbb{R} \setminus \mathbb{Q}$. A rational number p/q, where $p \in \mathbb{Z}$, $q \in \mathbb{N}$ and $(p, q) = 1$, is called a rational approximation of a if

$$\left| a - \frac{p}{q} \right| < \frac{1}{q^2}.$$

It is a well known fact of number theory that for every $a \in \mathbb{R} \setminus \mathbb{Q}$ there exists a sequence $(p_n/q_n)_{n\in\mathbb{N}}$ in $\mathbb{Q}$ such that $p_n \in \mathbb{Z}$, $q_n \in \mathbb{N}$, $(p_n, q_n) = 1$, $q_n \to +\infty$ and

$$\left| a - \frac{p_n}{q_n} \right| \le \frac{1}{q_n q_{n+1}} < \frac{1}{q_n^2}$$

for every $n \in \mathbb{N}$.

Let now $0 < a < 1$ be an irrational number and $0 < p/q < 1$ a rational approximation of a as above. Let also $\exp : \mathbb{R} \to S^1$ denote the universal covering map $\exp(t) = e^{2\pi i t}$. For every integer $0 \le k < q$ there exists exactly one point of the finite sequence

$e^{2\pi ina}$, $n = 1, 2, \ldots, q$, in the interval $\exp\Big(\Big[\frac{k}{q}, \frac{k+1}{q}\Big]\Big)$ on S^1. Indeed, assuming that $0 < a - \frac{p}{q} < \frac{1}{q^2}$ (the case $-\frac{1}{q^2} < a - \frac{p}{q} < 0$ being similar) we have

$$0 < na - \frac{np}{q} < \frac{n}{q^2} < \frac{1}{q}$$

for $n = 1, 2, \ldots, q$, and therefore $e^{2\pi ina} \in \exp\Big(\Big[\frac{np}{q}, \frac{np+1}{q}\Big]\Big)$. Since p and q are relatively prime, the finite sequences $\exp\Big(\Big[\frac{np}{q}, \frac{np+1}{q}\Big]\Big)$, $n = 1, 2, \ldots, q$ and $\exp\Big(\Big[\frac{k}{q}, \frac{k+1}{q}\Big]\Big)$, $k = 0, 1, \ldots, q-1$ of intervals coincide (with different order on S^1), and evidently have mutually disjoint interiors and cover S^1.

Theorem 3.1 (*Denjoy–Koksma Inequality*). *Let $f : S^1 \to S^1$ be an orientation preserving homeomorphism with irrational rotation number $e^{2\pi ia}$. Let $p/q \in \mathbb{Q}$ be a rational approximation of a. If $\phi : S^1 \to \mathbb{R}$ is a (not necessarily continuous) function of bounded variation $V(\phi)$, then*

$$\left|\sum_{n=0}^{q-1} \phi(f^n(z)) - q \int_{S^1} \phi d\mu\right| \le V(\phi)$$

for every f-invariant Borel probability measure μ and $z \in S^1$.

Proof. Since a is irrational, there exists an orientation preserving continuous surjection $h : S^1 \to S^1$ such that $h \circ f = r_a \circ h$, where r_a is the rotation by $2\pi a$, by Theorem 2.7. Thus, $h(f^n(z)) = h(z)e^{2\pi ina}$ for every $z \in S^1$ and $n \in \mathbb{Z}$. Also, $h_*\mu$ is invariant by r_a and is therefore the normalized Lebesgue measure on S^1. Let $z \in S^1$ and put $z_0 = z_q = z$ and $z_k \in S^1$ be such that $h(z_k) = h(z)e^{2\pi ik/q}$ for $1 \le k < q$. Denoting by $[z_k, z_{k+1}]$ the (positively oriented) maximal interval on S^1 with such endpoints z_k and z_{k+1} we have

$$\int_{[z_k, z_{k+1}]} d\mu = \int_{[h(z_k), h(z_{k+1})]} d(h_*\mu) = \frac{1}{q}$$

for all $0 \le k < q$. Since f is a homeomorphism, it suffices to prove the inequality

$$\left|\sum_{n=1}^{q} \phi(f^n(z)) - q \int_{S^1} \phi d\mu\right| \le V(\phi).$$

From the preceding observation, for every $n = 1, 2, \ldots, q$, there exists a unique interval I_n from the finite sequence of intervals $[z_k, z_{k+1}]$, $0 \le k < q$, such that $h(I_n)$ contains $h(z)e^{2\pi ina}$. Therefore, $f^n(z) \in I_n$ and

$$\begin{aligned}\left|\sum_{n=1}^{q} \phi(f^n(z)) - q \int_{S^1} \phi d\mu\right| &= \left|\sum_{n=1}^{q} \Big(\phi(f^n(z)) - q \int_{I_n} \phi d\mu\Big)\right| \\ &\le \sum_{n=1}^{q} q \left|\int_{I_n} \big(\phi(f^n(x)) - \phi\big) d\mu\right|\end{aligned}$$

$$\leq \sum_{n=1}^{q} \sup_{x \in I_n} |\phi(f^n(z)) - \phi(x)|$$
$$\leq \sum_{n=1}^{q} V(\phi|I_n) \leq V(\phi). \quad \square$$

Corollary 3.2. *An orientation preserving homeomorphism of S^1 with irrational rotation number is uniquely ergodic.*

Proof. Using the notations of Theorem 3.1 and choosing a sequence $(p_n/q_n)_{n\in\mathbb{N}}$ of rational approximations of a such that $q_n \to +\infty$, we have

$$\lim_{n\to+\infty} \frac{1}{q_n} \sum_{k=0}^{q_n-1} \phi \circ f^k = \int_{S^1} \phi d\mu$$

uniformly on S^1, for every function $\phi : S^1 \to \mathbb{R}$ of bounded variation. Since the subspace of continuous functions of bounded variation is dense in the space of continuous functions on S^1, it follows that the f-invariant Borel probability measure μ is unique. $\square$

We turn now to the study of orientation preserving C^1 diffeomorphisms of the circle with irrational rotation number.

Proposition 3.3 (*Denjoy*)**.** *Let $f : S^1 \to S^1$ be an orientation preserving C^1 diffeomorphism with irrational rotation number. Then,*

$$\int_{S^1} (\log f')d\mu = 0$$

where μ is the unique f-invariant Borel probability measure.

Proof. From the chain rule, for every $n \in \mathbb{N}$ we have $\log(f^n)' = \sum_{k=0}^{n-1}(\log f') \circ f^k$ and so

$$\lim_{n\to+\infty} \frac{1}{n} \log(f^n)' = \int_{S^1} \log f' d\mu$$

uniformly on S^1, by unique ergodicity.

If $\int_{S^1} \log f'd\mu > 0$, then $(f^n)' \to +\infty$ uniformly on S^1, from which it follows that

$$\int_{S^1} (f^n)'(z)dz = \int_0^1 (F^n)'(t)dt \to +\infty,$$

where $F : \mathbb{R} \to \mathbb{R}$ is a lift of f.

If $\int_{S^1} \log f'd\mu < 0$, then $(f^n)' \to 0$ uniformly on S^1, from which it follows that

$$\int_{S^1} (f^n)'(z)dz = \int_0^1 (F^n)'(t)dt \to 0.$$

However, in both cases we have

$$\int_{S^1} (f^n)'(z)dz = \int_0^1 (F^n)'(t)dt = F(1) - F(0) = 1. \quad \square$$

Corollary 3.4. *If $f : S^1 \to S^1$ is an orientation preserving C^1 diffeomorphism with irrational rotation number and unique invariant Borel probability measure μ, then,*

$$\lim_{n\to+\infty}\left(\int_{S^1}(f^n)'d\mu\right)^{1/n} = 1.$$

Proof. From the unique ergodicity, the chain rule and Proposition 3.3 it follows that $\lim_{n\to+\infty}\big((f^n)'\big)^{1/n} = 1$, uniformly on S^1. By Hölder's inequality

$$\liminf_{n\to+\infty}\left(\int_{S^1}(f^n)'d\mu\right)^{1/n} \geq \lim_{n\to+\infty}\int_{S^1}\big((f^n)'\big)^{1/n}d\mu = 1.$$

On the other hand, for every $n \in \mathbb{N}$ there exists $z_n \in S^1$ such that $(f^n)'(z_n) \geq (f^n)'(z)$ for all $z \in S^1$. Therefore,

$$\limsup_{n\to+\infty}\left(\int_{S^1}(f^n)'d\mu\right)^{1/n} \leq \lim_{n\to+\infty}\big((f^n)'(z_n)\big)^{1/n} = 1. \quad \square$$

Proposition 3.5 (*Denjoy*)**.** *Let $f : S^1 \to S^1$ be an orientation preserving C^1 diffeomorphism with irrational rotation number $e^{2\pi i a}$ and let $p/q \in \mathbb{Q}$ be a rational approximation of a. If $\log f'$ has bounded variation on S^1 and V is its total variation, then $e^{-V} \leq (f^{\pm q})' \leq e^V$ or equivalently $|\log(f^q)'| \leq V$.*

Proof. Since f is C^1 and S^1 is compact, and since $\log f'$ has bounded variation, by assumption, from the Denjoy–Koksma inequality we have

$$\left|\sum_{n=0}^{q-1}(\log f')(f^n(z)) - q\int_{S^1}(\log f')d\mu\right| \leq V.$$

By the chain rule and Proposition 3.3, this becomes $|\log(f^q)'(z)| \leq V$ for every $z \in S^1$. $\square$

Theorem 3.6 (*Denjoy*)**.** *Let $f : S^1 \to S^1$ be an orientation preserving C^1 diffeomorphism with irrational rotation number $e^{2\pi i a}$. If $\log f'$ has bounded variation on S^1, then f is topologically conjugate to r_a.*

Proof. Suppose that f is not topologically conjugate to r_a. There exists a Cantor set $K \subset S^1$ which is the unique minimal set of f, by Proposition 2.5. If I is a connected component of $S^1 \setminus K$, then $f^n(I)$, $n \in \mathbb{Z}$, is a sequence of disjoint open intervals. Let $(p_n/q_n)_{n\in\mathbb{N}}$ be a sequence of rational approximations of a such that $q_n \to +\infty$. From the mean value theorem and Proposition 3.5 we get $\lambda(f^{q_n}(I)) \geq e^{-V}\lambda(I)$, where λ denotes the normalized Lebesgue measure on S^1 and V is the total variation of $\log f'$. It follows that

$$1 \geq \lambda\left(\bigcup_{n=1}^{\infty} f^{q_n}(I)\right) = \sum_{n=1}^{\infty}\lambda(f^{q_n}(I)) = +\infty. \quad \square$$

In the next section we shall show that the preceding theorem of A. Denjoy is not true without the assumption on the bounded variation of $\log f'$ by constructing an explicit example. An orientation preserving C^1 diffeomorphism of the circle with irrational rotation number which is not topologically conjugate to a rotation will be called a Denjoy C^1 diffeomorphism.

4. Examples of Denjoy C^1 diffeomorphisms

From any orientation preserving C^1 diffeomorphism $g : S^1 \to S^1$ which is topologically conjugate to an irrational rotation we shall construct following M. Herman [5] a Denjoy C^1 diffeomorphism $f : S^1 \to S^1$ by inserting intervals at the points of one of its orbits and an orientation preserving continuous surjection $h : S^1 \to S^1$ of degree 1 such that $h \circ f = g \circ h$. We shall need some preparation.

Lemma 4.1. *The set of orientation preserving C^1 diffeomorphisms of S^1 with rational rotation number is dense in the space of all orientation preserving C^1 diffeomorphisms of S^1 endowed with the C^1 topology.*

Proof. Let $f : S^1 \to S^1$ be an orientation preserving C^1 diffeomorphism without periodic points and K be its unique minimal set according to Proposition 2.5. The set of points in K which are approximated by other points of K from both sides is uncountable. Let $z = e^{2\pi i t}$ be such a point. Let $F : \mathbb{R} \to \mathbb{R}$ be a lift of f. Since K is minimal, there is a sequence of positive integers $n_k \to +\infty$ and $m_k \in \mathbb{Z}$ such that $F^{n_k}(t) - m_k \to t$ and $F^{n_k}(t) - m_k < t$ for every $k \in \mathbb{N}$. For every $a \geq 0$ we consider the C^1 diffeomorphism $f_a = r_a \circ f$, where r_a is the rotation by $2\pi a$. Then a lift of f_a is the function $F_a(x) = F(x) + a,\ x \in \mathbb{R}$ and inductively we see that $F_a^n(x) \geq F^n(x) + a$ for all $n \in \mathbb{N}$, because if this holds for $n - 1$ then

$$F_a^n(x) = F_a(F_a^{n-1}(x)) \geq F_a(F^{n-1}(x) + a) \geq F_a(F^{n-1}(x)) = F^n(x) + a.$$

Since $F_a' = F', f_a,\ a \geq 0$, is a continuous curve of orientation preserving C^1 diffeomorphisms with respect to the C^1 topology. It follows now that for each $k \in \mathbb{N}$ there exists some $0 < a_k \leq t + m_k - F^{n_k}(t)$ such that $F_{a_k}^{n_k}(t) = t + m_k$. This means that z is a periodic point of f_{a_k} and since $a_k \to 0$, we have $f_{a_k} \to f$ in the C^1 topology. $\square$

Note that an orientation preserving C^1 diffeomorphism $f : S^1 \to S^1$ with rational rotation number always has a periodic point z such that $0 < (f^N)'(z) \leq 1$, where N is its period.

Proposition 4.2. *For every orientation preserving C^1 diffeomorphism $f : S^1 \to S^1$ there exists some $z \in S^1$ such that the sequence $((f^n)'(z))_{n\in\mathbb{Z}}$ is bounded.*

Proof. Let D denote the set of all orientation preserving C^1 diffeomorphisms of S^1 for which the conclusion holds. In view of Lemma 4.1 it suffices to prove that D is closed in the C^1 topology. Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in D converging to some f in the C^1 topology. Let $z_k \in S^1$ be such that $((f_k^n)'(z_k))_{n\in\mathbb{Z}}$ is bounded and let $M_k = \sup\{(f_k^n)'(z_k) : n \in \mathbb{Z}\}$. There exist $n_k \in \mathbb{Z}$ such that $\frac{1}{2}M_k < (f_k^{n_k})'(z_k) \leq M_k$. Passing to a subsequence, if necessary, we may assume that the sequence $((f_k^{n_k})(z_k))_{k\in\mathbb{N}}$ converges to some point

$z \in S^1$. We shall prove that the sequence $((f^n)'(z))_{n\in\mathbb{Z}}$ is bounded. Indeed, if this is not the case, there exists $n \in \mathbb{Z}$ such that $(f^n)'(z) > 2$ and so $(f_k^n)'(f_k^{n_k}(z_k)) > 2$ for large values of k. Therefore,

$$(f_k^{n+n_k})'(z_k) = (f_k^n)'(f_k^{n_k}(z_k)) \cdot (f_k^{n_k})'(z_k) > 2 \cdot \frac{M_k}{2} = M_k,$$

and this is a contradiction. $\square$

Let now $g : S^1 \to S^1$ be an orientation preserving C^1 diffeomorphism which is topologically conjugate to an irrational rotation and let $G : \mathbb{R} \to \mathbb{R}$ be a lift of g. By Proposition 4.2, there exists $t_0 \in \mathbb{R}$ such that the sequence $((G^n)'(t_0))_{n\in\mathbb{Z}}$ is bounded. Since $\{G^n(x) + m : m, n \in \mathbb{Z}\}$ is dense in $\mathbb{R}$ for all $x \in \mathbb{R}$, we may assume that $t_0 \neq G^n(0) + m$ for all $m, n \in \mathbb{Z}$, otherwise we replace g with a conjugate by a suitable rotation. Let $l_n > 0,\ n \in \mathbb{Z}$, be such that

(i) $\sum_{n\in\mathbb{Z}} l_n (G^n)'(t_0) = 1$

(ii) $\lim_{n\to\pm\infty} \frac{l_{n+1}}{l_n} = 1$ and

(iii) for a given $0 < \delta < \frac{1}{2}$ we have $\sup\left\{\left|\frac{l_{n+1}}{l_n} - 1\right| : n \in \mathbb{Z}\right\} < \delta < \frac{1}{2}$.

For instance if we choose suitable $\epsilon > 0$, $c > 0$ and sufficiently large $b > 0$, we can take

$$l_n = \frac{c}{(b + |n|)(\log(b + |n|))^{1+\epsilon}}.$$

We consider the functions $q : \mathbb{R} \to \mathbb{R}^+$ with

$$q(t) = \begin{cases} 0, & \text{if } t \neq G^n(t_0) + m \text{ for all } m,\ n \in \mathbb{Z}, \\ l_n (G^n)'(t_0), & \text{if } t = G^n(t_0) + m \text{ for some } m,\ n \in \mathbb{Z} \end{cases}$$

and $J : \mathbb{R} \to \mathbb{R}$ defined by

$$J(t) = \begin{cases} \displaystyle\sum_{0\le s\le t} q(s), & \text{if } t \geq 0, \\ \displaystyle -\sum_{t<s\le 0} q(s), & \text{if } t < 0. \end{cases}$$

The function J is strictly increasing, continuous except at the points of the set $\{G^n(t_0) + m : m, n \in \mathbb{Z}\}$, where it is only right continuous and from the left has jump $l_n(G^n)'(t_0)$ at the point $G^n(t_0) + m$. Moreover, $J(0) = 0$ and $J(t + 1) = J(t) + 1$ for every $t \in \mathbb{R}$, and so $J(k) = k$ for $k \in \mathbb{Z}$. The set $C = \overline{J(\mathbb{R})}$ is closed, perfect, totally disconnected, invariant under integer translations and has Lebesgue measure zero.

Let $I_{n,m} = [J(G^n(t_0) + m) - l_n(G^n)'(t_0), J(G^n(t_0) + m)],\ m,\ n \in \mathbb{Z}$ and $H : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$H(x) = \begin{cases} t, & \text{if } x = J(t) \text{ for some } t \in \mathbb{R}, \\ G^n(t_0) + m, & \text{if } x \in I_{n,m}. \end{cases}$$

The function H is continuous, non-decreasing, $H \circ J = id$ and $H(C) = \mathbb{R}$. Moreover, $H(0) = 0$ and $H(x + 1) = H(x) + 1$ for every $x \in \mathbb{R}$, and so $H(k) = k$ for $k \in \mathbb{Z}$.

Let $F_{n,0} : I_{n,0} \to I_{n+1,0}$ be the C^1 diffeomorphism defined by

$$F_{n,0}(x) = a_{n+1} + G'(G^n(t_0)) \int_{a_n}^{x} \left[1 + \frac{c_n}{l_n^2}(y - a_n)(b_n - y) \right] dy,$$

where we put for simplicity $a_n = J(G^n(t_0)) - l_n(G^n)'(t_0)$, $b_n = J(G^n(t_0))$ and

$$c_n = 6 \left| \frac{l_{n+1}}{l_n} - 1 \right| \frac{1}{((G^n)'(t_0))^2}.$$

Then $F'_{n,0}(a_n) = F'_{n,0}(b_n) = G'(G^n(t_0))$ and

$$\left| \frac{F'_{n,0}(x)}{G'(G^n(t_0))} - 1 \right| \le \frac{3}{2} \left| \frac{l_{n+1}}{l_n} - 1 \right| < \frac{3}{2}\delta < 1$$

for every $x \in I_{n,0}$, $n \in \mathbb{Z}$. Therefore

$$\lim_{n \to \pm\infty} \left(\sup \left\{ \left| \frac{F'_{n,0}(x)}{G'(G^n(t_0))} - 1 \right| : x \in I_{n,0} \right\} \right) = 0.$$

For every $m \in \mathbb{Z}$ we define $F_{n,m} : I_{n,m} \to I_{n+1,m}$ by $F_{n,m} = R_m \circ F_{n,0} \circ R_{-m}$, where R_m denotes translation by m. Then $F_{n,m}$ is an increasing C^1 diffeomorphism and $F'_{n,m}(x) = F'_{n,0}(x - m)$ for every $x \in I_{n,m}$. Also the function $\beta : \mathbb{R} \to (0, +\infty)$ defined by

$$\beta(x) = \begin{cases} G'(H(x)), & \text{if } x \in C, \\ F'_{n,m}(x), & \text{if } x \in I_{n,m} \text{ for some } m,\ n \in \mathbb{Z} \end{cases}$$

is continuous.

Since $\mathbb{R} \setminus C$ is dense in $\mathbb{R}$, there exists a unique increasing homeomorphism $F : \mathbb{R} \to \mathbb{R}$ which restricted to every $I_{n,m}$ coincides with $F_{n,m}$. If $x \in I_{n,m}$, then $x + 1 \in I_{n,m+1}$ and so $F(x + 1) = F(x) + 1$. Also, $H(F(x)) = G^{n+1}(t_0) + m = G(G^n(t_0) + m) = G(H(x))$. It follows by continuity that $F(x + 1) = F(x) + 1$ and $H(F(x)) = G(H(x))$ for all $x \in \mathbb{R}$. Therefore F is the lift of an orientation preserving homeomorphism $f : S^1 \to S^1$ and H is the lift of an orientation preserving continuous surjection $h : S^1 \to S^1$ of degree 1 such that $h \circ f = g \circ f$. So $\rho(f) = \rho(g)$, by Proposition 2.6. The set $K = p(C)$ is a minimal Cantor set of f, because $\{F^n(x) + m : m, n \in \mathbb{Z}\}$ is dense in C for every $x \in C$.

It remains to show that f is actually a C^1 diffeomorphism. Since F is increasing, it has bounded variation on every compact interval. Also, if $E \subset \mathbb{R}$ is a set of Lebesgue measure zero, then $F(\mathbb{R} \setminus C)$ has Lebesgue measure zero, because F is C^1 on $\mathbb{R} \setminus C$, and therefore $F(E)$ has Lebesgue measure zero, since C does. From Banach's theorem, F is absolutely continuous on every compact interval. It follows that

$$F(x) = F(0) + \int_0^x F'(t)dt = F(0) + \int_0^x \beta(t)dt$$

for every $x \in \mathbb{R}$, because $F' = \beta$ almost everywhere. Hence F is C^1 and $F' = \beta$ everywhere on $\mathbb{R}$. Consequently, f is a Denjoy C^1 diffeomorphism. Note that $f'(z) =$

$g'(h(z))$ for every $z \in K$. We shall return to this property of the derivative of f at the end of Section 6.

5. The lengths of the complementary intervals

Let $f : S^1 \to S^1$ be a Denjoy C^1 diffeomorphism with unique minimal set K. The set $\mathcal{C}$ of the connected components of $S^1 \setminus K$ is countable and $\sum_{I \in \mathcal{C}} \lambda(I) \leq 1$, where λ denotes the normalized Lebesgue measure on S^1. So, there exists a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of real numbers such that

(i) $\sum_{n=1}^{\infty} \lambda_n \leq 1$,
(ii) $\lambda_{n+1} < \lambda_n$ for every $n \in \mathbb{N}$, and
(iii) for every $n \in \mathbb{N}$ there exists some (possibly not unique) $I \in \mathcal{C}$ such that $\lambda_n = \lambda(I)$.

The sequence $(\lambda_n)_{n\in\mathbb{N}}$ is called the spectrum of K. McDuff's theorem can be stated as follows.

Theorem 5.1. *The sequence of ratios* $\left(\frac{\lambda_n}{\lambda_{n+1}}\right)_{n\in\mathbb{N}}$ *is bounded and has* 1 *as a limit point.*

That it is bounded can be proved easily as follows. Let $I \in \mathcal{C}$. For every $n \in \mathbb{N}$ there exists some non-negative integer m_0 such that $\lambda_n \leq \lambda(f^{m_0}(I))$ and $\lambda(f^m(I)) \leq \lambda_{n+1}$ for all $m > m_0$. Let $J = f^{m_0}(I)$. Then,

$$\frac{\lambda_n}{\lambda_{n+1}} \leq \frac{\lambda(J)}{\lambda(f(J))} \leq \frac{1}{\tau},$$

where $\tau = \inf\{f'(z) : z \in S^1\}$, from the mean value theorem.

It follows immediately from Theorem 5.1 that the standard ternary Cantor set is not the minimal set of any Denjoy C^1 diffeomorphism, since the sequence of ratios is constant with all terms equal to 3.

Theorem 5.1 says in other words that $\liminf_{n\to+\infty} \frac{\lambda_n}{\lambda_{n+1}} = 1$. What about lim sup?

McDuff's Question. Is it true that

$$\lim_{n\to+\infty} \frac{\lambda_n}{\lambda_{n+1}} = 1?$$

This is a natural question asked more than three decades ago by Dusa McDuff and remains still unanswered to the author's knowledge. Note that this is the case if in the example of Section 4 we choose g to be an irrational rotation. In the next section we shall show that an affirmative answer is implied by the solvability of a cohomological equation on K.

Proof of Theorem 5.1. We shall prove the conclusion by contradiction. We shall assume that $\liminf_{n\to+\infty} \frac{\lambda_n}{\lambda_{n+1}} > 1$ and prove that then for every $\epsilon > 0$ there exists an open set $U \subset S^1$ such that $U \cap K \neq \varnothing$ and $\lambda(f^n(I)) < \epsilon$ for every non-negative integer n and every $I \in \mathcal{C}$ with $I \subset U$. This is certainly a contradiction, because choosing any $I \in \mathcal{C}$ and $0 < \epsilon < \lambda(I)$ there exists a non-negative integer n (actually infinitely many) such that $f^{-n}(I) \subset U$ and at the same time $f^{-n}(I) \in \mathcal{C}$.

We start now the proof of the above assertion. Our assumption means that there exists $\rho > 0$ such that

$$1+\rho < \frac{\lambda_n}{\lambda_{n+1}} \le \frac{1}{\tau}$$

for every $n \in \mathbb{N}$, and so

$$(1+\rho)^m < \frac{\lambda_n}{\lambda_{n+1}} \cdot \frac{\lambda_{n+1}}{\lambda_{n+2}} \cdots \frac{\lambda_{n+m-1}}{\lambda_{n+m}} = \frac{\lambda_n}{\lambda_{n+m}} \le \frac{1}{\tau^m}$$

for every $m,\ n \in \mathbb{N}$.

For $I \in \mathcal{C}$ let $d(I) = n$ in case $\lambda(I) = \lambda_n$. Obviously, $d : \mathcal{C} \to \mathbb{N}$ is a surjective function, by definition, but may be not injective. As an intermediate step we shall prove now that K can be covered by a finite number of disjoint open intervals $A_1, \dots, A_r$, for some $r \in \mathbb{N}$, such that if $I,\ I' \in \mathcal{C}$ and $I,\ I' \subset A_j$ for some $1 \le j \le r$, then $d(I) \le d(I')$ implies $d(f(I)) \le d(f(I'))$. Let $m \in \mathbb{N}$ be such that $(1+\rho)^m > \max\{\frac{1}{\tau}, \|f'\|\}$. If $I \in \mathcal{C}$ is such that $d(I) = n > m$, then from the mean value theorem we have

$$\begin{aligned}\lambda_{n+m} &< \frac{\lambda_n}{(1+\rho)^m} < \tau\lambda_n = \tau\lambda(I) \le \lambda(f(I)) \le \|f'\|\lambda(I)\\ &= \|f'\|\lambda_n < \frac{\lambda_{n-m}}{\lambda_n} \cdot \lambda_n = \lambda_{n-m}.\end{aligned}$$

Consequently, $|d(f(I)) - d(I)| < m$. Let also $0 < \delta < \rho\tau^m$. Since f is C^1 and K is a Cantor set, there exists an open cover $\{A_1, \dots, A_r\}$ of K consisting of disjoint open intervals such that $|f'(z_1) - f'(z_2)| < \delta$ for $z_1, z_2 \in A_j,\ 1 \le j \le r$. Moreover, we may choose the open cover $\{A_1, \dots, A_r\}$ so fine that any connected component of $S^1 \setminus K$ contained in some $A_j,\ 1 \le j \le r$ has length smaller than λ_m. If $I,\ I' \in \mathcal{C}$ are contained in A_j for some $1 \le j \le r$, then $d(I),\ d(I') > m$ and from the mean value theorem it follows that

$$\left| \frac{\lambda(f(I))}{\lambda(I)} - \frac{\lambda(f(I'))}{\lambda(I')} \right| < \delta.$$

Suppose that $n = d(I) \le d(I')$. There is some $k \in \mathbb{Z}$ such that $d(f(I)) = n+k$. From the above observations we have $|k| = |d(f(I)) - d(I)| < m$ and so $\lambda_{n+k} > \lambda_{n+m}$. It follows that

$$\begin{aligned}\lambda(f(I')) &< \lambda_{n+k} + \delta\lambda_n < \lambda_{n+k} + \rho\tau^m\lambda_n \le \lambda_{n+k} + \rho\lambda_{n+m}\\ &< \lambda_{n+k}(1+\rho) < \lambda_{n+k-1}.\end{aligned}$$

Therefore $\lambda(f(I')) \le \lambda_{n+k}$, which means that $d(f(I)) \le d(f(I'))$.

Let now $\epsilon > 0$. Taking ϵ smaller, if necessary, we may assume that any $J \in \mathcal{C}$ with $\lambda(J) < \epsilon$ is contained in A_j for some $1 \le j \le r$. Let $\mathcal{C}_\epsilon$ denote the set of all $J \in \mathcal{C}$ such that $\lambda(f^n(J)) < \epsilon$ for every non-negative integer n. Note that for every $J \in \mathcal{C}$ there exists $n_0 \in \mathbb{N}$ such that $\lambda(f^n(J)) < \epsilon$ for all $n \ge n_0$, and therefore $f^{n_0}(J) \in \mathcal{C}_\epsilon$. There exists an open interval $U \subset S^1$ such that $U \cap K \ne \varnothing$ and one of the connected components I_0 of $S^1 \setminus K$ contained in U is of maximal length and belongs to $\mathcal{C}_\epsilon$.

We shall prove by induction the following claims:

(P_n) $f^n(U) \cap K \subset A_j$ for some $1 \le j \le r$ and
(Q_n) $d(f^n(I)) \ge d(f^n(I_0))$ for every $I \in \mathcal{C}$ with $I \subset U$.

Once we have proved the claims the conclusion is obvious, since by (Q_n) we shall have

$$\lambda(f^n(I)) \le \lambda(f^n(I_0)) < \epsilon$$

for every non-negative integer n. We proceed to prove the claims. Firstly, (Q_0) holds by the choice of U. If (P_n) and (Q_n) hold, it follows from the property of the open cover $\{A_1, \dots, A_r\}$ we proved in the intermediate step that (Q_{n+1}) holds. So it suffices to prove that (Q_n) implies (P_n). If (Q_n) is true and $I \in \mathcal{C}$ with $I \subset U$, then $\lambda(f^n(I)) \le \lambda(f^n(I_0)) < \epsilon$. Thus, $f^n(I) \subset A_j$ for some $1 \le j \le r$. Since U is connected and $A_1, \dots, A_r$ are disjoint, there exists some $1 \le j \le r$ such that $f^n(I) \subset A_j$ for every $I \in \mathcal{C}$ with $I \subset U$. Hence $f^n(U) \cap K = f^n(U \cap K) \subset A_j$. $\square$

6. The derivative of Denjoy C^1 diffeomorphisms

In this last section we shall relate McDuff's question to the problem of the solvability of the cohomological equation $\log f' = u - u \circ f$ on the unique Cantor minimal set K of a Denjoy C^1 diffeomorphism f. This depends on the behavior of the sequence of the derivatives of the iterates of f on K. It should be noted that in any case there is no continuous function $u : S^1 \to \mathbb{R}$ such that $\log f' = u - u \circ f$ on S^1. Indeed, if $I \subset S^1 \setminus K$ is a closed interval, then $(f^n(I))_{n\in\mathbb{Z}}$ is a family of mutually disjoint closed intervals. So,

$$\sum_{n=1}^{\infty} \int_I (f^n)' dz = \sum_{n=1}^{\infty} \int_{f^n(I)} dz = \lambda \left(\bigcup_{n=1}^{\infty} f^n(I) \right) \le 1,$$

where λ denotes the normalized Lebesgue measure on S^1. It follows that the series $\sum_{n=1}^{\infty} (f^n)'(z)$ converges for λ-almost all $z \in I$ and therefore $\lim_{n\to+\infty} (f^n)'(z) = 0$, λ-almost everywhere on I. Since $\lambda(I) > 0$, there exists a point $z \in S^1 \setminus K$ such that $\lim_{n\to+\infty} (f^n)'(z) = 0$. It follows from this that there exists no continuous function $u : S^1 \to \mathbb{R}$ such that $\log f' = u - u \circ f$ on S^1.

We proceed now to the description of the relation between the lengths of the complementary intervals of the unique Cantor minimal set of a Denjoy C^1 diffeomorphism $f : S^1 \to S^1$ with unique minimal set K and the behavior of its derivative on K. Let $(\lambda_n)_{n\in\mathbb{N}}$ be the spectrum of K. By Theorem 5.1, $\liminf_{n\to+\infty} \frac{\lambda_n}{\lambda_{n+1}} = 1$. Let

$$\sigma = \limsup_{n\to+\infty} \frac{\lambda_n}{\lambda_{n+1}}.$$

Proposition 6.1. *There exists a point $z_0 \in K$ such that $(f^n)'(z_0) \le \frac{1}{\sigma}$ for every $n \in \mathbb{N}$ and there exists a point $z_1 \in K$ such that $f'(z_1) \ge \sigma$.*

Proof. There exist positive integers $n_k \to +\infty$ such that $\sigma = \lim_{k\to+\infty} \frac{\lambda_{n_k}}{\lambda_{n_k+1}}$. For every $k \in \mathbb{N}$ there exists some $I_k \in \mathcal{C}$ such that $\lambda_{n_k} \le \lambda(I_k)$ and $\lambda(f^n(I_k)) \le \lambda_{n_k+1}$ for every $n \in \mathbb{N}$. Since $\lim_{k\to+\infty} \lambda(f(I_k)) = 0$, there exists an accumulation point $y \in K$ of $(f(I_k))_{k\in\mathbb{N}}$. So, $z_0 = f^{-1}(y)$ is an accumulation point of $(I_k)_{k\in\mathbb{N}}$, and passing to a subsequence, if necessary, we may assume that $\lim_{k\to+\infty}(\sup\{\mathrm{dist}(z_0, z) : z \in I_k\}) = 0$. On the other hand, from the mean value theorem we have

$$(f^n)'(z_0) = \lim_{k\to+\infty} \frac{\lambda(f^n(I_k))}{\lambda(I_k)}$$

for every $n \in \mathbb{N}$, because f is C^1. For every $\epsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that

$$\sigma - \epsilon < \frac{\lambda_{n_k}}{\lambda_{n_k+1}} \le \frac{\lambda(I_k)}{\lambda(f^n(I_k))}$$

for every $k \ge k_0$ and every $n \in \mathbb{N}$. Therefore, $(f^n)'(z_0) \le \frac{1}{\sigma-\epsilon}$ for every $\epsilon > 0$ and $n \in \mathbb{N}$, and the first assertion follows.

For the second assertion let $I \in \mathcal{C}$. For every $k \in \mathbb{N}$ there exists $m_k \in \mathbb{N}$ such that

$$\lambda(f^{-m_k}(I)) \le \lambda_{n_k+1} < \lambda_{n_k} \le \lambda(f^{-m_k+1}(I)).$$

By the mean value theorem, there exists $z_k \in f^{-m_k}(I)$ such that

$$\lambda(f^{-m_k+1}(I)) = f'(z_k)\lambda(f^{-m_k}(I)).$$

For every $\epsilon > 0$ there exists some $k_0 \in \mathbb{N}$ such that

$$f'(z_k) = \frac{\lambda(f^{-m_k+1}(I))}{\lambda(f^{-m_k}(I))} \ge \frac{\lambda_{n_k}}{\lambda_{n_k+1}} > \sigma - \epsilon$$

for every $k \ge k_0$. Since $\lim_{k\to+\infty} \lambda(f^{-m_k}(I)) = 0$, the sequence $(z_k)_{k\in\mathbb{N}}$ has an accumulation point $z \in K$. It follows that for every $\epsilon > 0$ there exists $z \in K$ such that $f'(z) \ge \sigma - \epsilon$, because f is C^1. For the same reason and the compactness of K we conclude that there exists a point $z_1 \in K$ such that $f'(z_1) \ge \sigma$. □

Proposition 6.2. *There exists a point $z_2 \in K$ such that $(f^n)'(z_2) \ge 1$ for every $n \in \mathbb{N}$.*

Proof. Suppose on the contrary that for every $z \in K$ there exists $n(z) \in \mathbb{N}$ such that $(f^{n(z)})'(z) < 1$. Since f is C^1, there exists an open neighborhood V_z of z in S^1 such that $(f^{n(z)})'(y) < 1$ for all $y \in V_z$. The open cover $\{V_z : z \in K\}$ has a finite refinement $\{I_1, \ldots, I_m\}$ consisting of open disjoint intervals covering K, because K is a Cantor set. For each integer $1 \le j \le m$ there exists some $n_j \in \mathbb{N}$ such that $(f^{n_j})'(y) < 1$ for all $y \in I_j$. Now $S^1 \setminus \bigcup_{j=1}^m I_j$ is a disjoint union of closed intervals $A_1, \ldots, A_m$. Each one of them is contained in a connected component of $S^1 \setminus K$ and is therefore wandering with respect to f. Let $N = \max\{n_j : 1 \le j \le m\}$ and $M = \sup\{f'(z) : z \in S^1\}$. Then $M \ge 1$, by Proposition 6.1. Let R be a connected component of $S^1 \setminus K$ such that $f^{-n}(A_1) \subset R$ for some $n \in \mathbb{N}$ and

$$\lambda(R) < \frac{1}{M^N} \cdot \min\{\lambda(A_1), \ldots, \lambda(A_m)\}.$$

Then there exists some integer $1 \le j \le m$ such that $R \subset I_j$. From the mean value theorem, there exists a point $\xi \in R$ such that

$$\frac{\lambda(f^{n_j}(R))}{\lambda(R)} = (f^{n_j})'(\xi) < 1$$

and so $\lambda(f^{n_j}(R)) < \lambda(R)$. Taking $f^{n_j}(R)$ in the place of R and repeating the above argument inductively we construct a sequence $(r_k)_{k\in\mathbb{N}}$ whose terms are elements of $\{n_1, \dots, n_m\}$ such that

$$\lambda(f^{r_1+\cdots+r_k}(R)) < \cdots < \lambda(f^{r_1}(R)) < \lambda(R)$$

for every $k \in \mathbb{N}$. For every $j \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that

$$r_1 + \cdots + r_k \le j < r_1 + \cdots + r_k + r_{k+1}$$

and then we have

$$\begin{aligned}\lambda(f^j(R)) &= \lambda(f^{j-(r_1+\cdots+r_k)}(f^{r_1+\cdots+r_k}(R))) \le M^N \cdot \lambda(f^{r_1+\cdots+r_k}(R)) \\ &< M^n \cdot \lambda(R) < \lambda(A_1).\end{aligned}$$

This contradicts our choice of R so that $A_1 \subset f^n(R)$ for some $n \in \mathbb{N}$. $\square$

Theorem 6.3. *If $\sigma > 1$, there exists a point $z \in K$ such that* $\inf\{(f^n)'(z) : n \in \mathbb{N}\} = 0$.

Proof. We shall prove that this holds for every point $z \in K$ such that $(f^n)'(z) \le \frac{1}{\sigma}$ for all $n \in \mathbb{N}$. The existence of this kind of points has been proved in Proposition 6.1. Suppose on the contrary that $a = \inf\{\log(f^n)'(z) : n \in \mathbb{N}\} > -\infty$. There exists some $n_0 \in \mathbb{N}$ such that

$$0 \le \log(f^{n_0})'(z) - a < \frac{1}{2}\log\sigma$$

and therefore for every $n \in \mathbb{N}$ we have

$$\log(f^n)'(f^{n_0}(z)) = \log(f^{n+n_0})'(z) - \log(f^{n_0})'(z) > -\frac{1}{2}\log\sigma.$$

Since K is minimal, there exists a sequence of positive integers $n_k \to +\infty$ such that $f^{-n_0}(z) = \lim_{k\to+\infty} f^{n_k}(z)$. Now we have

$$\begin{aligned}&\left|\log(f^{n_0+n_k})'(f^{n_0}(z)) - \log(f^{n_0+n_k})'(z)\right| \\ &\quad\le \sum_{j=0}^{n_0-1} \left|\log f'(f^{n_0+n_k+j}(z)) - \log f'(f^j(z))\right|.\end{aligned}$$

From the uniform continuity of $\log f'$ and the finite number of iterates $f, \dots, f^{n_0-1}$ it follows that for every $\epsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that

$$\sum_{j=0}^{n_0-1} \left|\log f'(f^{n_0+n_k+j}(z)) - \log f'(f^j(z))\right| < \epsilon$$

for every $k \ge k_0$. Consequently,

$$-\frac{1}{2}\log\sigma < \log(f^{n_0+n_k})'(f^{n_0}(z)) < \epsilon + \log(f^{n_0+n_k})'(z) \le \epsilon - \log\sigma$$

for every $k \ge k_0$. This means that $0 \le \frac{1}{2}\log\sigma < \epsilon$ for every $\epsilon > 0$, which contradicts our assumption that $\sigma > 1$. $\square$

Corollary 6.4. *If $\sigma > 1$, there is a point $z \in K$ such that* $\sup\{(f^n)'(z) : n \in \mathbb{N}\} = +\infty$.

Proof. From Proposition 6.2 there is a point $z \in K$ such that $(f^n)'(z) \geq 1$ for every $n \in \mathbb{N}$. By Theorem 6.3, there is a point $z_0 \in K$ such that $\inf\{(f^n)'(z_0) : n \in \mathbb{N}\} = 0$. For every $M \geq 1$ there is $N \in \mathbb{N}$ such that $(f^N)'(z_0) < \frac{1}{M}$. Since K is minimal, there is $n_0 \in \mathbb{N}$ such that $(f^N)'(f^{n_0}(z)) < \frac{1}{M}$. It follows that

$$(f^{n_0})'(z) = \frac{(f^{N+n_0})'(z)}{(f^N)'(f^{n_0}(z))} > M. \quad \square$$

From Theorem 6.3 we get immediately the following.

Corollary 6.5. *If $\sigma > 1$, there exists no continuous function $u : K \to \mathbb{R}$ such that* $\log f' = u - u \circ f$. $\square$

The question now arises whether there exists a continuous function $u : K \to \mathbb{R}$ such that $\log f' = u - u \circ f$ on K. In other words, is $\log f'$ a continuous coboundary on K (see p. 100 in [7])? Roughly speaking, in case $\log f'$ is a coboundary on K, the unique f-invariant Borel probability measure, whose support is K, is of "geometric nature".

The solvability of the cohomological equation $\log f' = u - u \circ f$ on K is C^1-invariant. More precisely, let $h : S^1 \to S^1$ be a C^1 diffeomorphism and $g = h \circ f \circ h^{-1}$. Then g is also a Denjoy C^1 diffeomorphism with Cantor minimal set $h(K)$. If there exists a continuous function $u : K \to \mathbb{R}$ such that $\log f' = u - u \circ f$ on K, then for $w = u \circ h^{-1} + \log(h^{-1})'$ we have $\log g' = w - w \circ g$ on $h(K)$.

The examples presented in Section 4 show that there are Denjoy C^1 diffeomorphisms for which the answer to the above question is affirmative and others for which is negative. More precisely, starting with an orientation preserving C^1 diffeomorphism $g : S^1 \to S^1$ which is topologically conjugate to an irrational rotation, we constructed in Section 4 a Denjoy C^1 diffeomorphism $f : S^1 \to S^1$ and an orientation preserving continuous surjection $h : S^1 \to S^1$ of degree 1 such that $h \circ f = g \circ h$. The construction has the additional feature that $f'(z) = g'(h(z))$ for all z in the Cantor minimal set K of f. So, if we choose in the beginning g to be an irrational rotation, then $f'(z) = 1$ for every $z \in K$ and $\log f'$ is a continuous coboundary on K. However, if we choose g with the property that there exists $z_0 \in S^1$ such that the sequence $(\log(g^n)'(z_0))_{n\in\mathbb{Z}}$ is unbounded, then $\log f'$ is not a continuous coboundary on K. This property of g is equivalent to saying that g is not C^1 conjugate to an irrational rotation. For a proof of this we refer to Theorem 6.1.1 on p. 48 of [5] or Lemma 3.2 on p. 53 of [3]. Examples of this kind of C^1 diffeomorphisms are included in the family $g_a : S^1 \to S^1, 0 < |a| < \frac{1}{2\pi}$, having corresponding lifts $G_a : \mathbb{R} \to \mathbb{R}$ given by the formula $G_a(x) = x + a + a \sin 2\pi x$. The set

$$P = \left\{a \in \left(-\frac{1}{2\pi}, \frac{1}{2\pi}\right) : \rho(G_a) \text{ is irrational}\right\}$$

is perfect, nowhere dense and g_a is not C^1 conjugate to an irrational rotation for a in a dense subset of P (see the corollary on p. 73 in [3]).

The following problem now arises.

Problem. Characterize the class of Denjoy C^1 diffeomorphisms for which the logarithm of the derivative is a continuous coboundary on their Cantor minimal set.

According to the theory developed in [1], which generalizes the Gottschalk–Hedlund theorem, the solvability of the above cohomological equation is closely related to the Cesaro summability of the distortion at certain intervals.

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